Incomplete Block Designs

In certain experiments using randomized block designs, we may not be able to run all the treatment combinations in each block. Situation like this usually occur because of shortages of experimental apparatus or facilities or the physical size of the block. For example, we wish to compare brands of tires using the wheels as the experimental units with regard to a car being a block. Therefore all brands of tires cannot be compared in each car. For this type of problem it is possible to use randomized block designs in which every treatment is not present in every block. These designs are known as Randomized Incomplete Block Designs.

There are several special types of incomplete block designs, that have envolved in the process of refining the art of experimental design. These types being characterized by some common feature or property. One such type of design is the balanced incomplete block design (BIBD) introduced by Yates (1936a)

Balanced Incomplete Block Design (BIBD):

When all treatment comparisons are equally important then the treatment combinations used in each block should be selected in a balanced manner, so that any pair of treatments occur together the same number of times (say λ). Thus, a balanced incomplete block design (BIBD) is an incomplete block design in which any two treatments appear together an equal number of times.

⇒ An complete definition of BIBD:
An incomplete block designs is said to be a balanced incomplete block design (BIBD) if it satisfies the following conditions:
(i) The experimental material is divided into "b" blocks of "K" units each, different treatments being applied to the units in the same block
(ii) There are "t" treatments each of which occurs in "r" blocks / replications.
(iii) Any two treatments occur together exactly λ times.
The quantities t, b, r, K, and λ are called the parameters of the BIBD.

⇒ Notations:

t = Number of treatments.

b = Number of blocks.

K = Block size

r = Number of replications.

λ = time of occurrence together of any two treatments.

⇒ Properties of BIBD:

The following relations hold among the parameters and even these are necessary conditions for the existence of a BIBD.

(i) $K < t$ ⟹ Incomplete block design.

(ii) $rt = Kb$

(iii) $\lambda(t-1) = r(K-1)$

$$\boxed{\lambda = \frac{r(K-1)}{(t-1)}}$$ ⟹ formula to find λ.

(iv) $r > \lambda$

(v) $b \geq t$

⇒ Construction of BIBD:

Suppose that there are 4 treatments and that each block can hold exactly 3 treatments.

Block size is less than number of treatments so it is a incomplete block design.

A BIBD may be constructed by taking $\binom{t}{K}$ blocks and assigning a different combination of treatments to each block.

So

$$b = \binom{t}{K} = \binom{4}{3} = 4$$

$$r = Kb/t = (3)(4)/4 = 12/4 = 3$$

$$t = 4$$

$$K = 3$$

$$\lambda = \frac{r(K-1)}{(t-1)}$$

$$= \frac{3(3-1)}{4-1} = \frac{6}{3} = 2$$

$t = 4$, $b = 4$, $r = 3$, $K = 3$; $\lambda = 2$

B1	B2	B3	B4
t_1	t_4	t_3	t_2
t_2	t_1	t_4	t_3
t_3	t_2	t_1	t_4

In this design every pair of treatment appear together 2 times as $\lambda = 2$.
After than we can arranged it for analysis:

	B1	B2	B3	B4
t_1	y_{11}	y_{12}	y_{13}	—
t_2	y_{21}	y_{22}	—	y_{24}
t_3	y_{31}	—	y_{33}	y_{34}
t_4	—	y_{42}	y_{43}	y_{44}

★ Home-work.

Construct layout of BIBD for

(i) $t = 3$ & $K = 2$

(ii) $t = 5$ & $K = 4$

General Procedure for the numerical of BIBD

1- Formulation of Hypothesis:

H_0: All the treatment's effects are equal

H_1: Not all the treatment's effects are equal

2. Level of Significance:

$\alpha = ?$

3- Test Statistics:

$$F_{cal} = \frac{MST\,(Adjusted)}{MSE}$$

4- Calculation:

ANOVA Table

SOV	d.f	SS	MS	F
Treatments (adjusted for blocks)	$t-1$	$\frac{K\sum_{i=1}^{t} Q_i^2}{\lambda t}$	$SST_a / t-1$	MST_a / MSE
Blocks	$b-1$	$\frac{\sum B_j^2}{K} - \frac{(T_{..})^2}{n}$		
Error	By subtraction	By Subtraction	$SSE / it\ d.f$	
Total	$n-1$	$\sum\sum_{i,j} Y_{ij}^2 - \frac{(T_{..})^2}{n}$		

where

$$n = rt = Kb$$

& Q_i is the adjusted total for the ith treatment, which is computed as:

$$Q_i = T_{i.} - \frac{1}{K}\left(\text{Sum of all the blocks containing ith treatment}\right)$$

5- Critical Region:

If $F_{cal} \geq F_{\alpha(v_1, v_2)}$ then reject H_0 otherwise donot reject H_0.

6- Decision:

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Numerical:

A chemical engineer investigate the four catalysts by using batches of raw material as blocks to observing the reaction time. However, each batch is only large enough to permit three catalysts to be run. The balanced incomplete block design for this experiment, along with the observations recorded is shown in the following table.

Treatment (Catalyst)	Block (Batch of raw material) 1	2	3	4
1	73	74	–	71
2	–	75	67	72
3	73	75	68	–
4	75	–	72	75

Solution:

① H_0: The effect of all the catalysts are same

H_1: Not all the catalyst's effects are same.

②

$\alpha = 0.05$

③

$$F_{cal} = \frac{MST(\text{adjusted})}{MSE}$$

④

Treatment	Block 1	2	3	4	Treatment total ($T_{i.}$)	Q_i	Q_i^2
1	73	74	–	71	218	-3	9
2	–	75	67	72	214	-2.33	5.43
3	73	75	68	–	216	-1.33	1.77
4	75	–	72	75	222	6.67	44.49
Block total (B_j)	221	224	207	218	870	0	60.69

$b=4$, $r=3$, $K=3$, $t=4$

$$n = tr = bK$$

So

$$n = tr = (4)(3) = 12$$

$$\lambda = \frac{r(K-1)}{(t-1)} = \frac{3(3-1)}{(4-1)} = \frac{6}{3} = 2$$

(Total) $SST = \sum\sum Y_{ij}^2 - \frac{(T_{..})^2}{n}$

$$= (73)^2 + (74)^2 + \cdots + (75)^2 - \frac{(870)^2}{12}$$

$$= 63156 - 63075$$

$$= 81$$

$$SSB = \frac{\sum B_j^2}{K} - \frac{(T_{..})^2}{n}$$

$$= \frac{(221)^2 + (224)^2 + (207)^2 + (218)^2}{3} - \frac{(870)^2}{12}$$

$$= 63130 - 63075$$

$$= 55$$

To compute the treatment sum of squares adjusted for blocks, we first determine the adjusted treatment totals. by following formula.

$Q_i = T_i - \frac{1}{K}$ (Sum of all the blocks containing ith treatment).

$$Q_1 = 218 - \frac{1}{3}(221 + 224 + 218)$$

$$= 218 - 221$$

$$= -3$$

$$Q_2 = 214 - \frac{1}{3}(224 + 207 + 218)$$

$$= 214 - 216.33$$

$$= -2.33$$

$$Q_3 = 216 - \frac{1}{3}(221 + 224 + 207)$$

$$= 216 - 217.33$$

$$= -1.33$$

$$Q_4 = 222 - \frac{1}{3}(221 + 207 + 218)$$

$$= 222 - 215.33$$

$$= 6.67$$

$$SST_{(adjusted)} = \frac{K \sum Q_i^2}{\lambda t}$$

$$= \frac{3(60.69)}{(2)(4)} = 22.76$$

$$SSE = SST - SSB - SST_{(adjusted)}$$

$$= 81 - 55 - 22.76$$

$$= 3.24$$

ANOVA Table

SOV	d.f	SS	MS	F
Treatment (adjusted)	4-1=3	22.76	7.59	11.68 > 5.41
Block	4-1=3	55	—	
Error	11-3-3=5	3.24	0.65	
Total	12-1=11	81		

(5) $F_{\alpha(v_1, v_2)} = F_{0.05(3,5)} = 5.41$

If $F_{cal} \geq 5.41$ then reject H_0 otherwise don't reject H_0.

(6) As $11.68 > 5.41$, so we reject our null hypothesis and conclude that not all the catalyst's effects are same.