CURVILINEAR MOTION: RECTANGULAR COMPONENTS (Sections 12.4-12.5)

Today's Objectives:

Students will be able to:

a) Describe the motion of a particle traveling along a curved path.

b) Relate kinematic quantities in terms of the rectangular components of the vectors.

In-Class Activities:

- Applications
- General curvilinear motion
- Rectangular components of kinematic vectors
- Group problem solving

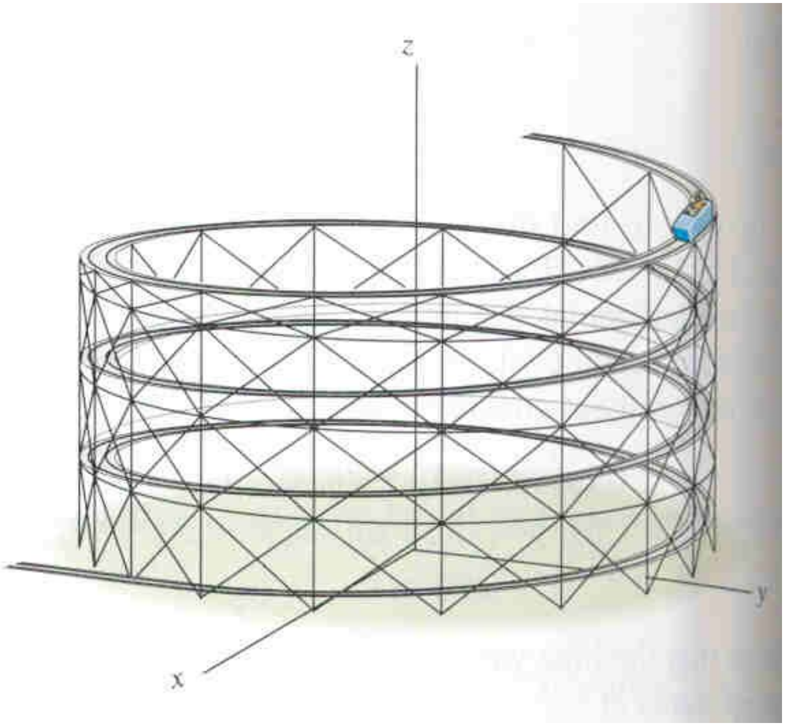
APPLICATIONS

The path of motion of each plane in this formation can be tracked with radar and their x, y, and z coordinates (relative to a point on earth) recorded as a function of time.

How can we determine the velocity or acceleration of each plane at any instant?

Should they be the same for each aircraft?

APPLICATIONS (continued)



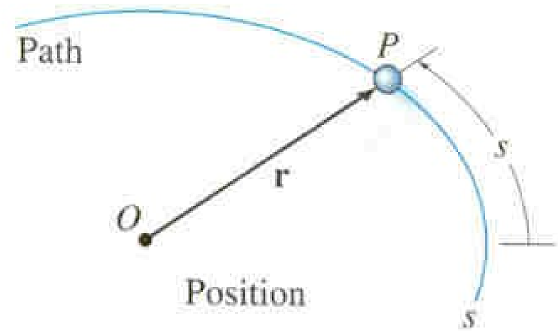
A roller coaster car travels down a fixed, helical path at a constant speed.

How can we determine its position or acceleration at any instant?

If you are designing the track, why is it important to be able to predict the acceleration of the car?

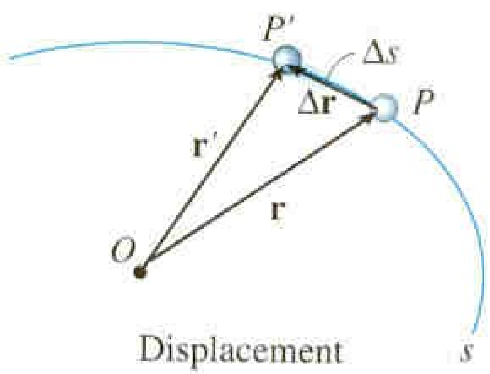
POSITION AND DISPLACEMENT

A particle moving along a curved path undergoes curvilinear motion. Since the motion is often three-dimensional, vectors are used to describe the motion.



A particle moves along a curve defined by the path function, s.

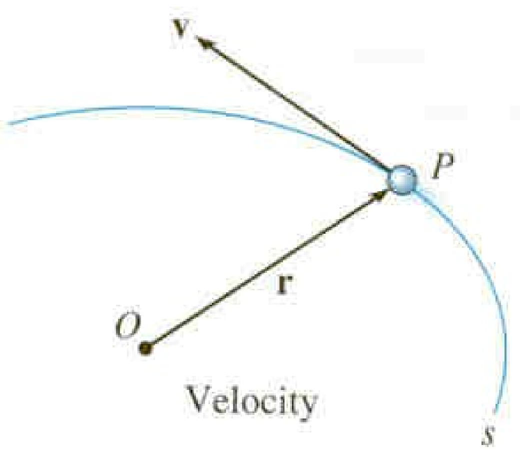
The position of the particle at any instant is designated by the vector $\boldsymbol{r} = \boldsymbol{r}(t)$. Both the magnitude and direction of $\boldsymbol{r}$ may vary with time.



If the particle moves a distance Δs along the curve during time interval Δt, the displacement is determined by vector subtraction: $\Delta \boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r}$

VELOCITY

Velocity represents the rate of change in the position of a particle.



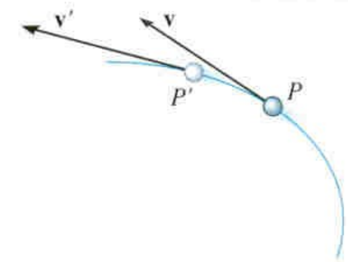
The average velocity of the particle during the time increment Δt is $\boldsymbol{v}_{avg} = \Delta \boldsymbol{r}/\Delta t$.

The instantaneous velocity is the time-derivative of position $\boldsymbol{v} = d\boldsymbol{r}/dt$.

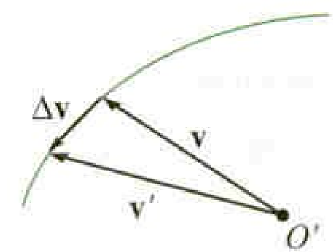
The velocity vector, $\boldsymbol{v}$, is always tangent to the path of motion.

The magnitude of $\boldsymbol{v}$ is called the speed. Since the arc length Δs approaches the magnitude of $\Delta \boldsymbol{r}$ as $t \rightarrow 0$, the speed can be obtained by differentiating the path function ($v = ds/dt$). Note that this is not a vector!

ACCELERATION

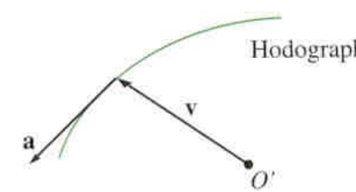


Acceleration represents the rate of change in the velocity of a particle.



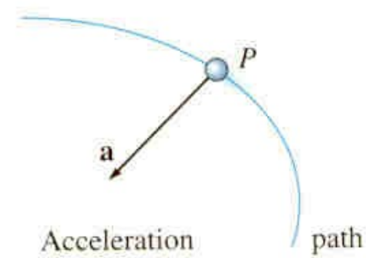
If a particle's velocity changes from ***v*** to ***v'*** over a time increment Δt, the average acceleration during that increment is:

$$\boldsymbol{a}_{avg} = \Delta \boldsymbol{v}/\Delta t = (\boldsymbol{v} - \boldsymbol{v}')/\Delta t$$



The instantaneous acceleration is the time-derivative of velocity:

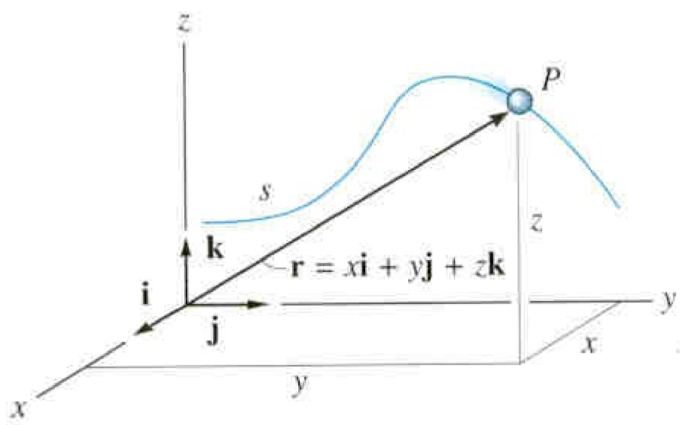
$$\boldsymbol{a} = d\boldsymbol{v}/dt = d^2\boldsymbol{r}/dt^2$$



A plot of the locus of points defined by the arrowhead of the velocity vector is called a hodograph. The acceleration vector is tangent to the hodograph, but not, in general, tangent to the path function.

RECTANGULAR COMPONENTS: POSITION

It is often convenient to describe the motion of a particle in terms of its x, y, z or rectangular components, relative to a fixed frame of reference.



Position

The position of the particle can be defined at any instant by the position vector

$$\boldsymbol{r} = \boldsymbol{x}\,\boldsymbol{i} + \text{y}\,\boldsymbol{j} + \text{z}\,\boldsymbol{k}\ .$$

The x, y, z components may all be functions of time, i.e., x = x(t), y = y(t), and z = z(t) .

The magnitude of the position vector is: $r = (x^2 + y^2 + z^2)^{0.5}$

The direction of $\boldsymbol{r}$ is defined by the unit vector: $\boldsymbol{\lambda}_r = (1/r)\boldsymbol{r}$

Note: I'm used to **λ** for a unit vector book uses **u**

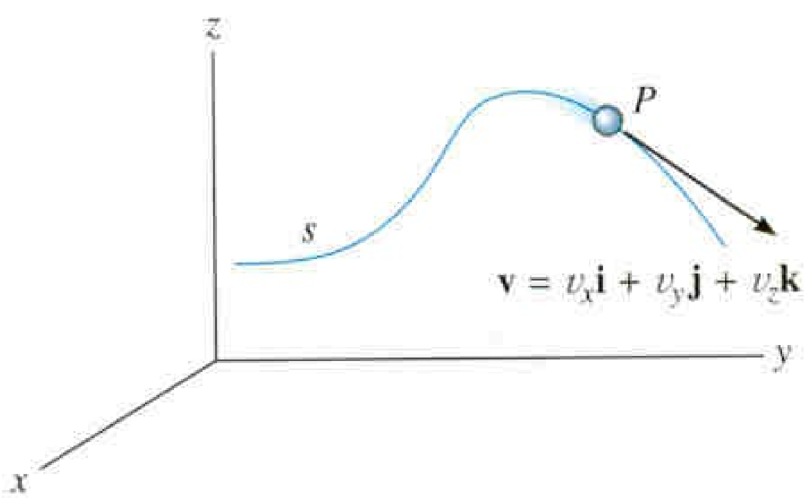
RECTANGULAR COMPONENTS: VELOCITY

The velocity vector is the time derivative of the position vector:

$$\boldsymbol{v} = d\boldsymbol{r}/dt = d(x\boldsymbol{i})/dt + d(y\boldsymbol{j})/dt + d(z\boldsymbol{k})/dt$$

Since the unit vectors ***i***, ***j***, ***k*** are constant in magnitude and direction, this equation reduces to $v = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}$

where $v_x = \dot{x} = dx/dt$, $v_y = \dot{y} = dy/dt$, $v_z = \dot{z} = dz/dt$



Velocity

The magnitude of the velocity vector is

$$v = [(v_x)^2 + (v_y)^2 + (v_z)^2]^{0.5}$$

The direction of ***v*** is tangent to the path of motion.

RECTANGULAR COMPONENTS: ACCELERATION

The acceleration vector is the time derivative of the velocity vector (second derivative of the position vector):

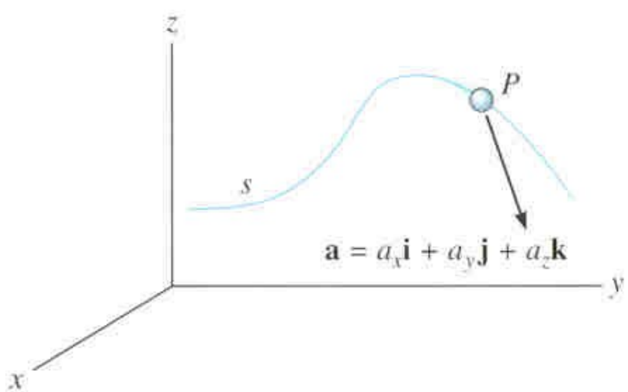
$$\boldsymbol{a} = d\boldsymbol{v}/dt = d^2\boldsymbol{r}/dt^2 = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$$

where $a_x = \dot{v}_x = \ddot{x} = dv_x/dt$, $a_y = \dot{v}_y = \ddot{y} = dv_y/dt$,

$a_z = \dot{v}_z = \ddot{z} = dv_z/dt$

The magnitude of the acceleration vector is

$$a = [(a_x)^2 + (a_y)^2 + (a_z)^2]^{0.5}$$



Acceleration

The direction of ***a*** is usually not tangent to the path of the particle.

EXAMPLE

Given: The motion of two particles (A and B) is described by the position vectors

$\boldsymbol{r}_A = [3t\,\boldsymbol{i} + 9t(2 - t)\,\boldsymbol{j}]$ m

$\boldsymbol{r}_B = [3(t^2 - 2t + 2)\,\boldsymbol{i} + 3(t - 2)\,\boldsymbol{j}]$ m

Find: The point at which the particles collide and their speeds just before the collision.

Plan:

1) The particles will collide when their position vectors are equal, or $\boldsymbol{r}_A = \boldsymbol{r}_B$.

2) Their speeds can be determined by differentiating the position vectors.

EXAMPLE (continued)

Solution:

1) The point of collision requires that $\boldsymbol{r}_A = \boldsymbol{r}_B$, so $x_A = x_B$ and $y_A = y_B$.

x-components: $3t = 3(t^2 - 2t + 2)$

Simplifying: $t^2 - 3t + 2 = 0$

Solving: $t = \{3 \pm [3^2 - 4(1)(2)]^{0.5}\}/2(1)$

=> $t = 2$ or 1 s

y-components: $9t(2 - t) = 3(t - 2)$

Simplifying: $3t^2 - 5t - 2 = 0$

Solving: $t = \{5 \pm [5^2 - 4(3)(-2)]^{0.5}\}/2(3)$

=> $t = 2$ or $-1/3$ s

So, the particles collide when $t = 2$ s. Substituting this value into $\boldsymbol{r}_A$ or $\boldsymbol{r}_B$ yields

$$x_A = x_B = 6 \text{ m} \quad \text{and} \quad y_A = y_B = 0$$

EXAMPLE (continued)

2) Differentiate $\boldsymbol{r_A}$ and $\boldsymbol{r_B}$ to get the velocity vectors.

$$\boldsymbol{v_A} = d\boldsymbol{r_A}/dt = \dot{x}_A \, \boldsymbol{i} + \dot{y}_A \, \boldsymbol{j} = [3\boldsymbol{i} + (18 - 18t)\boldsymbol{j}] \text{ m/s}$$

At t = 2 s: $\boldsymbol{v_A} = [3\boldsymbol{i} - 18\boldsymbol{j}]$ m/s

$$\boldsymbol{v_B} = d\boldsymbol{r_B}/dt = \dot{x}_B \boldsymbol{i} + \dot{y}_B \boldsymbol{j} = [(6t - 6)\boldsymbol{i} + 3\boldsymbol{j}] \text{ m/s}$$

At t = 2 s: $\boldsymbol{v_B} = [6\boldsymbol{i} + 3\boldsymbol{j}]$ m/s

Speed is the magnitude of the velocity vector.

$$v_A = (3^2 + 18^2)^{0.5} = 18.2 \text{ m/s}$$

$$v_B = (6^2 + 3^2)^{0.5} = 6.71 \text{ m/s}$$

GROUP PROBLEM SOLVING

Given: A particle travels along a path described by the parabola $y = 0.5x^2$. The x-component of velocity is given by $v_x = (5t)$ ft/s. When $t = 0$, $x = y = 0$.

Find: The particle's distance from the origin and the magnitude of its acceleration when $t = 1$ s.

Plan: Note that v_x is given as a function of time.

1) Determine the x-component of position and acceleration by integrating and differentiating v_x, respectively.
2) Determine the y-component of position from the parabolic equation and differentiate to get a_y.
3) Determine the magnitudes of the position and acceleration vectors.

GROUP PROBLEM SOLVING (continued)

Solution:

1) x-components:

Velocity: $v_x = \dot{x} = dx/dt = (5t)$ ft/s

Position: $\int_0^x dx = \int_0^t 5t\,dt \Rightarrow x = (5/2)t^2 = (2.5t^2)$ ft

Acceleration: $a_x = \ddot{x} = \dot{v}_x = d/dt\,(5t) = 5$ ft/s^2

2) y-components:

Position: $y = 0.5x^2 = 0.5(2.5t^2)^2 = (3.125t^4)$ ft

Velocity: $v_y = dy/dt = d\,(3.125t^4)\,/dt = (12.5t^3)$ ft/s

Acceleration: $a_y = \dot{v}_y = d\,(12.5t^3)\,/dt = (37.5t^2)$ ft/s^2

GROUP PROBLEM SOLVING (continued)

3) The distance from the origin is the magnitude of the position vector:

$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} = [2.5t^2\,\boldsymbol{i} + 3.125t^4\,\boldsymbol{j}]$ ft

At t = 1 s, $\boldsymbol{r} = (2.5\,\boldsymbol{i} + 3.125\,\boldsymbol{j})$ ft

Distance: $d = r = (2.5^2 + 3.125^2)^{0.5} = 4.0$ ft

The magnitude of the acceleration vector is calculated as:

Acceleration vector: $\boldsymbol{a} = [5\,\boldsymbol{i} + 37.5t^2\,\boldsymbol{j}]$ ft/s^2

Magnitude: $a = (5^2 + 37.5^2)^{0.5} = 37.8$ ft/s^2

Example

For a short time, the path of the plane in Fig. 12–19*a* is described by $y = (0.001x^2)$ m. If the plane is rising with a constant upward velocity of 10 m/s, determine the magnitudes of the velocity and acceleration of the plane when it reaches an altitude of $y = 100$ m.

SOLUTION

When $y = 100$ m, then $100 = 0.001x^2$ or $x = 316.2$ m. Also, due to constant velocity $v_y = 10$ m/s, so

$$y = v_y t; \qquad 100 \text{ m} = (10 \text{ m/s})\, t \qquad t = 10 \text{ s}$$

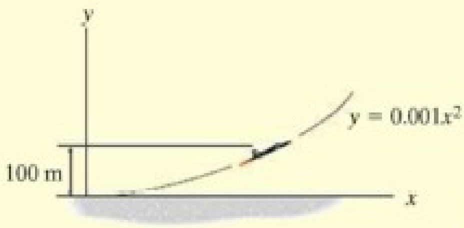
Velocity. Using the chain rule (see Appendix C) to find the relationship between the velocity components, we have

$$y = 0.001x^2$$

$$v_y = \dot{y} = \frac{d}{dt}(0.001x^2) = (0.002x)\dot{x} = 0.002xv_x \qquad (1)$$

Thus

$$10 \text{ m/s} = 0.002(316.2 \text{ m})(v_x)$$
$$v_x = 15.81 \text{ m/s}$$



(a)

The magnitude of the velocity is therefore

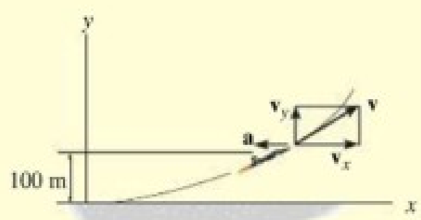
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(15.81 \text{ m/s})^2 + (10 \text{ m/s})^2} = 18.7 \text{ m/s} \qquad \textit{Ans.}$$

Acceleration. Using the chain rule, the time derivative of Eq. (1) gives the relation between the acceleration components.

$$a_y = \dot{v}_y = (0.002\dot{x})\dot{x} + 0.002x(\ddot{x}) = 0.002(v_x^2 + xa_x)$$

When $x = 316.2$ m, $v_x = 15.81$ m/s, $\dot{v}_y = a_y = 0$,

$$0 = 0.002\left[(15.81 \text{ m/s})^2 + 316.2 \text{ m}(a_x)\right]$$
$$a_x = -0.791 \text{ m/s}^2$$



(b)

Fig. 12–19

The magnitude of the plane's acceleration is therefore

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-0.791 \text{ m/s}^2)^2 + (0 \text{ m/s}^2)^2}$$
$$= 0.791 \text{ m/s}^2 \qquad \textit{Ans.}$$

These results are shown in Fig. 12–19*b*.

End of the Lecture

Let Learning Continue