chapter

9

Sinusoids and Phasors

He who knows not, and knows not that he knows not, is a fool—shun him. He who knows not, and knows that he knows not, is a child—teach him. He who knows, and knows not that he knows, is asleep—wake him up. He who knows, and knows that he knows, is wise—follow him.

—Persian Proverb

Enhancing Your Skills and Your Career

ABET EC 2000 criteria (3.d), ***"an ability to function on multi-disciplinary teams."***

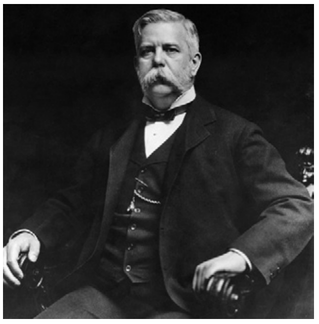
The "ability to function on multidisciplinary teams" is inherently critical for the working engineer. Engineers rarely, if ever, work by themselves. Engineers will always be part of some team. One of the things I like to remind students is that you do not have to like everyone on a team; you just have to be a successful part of that team.

Most frequently, these teams include individuals from a variety of engineering disciplines, as well as individuals from nonengineering disciplines such as marketing and finance.

Students can easily develop and enhance this skill by working in study groups in every course they take. Clearly, working in study groups in nonengineering courses, as well as engineering courses outside your discipline, will also give you experience with multidisciplinary teams.

Photo by Charles Alexander

Historical



George Westinghouse. Photo © Bettmann/Corbis

Nikola Tesla (1856–1943) and **George Westinghouse** (1846–1914) helped establish alternating current as the primary mode of electricity transmission and distribution.

Today it is obvious that ac generation is well established as the form of electric power that makes widespread distribution of electric power efficient and economical. However, at the end of the 19th century, which was the better—ac or dc—was hotly debated and had extremely outspoken supporters on both sides. The dc side was led by Thomas Edison, who had earned a lot of respect for his many contributions. Power generation using ac really began to build after the successful contributions of Tesla. The real commercial success in ac came from George Westinghouse and the outstanding team, including Tesla, he assembled. In addition, two other big names were C. F. Scott and B. G. Lamme.

The most significant contribution to the early success of ac was the patenting of the polyphase ac motor by Tesla in 1888. The induction motor and polyphase generation and distribution systems doomed the use of dc as the prime energy source.

9.1 Introduction

Thus far our analysis has been limited for the most part to dc circuits: those circuits excited by constant or time-invariant sources. We have restricted the forcing function to dc sources for the sake of simplicity, for pedagogic reasons, and also for historic reasons. Historically, dc sources were the main means of providing electric power up until the late 1800s. At the end of that century, the battle of direct current versus alternating current began. Both had their advocates among the electrical engineers of the time. Because ac is more efficient and economical to transmit over long distances, ac systems ended up the winner. Thus, it is in keeping with the historical sequence of events that we considered dc sources first.

We now begin the analysis of circuits in which the source voltage or current is time-varying. In this chapter, we are particularly interested in sinusoidally time-varying excitation, or simply, excitation by a *sinusoid.*

> A **sinusoid** is a signal that has the form of the sine or cosine function.

A sinusoidal current is usually referred to as *alternating current* (*ac*). Such a current reverses at regular time intervals and has alternately positive and negative values. Circuits driven by sinusoidal current or voltage sources are called *ac circuits.*

We are interested in sinusoids for a number of reasons. First, nature itself is characteristically sinusoidal. We experience sinusoidal variation in the motion of a pendulum, the vibration of a string, the ripples on the ocean surface, and the natural response of underdamped second-order systems, to mention but a few. Second, a sinusoidal signal is easy to generate and transmit. It is the form of voltage generated throughout the world and supplied to homes, factories, laboratories, and so on. It is the dominant form of signal in the communications and electric power industries. Third, through Fourier analysis, any practical periodic signal can be represented by a sum of sinusoids. Sinusoids, therefore, play an important role in the analysis of periodic signals. Lastly, a sinusoid is easy to handle mathematically. The derivative and integral of a sinusoid are themselves sinusoids. For these and other reasons, the sinusoid is an extremely important function in circuit analysis.

A sinusoidal forcing function produces both a transient response and a steady-state response, much like the step function, which we studied in Chapters 7 and 8. The transient response dies out with time so that only the steady-state response remains. When the transient response has become negligibly small compared with the steady-state response, we say that the circuit is operating at sinusoidal steady state. It is this *sinusoidal steady-state response* that is of main interest to us in this chapter.

We begin with a basic discussion of sinusoids and phasors. We then introduce the concepts of impedance and admittance. The basic circuit laws, Kirchhoff's and Ohm's, introduced for dc circuits, will be applied to ac circuits. Finally, we consider applications of ac circuits in phase-shifters and bridges.

9.2 Sinusoids

Consider the sinusoidal voltage

$$v(t) = V_m \sin \omega t \tag{9.1}$$

where

V_m = the *amplitude* of the sinusoid
ω = the *angular frequency* in radians/s
ωt = the *argument* of the sinusoid

The sinusoid is shown in Fig. 9.1(a) as a function of its argument and in Fig. 9.1(b) as a function of time. It is evident that the sinusoid repeats itself every T seconds; thus, T is called the *period* of the sinusoid. From the two plots in Fig. 9.1, we observe that $\omega T = 2\pi$,

$$T = \frac{2\pi}{\omega} \tag{9.2}$$

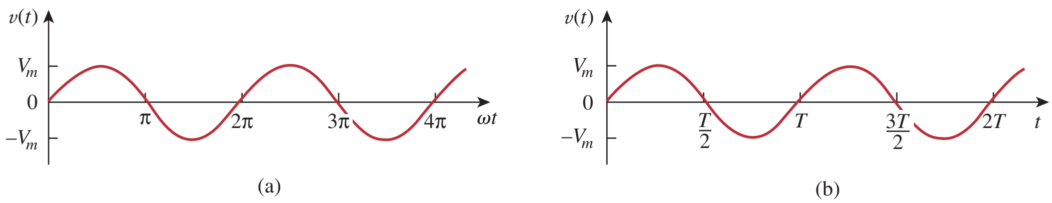
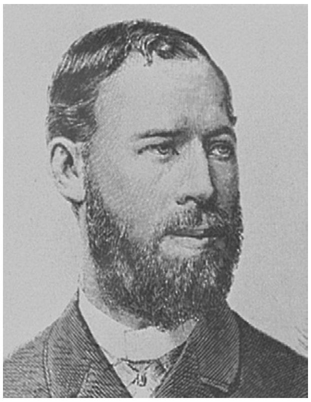


Figure 9.1
A sketch of $V_m \sin \omega t$: (a) as a function of ωt, (b) as a function of t.

Historical



The Burndy Library Collection at The Huntington Library, San Marino, California.

Heinrich Rudorf Hertz (1857–1894), a German experimental physicist, demonstrated that electromagnetic waves obey the same fundamental laws as light. His work confirmed James Clerk Maxwell's celebrated 1864 theory and prediction that such waves existed.

Hertz was born into a prosperous family in Hamburg, Germany. He attended the University of Berlin and did his doctorate under the prominent physicist Hermann von Helmholtz. He became a professor at Karlsruhe, where he began his quest for electromagnetic waves. Hertz successfully generated and detected electromagnetic waves; he was the first to show that light is electromagnetic energy. In 1887, Hertz noted for the first time the photoelectric effect of electrons in a molecular structure. Although Hertz only lived to the age of 37, his discovery of electromagnetic waves paved the way for the practical use of such waves in radio, television, and other communication systems. The unit of frequency, the hertz, bears his name.

The fact that $v(t)$ repeats itself every T seconds is shown by replacing t by $t + T$ in Eq. (9.1). We get

$$\begin{aligned} v(t+T) &= V_m \sin\omega(t+T) = V_m \sin\omega\left(t + \frac{2\pi}{\omega}\right) \\ &= V_m \sin(\omega t + 2\pi) = V_m \sin\omega t = v(t) \end{aligned} \tag{9.3}$$

Hence,

$$v(t+T) = v(t) \tag{9.4}$$

that is, v has the same value at $t + T$ as it does at t and $v(t)$ is said to be *periodic*. In general,

> A **periodic function** is one that satisfies $f(t) = f(t + nT)$, for all t and for all integers n.

As mentioned, the *period* T of the periodic function is the time of one complete cycle or the number of seconds per cycle. The reciprocal of this quantity is the number of cycles per second, known as the *cyclic frequency* f of the sinusoid. Thus,

$$f = \frac{1}{T} \tag{9.5}$$

From Eqs. (9.2) and (9.5), it is clear that

$$\omega = 2\pi f \tag{9.6}$$

While ω is in radians per second (rad/s), f is in hertz (Hz).

The unit of f is named after the German physicist Heinrich R. Hertz (1857–1894).

Let us now consider a more general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \phi) \tag{9.7}$$

where $(\omega t + \phi)$ is the argument and ϕ is the *phase*. Both argument and phase can be in radians or degrees.

Let us examine the two sinusoids

$$v_1(t) = V_m \sin \omega t \quad \text{and} \quad v_2(t) = V_m \sin(\omega t + \phi) \tag{9.8}$$

shown in Fig. 9.2. The starting point of v_2 in Fig. 9.2 occurs first in time. Therefore, we say that v_2 *leads* v_1 by ϕ or that v_1 *lags* v_2 by ϕ. If $\phi \neq 0$, we also say that v_1 and v_2 are *out of phase*. If $\phi = 0$, then v_1 and v_2 are said to be *in phase*; they reach their minima and maxima at exactly the same time. We can compare v_1 and v_2 in this manner because they operate at the same frequency; they do not need to have the same amplitude.

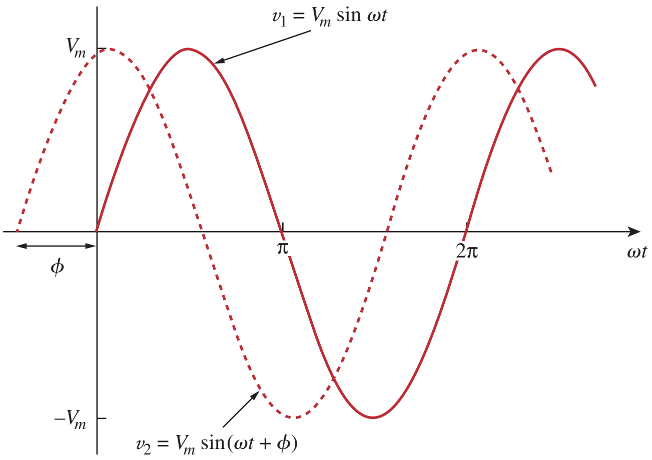


Figure 9.2
Two sinusoids with different phases.

A sinusoid can be expressed in either sine or cosine form. When comparing two sinusoids, it is expedient to express both as either sine or cosine with positive amplitudes. This is achieved by using the following trigonometric identities:

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \end{aligned} \tag{9.9}$$

With these identities, it is easy to show that

$$\begin{aligned} \sin(\omega t \pm 180^\circ) &= -\sin \omega t \\ \cos(\omega t \pm 180^\circ) &= -\cos \omega t \\ \sin(\omega t \pm 90^\circ) &= \pm \cos \omega t \\ \cos(\omega t \pm 90^\circ) &= \mp \sin \omega t \end{aligned} \tag{9.10}$$

Using these relationships, we can transform a sinusoid from sine form to cosine form or vice versa.

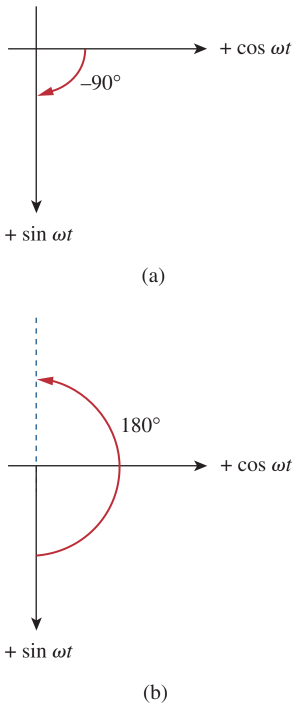


Figure 9.3
A graphical means of relating cosine and sine: (a) $\cos(\omega t - 90^\circ) = \sin\omega t$, (b) $\sin(\omega t + 180^\circ) = -\sin\omega t$.

A graphical approach may be used to relate or compare sinusoids as an alternative to using the trigonometric identities in Eqs. (9.9) and (9.10). Consider the set of axes shown in Fig. 9.3(a). The horizontal axis represents the magnitude of cosine, while the vertical axis (pointing down) denotes the magnitude of sine. Angles are measured positively counterclockwise from the horizontal, as usual in polar coordinates. This graphical technique can be used to relate two sinusoids. For example, we see in Fig. 9.3(a) that subtracting 90° from the argument of $\cos\omega t$ gives $\sin\omega t$, or $\cos(\omega t - 90^\circ) = \sin\omega t$. Similarly, adding 180° to the argument of $\sin\omega t$ gives $-\sin\omega t$, or $\sin(\omega t + 180^\circ) = -\sin\omega t$, as shown in Fig. 9.3(b).

The graphical technique can also be used to add two sinusoids of the same frequency when one is in sine form and the other is in cosine form. To add $A\cos\omega t$ and $B\sin\omega t$, we note that A is the magnitude of $\cos\omega t$ while B is the magnitude of $\sin\omega t$, as shown in Fig. 9.4(a). The magnitude and argument of the resultant sinusoid in cosine form is readily obtained from the triangle. Thus,

$$A\cos\omega t + B\sin\omega t = C\cos(\omega t - \theta) \tag{9.11}$$

where

$$C = \sqrt{A^2 + B^2}, \qquad \theta = \tan^{-1}\frac{B}{A} \tag{9.12}$$

For example, we may add $3\cos\omega t$ and $-4\sin\omega t$ as shown in Fig. 9.4(b) and obtain

$$3\cos\omega t - 4\sin\omega t = 5\cos(\omega t + 53.1^\circ) \tag{9.13}$$

Compared with the trigonometric identities in Eqs. (9.9) and (9.10), the graphical approach eliminates memorization. However, we must not confuse the sine and cosine axes with the axes for complex numbers to be discussed in the next section. Something else to note in Figs. 9.3 and 9.4 is that although the natural tendency is to have the vertical axis point up, the positive direction of the sine function is down in the present case.

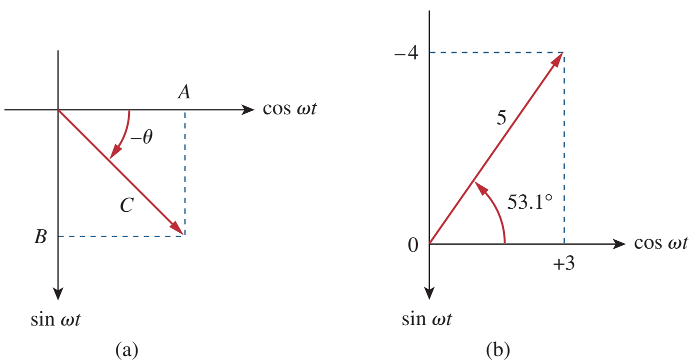


Figure 9.4
(a) Adding $A\cos\omega t$ and $B\sin\omega t$, (b) adding $3\cos\omega t$ and $-4\sin\omega t$.

Example 9.1

Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12\cos(50t + 10^\circ)$$

Solution:

The amplitude is $V_m = 12$ V.
The phase is $\phi = 10^\circ$.
The angular frequency is $\omega = 50$ rad/s.
The period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{50} = 0.1257$ s.
The frequency is $f = \dfrac{1}{T} = 7.958$ Hz.

Practice Problem 9.1

Given the sinusoid $30\sin(4\pi t - 75^\circ)$, calculate its amplitude, phase, angular frequency, period, and frequency.

Answer: 30, -75°, 12.57 rad/s, 0.5 s, 2 Hz.

Example 9.2

Calculate the phase angle between $v_1 = -10\cos(\omega t + 50^\circ)$ and $v_2 = 12\sin(\omega t - 10^\circ)$. State which sinusoid is leading.

Solution:
Let us calculate the phase in three ways. The first two methods use trigonometric identities, while the third method uses the graphical approach.

■ METHOD 1 In order to compare v_1 and v_2, we must express them in the same form. If we express them in cosine form with positive amplitudes,

$$v_1 = -10\cos(\omega t + 50^\circ) = 10\cos(\omega t + 50^\circ - 180^\circ)$$
$$v_1 = 10\cos(\omega t - 130^\circ) \quad \text{or} \quad v_1 = 10\cos(\omega t + 230^\circ) \tag{9.2.1}$$

and

$$v_2 = 12\sin(\omega t - 10^\circ) = 12\cos(\omega t - 10^\circ - 90^\circ)$$
$$v_2 = 12\cos(\omega t - 100^\circ) \tag{9.2.2}$$

It can be deduced from Eqs. (9.2.1) and (9.2.2) that the phase difference between v_1 and v_2 is 30°. We can write v_2 as

$$v_2 = 12\cos(\omega t - 130^\circ + 30^\circ) \quad \text{or} \quad v_2 = 12\cos(\omega t + 260^\circ) \tag{9.2.3}$$

Comparing Eqs. (9.2.1) and (9.2.3) shows clearly that v_2 leads v_1 by 30°.

■ METHOD 2 Alternatively, we may express v_1 in sine form:

$$v_1 = -10\cos(\omega t + 50^\circ) = 10\sin(\omega t + 50^\circ - 90^\circ)$$
$$= 10\sin(\omega t - 40^\circ) = 10\sin(\omega t - 10^\circ - 30^\circ)$$

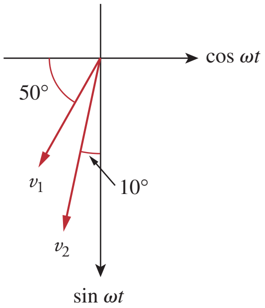


Figure 9.5
For Example 9.2.

But $v_2 = 12 \sin(\omega t - 10^\circ)$. Comparing the two shows that v_1 lags v_2 by 30°. This is the same as saying that v_2 leads v_1 by 30°.

■ **METHOD 3** We may regard v_1 as simply $-10 \cos\omega t$ with a phase shift of $+50^\circ$. Hence, v_1 is as shown in Fig. 9.5. Similarly, v_2 is $12 \sin\omega t$ with a phase shift of -10°, as shown in Fig. 9.5. It is easy to see from Fig. 9.5 that v_2 leads v_1 by 30°, that is, $90^\circ - 50^\circ - 10^\circ$.

Practice Problem 9.2

Find the phase angle between

$$i_1 = -4 \sin(377t + 55^\circ) \qquad \text{and} \qquad i_2 = 5 \cos(377t - 65^\circ)$$

Does i_1 lead or lag i_2?

Answer: 210°, i_1 leads i_2.

9.3 Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions.

> A **phasor** is a complex number that represents the amplitude and phase of a sinusoid.

Phasors provide a simple means of analyzing linear circuits excited by sinusoidal sources; solutions of such circuits would be intractable otherwise. The notion of solving ac circuits using phasors was first introduced by Charles Steinmetz in 1893. Before we completely define phasors and apply them to circuit analysis, we need to be thoroughly familiar with complex numbers.

Charles Proteus Steinmetz (1865–1923) was a German-Austrian mathematician and electrical engineer.

Appendix B presents a short tutorial on complex numbers.

A complex number z can be written in rectangular form as

$$z = x + jy \tag{9.14a}$$

where $j = \sqrt{-1}$; x is the real part of z; y is the imaginary part of z. In this context, the variables x and y do not represent a location as in two-dimensional vector analysis but rather the real and imaginary parts of z in the complex plane. Nevertheless, we note that there are some resemblances between manipulating complex numbers and manipulating two-dimensional vectors.

The complex number z can also be written in polar or exponential form as

$$z = r\underline{/\phi} = re^{j\phi} \tag{9.14b}$$

Historical

Charles Proteus Steinmetz (1865–1923), a German-Austrian mathematician and engineer, introduced the phasor method (covered in this chapter) in ac circuit analysis. He is also noted for his work on the theory of hysteresis.

Steinmetz was born in Breslau, Germany, and lost his mother at the age of one. As a youth, he was forced to leave Germany because of his political activities just as he was about to complete his doctoral dissertation in mathematics at the University of Breslau. He migrated to Switzerland and later to the United States, where he was employed by General Electric in 1893. That same year, he published a paper in which complex numbers were used to analyze ac circuits for the first time. This led to one of his many textbooks, *Theory and Calculation of ac Phenomena*, published by McGraw-Hill in 1897. In 1901, he became the president of the American Institute of Electrical Engineers, which later became the IEEE.

where r is the magnitude of z, and ϕ is the phase of z. We notice that z can be represented in three ways:

$$\begin{aligned} z &= x + jy && \text{Rectangular form} \\ z &= r\underline{/\phi} && \text{Polar form} \\ z &= re^{j\phi} && \text{Exponential form} \end{aligned} \qquad \textbf{(9.15)}$$

The relationship between the rectangular form and the polar form is shown in Fig. 9.6, where the x axis represents the real part and the y axis represents the imaginary part of a complex number. Given x and y, we can get r and ϕ as

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x} \qquad \textbf{(9.16a)}$$

On the other hand, if we know r and ϕ, we can obtain x and y as

$$x = r\cos\phi, \qquad y = r\sin\phi \qquad \textbf{(9.16b)}$$

Thus, z may be written as

$$\boxed{z = x + jy = r\underline{/\phi} = r(\cos\phi + j\sin\phi)} \qquad \textbf{(9.17)}$$

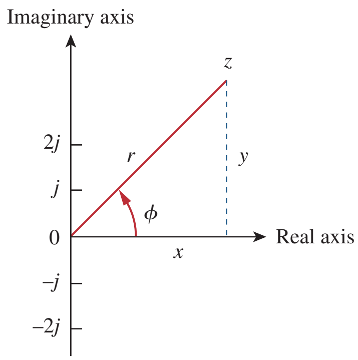


Figure 9.6
Representation of a complex number $z = x + jy = r\underline{/\phi}$.

Addition and subtraction of complex numbers are better performed in rectangular form; multiplication and division are better done in polar form. Given the complex numbers

$$z = x + jy = r\underline{/\phi}, \qquad z_1 = x_1 + jy_1 = r_1\underline{/\phi_1}$$
$$z_2 = x_2 + jy_2 = r_2\underline{/\phi_2}$$

the following operations are important.

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \qquad \textbf{(9.18a)}$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \tag{9.18b}$$

Multiplication:

$$z_1 z_2 = r_1 r_2 \underline{/\phi_1 + \phi_2} \tag{9.18c}$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \underline{/\phi_1 - \phi_2} \tag{9.18d}$$

Reciprocal:

$$\frac{1}{z} = \frac{1}{r} \underline{/-\phi} \tag{9.18e}$$

Square Root:

$$\sqrt{z} = \sqrt{r} \underline{/\phi/2} \tag{9.18f}$$

Complex Conjugate:

$$z^* = x - jy = r\underline{/-\phi} = re^{-j\phi} \tag{9.18g}$$

Note that from Eq. (9.18e),

$$\frac{1}{j} = -j \tag{9.18h}$$

These are the basic properties of complex numbers we need. Other properties of complex numbers can be found in Appendix B.

The idea of phasor representation is based on Euler's identity. In general,

$$\boxed{e^{\pm j\phi} = \cos\phi \pm j\sin\phi} \tag{9.19}$$

which shows that we may regard $\cos\phi$ and $\sin\phi$ as the real and imaginary parts of $e^{j\phi}$; we may write

$$\cos\phi = \text{Re}(e^{j\phi}) \tag{9.20a}$$

$$\sin\phi = \text{Im}(e^{j\phi}) \tag{9.20b}$$

where Re and Im stand for the *real part of* and the *imaginary part of*. Given a sinusoid $v(t) = V_m \cos(\omega t + \phi)$, we use Eq. (9.20a) to express $v(t)$ as

$$v(t) = V_m \cos(\omega t + \phi) = \text{Re}(V_m e^{j(\omega t + \phi)}) \tag{9.21}$$

or

$$v(t) = \text{Re}(V_m e^{j\phi} e^{j\omega t}) \tag{9.22}$$

Thus,

$$\boxed{v(t) = \text{Re}(\mathbf{V} e^{j\omega t})} \tag{9.23}$$

where

$$\mathbf{V} = V_m e^{j\phi} = V_m \underline{/\phi} \tag{9.24}$$

V is thus the *phasor representation* of the sinusoid $v(t)$, as we said earlier. In other words, a phasor is a complex representation of the magnitude and phase of a sinusoid. Either Eq. (9.20a) or Eq. (9.20b) can be used to develop the phasor, but the standard convention is to use Eq. (9.20a).

A phasor may be regarded as a mathematical equivalent of a sinusoid with the time dependence dropped.

One way of looking at Eqs. (9.23) and (9.24) is to consider the plot of the *sinor* $\mathbf{V}e^{j\omega t} = V_m e^{j(\omega t+\phi)}$ on the complex plane. As time increases, the sinor rotates on a circle of radius V_m at an angular velocity ω in the counterclockwise direction, as shown in Fig. 9.7(a). We may regard $v(t)$ as the projection of the sinor $\mathbf{V}e^{j\omega t}$ on the real axis, as shown in Fig. 9.7(b). The value of the sinor at time $t = 0$ is the phasor **V** of the sinusoid $v(t)$. The sinor may be regarded as a rotating phasor. Thus, whenever a sinusoid is expressed as a phasor, the term $e^{j\omega t}$ is implicitly present. It is therefore important, when dealing with phasors, to keep in mind the frequency ω of the phasor; otherwise we can make serious mistakes.

If we use sine for the phasor instead of cosine, then $v(t) = V_m \sin(\omega t + \phi) = \text{Im}(V_m e^{j(\omega t+\phi)})$ and the corresponding phasor is the same as that in Eq. (9.24).

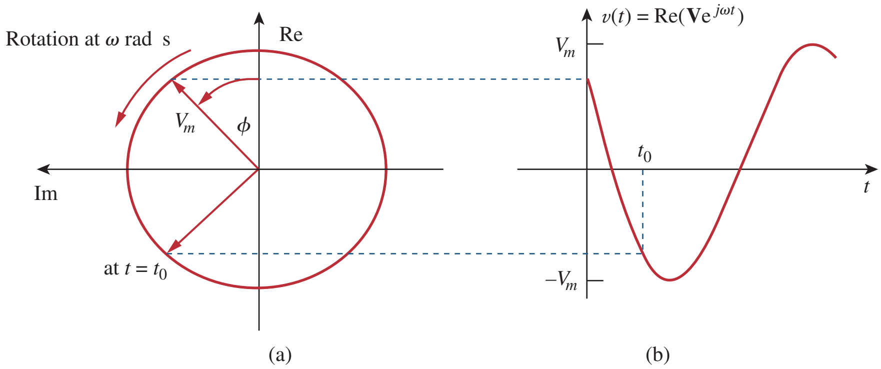


Figure 9.7
Representation of $\mathbf{V}e^{j\omega t}$: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

Equation (9.23) states that to obtain the sinusoid corresponding to a given phasor **V**, multiply the phasor by the time factor $e^{j\omega t}$ and take the real part. As a complex quantity, a phasor may be expressed in rectangular form, polar form, or exponential form. Since a phasor has magnitude and phase ("direction"), it behaves as a vector and is printed in boldface. For example, phasors $\mathbf{V} = V_m\underline{/\phi}$ and $\mathbf{I} = I_m\underline{/-\theta}$ are graphically represented in Fig. 9.8. Such a graphical representation of phasors is known as a *phasor diagram.*

We use lightface italic letters such as z to represent complex numbers but boldface letters such as **V** to represent phasors, because phasors are vectorlike quantities.

Equations (9.21) through (9.23) reveal that to get the phasor corresponding to a sinusoid, we first express the sinusoid in the cosine form so that the sinusoid can be written as the real part of a complex number. Then we take out the time factor $e^{j\omega t}$, and whatever is left is the phasor corresponding to the sinusoid. By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m\underline{/\phi} \qquad \textbf{(9.25)}$$

(Time-domain representation) (Phasor-domain representation)

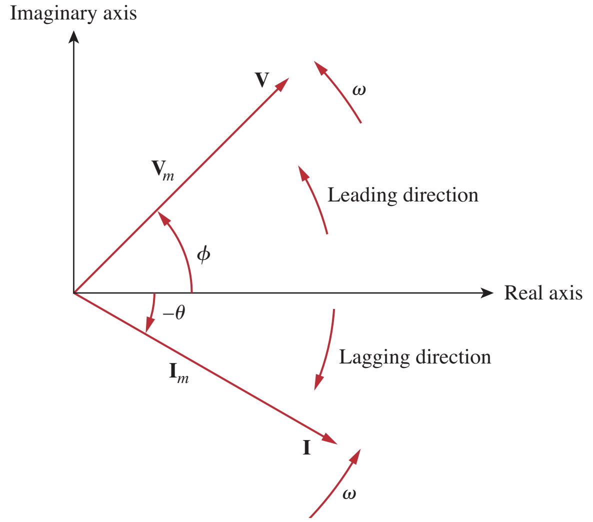


Figure 9.8
A phasor diagram showing $\mathbf{V} = V_m \underline{/\phi}$ and $\mathbf{I} = I_m \underline{/-\theta}$.

Given a sinusoid $v(t) = V_m \cos(\omega t + \phi)$, we obtain the corresponding phasor as $\mathbf{V} = V_m \underline{/\phi}$. Equation (9.25) is also demonstrated in Table 9.1, where the sine function is considered in addition to the cosine function. From Eq. (9.25), we see that to get the phasor representation of a sinusoid, we express it in cosine form and take the magnitude and phase. Given a phasor, we obtain the time domain representation as the cosine function with the same magnitude as the phasor and the argument as ωt plus the phase of the phasor. The idea of expressing information in alternate domains is fundamental to all areas of engineering.

TABLE 9.1

Sinusoid-phasor transformation.

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \underline{/\phi}$
$V_m \sin(\omega t + \phi)$	$V_m \underline{/\phi - 90^\circ}$
$I_m \cos(\omega t + \theta)$	$I_m \underline{/\theta}$
$I_m \sin(\omega t + \theta)$	$I_m \underline{/\theta - 90^\circ}$

Note that in Eq. (9.25) the frequency (or time) factor $e^{j\omega t}$ is suppressed, and the frequency is not explicitly shown in the phasor domain representation because ω is constant. However, the response depends on ω. For this reason, the phasor domain is also known as the *frequency domain.*

From Eqs. (9.23) and (9.24), $v(t) = \text{Re}(\mathbf{V}e^{j\omega t}) = V_m \cos(\omega t + \phi)$, so that

$$\begin{aligned} \frac{dv}{dt} &= -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90^\circ) \\ &= \text{Re}(\omega V_m e^{j\omega t} e^{j\phi} e^{j90^\circ}) = \text{Re}(j\omega \mathbf{V} e^{j\omega t}) \end{aligned} \tag{9.26}$$

This shows that the derivative $v(t)$ is transformed to the phasor domain as $j\omega\mathbf{V}$

$$\underset{\text{(Time domain)}}{\frac{dv}{dt}} \quad \Leftrightarrow \quad \underset{\text{(Phasor domain)}}{j\omega\mathbf{V}} \tag{9.27}$$

Differentiating a sinusoid is equivalent to multiplying its corresponding phasor by $j\omega$.

Similarly, the integral of $v(t)$ is transformed to the phasor domain as $\mathbf{V}/j\omega$

$$\underset{\text{(Time domain)}}{\int v\,dt} \quad \Leftrightarrow \quad \underset{\text{(Phasor domain)}}{\frac{\mathbf{V}}{j\omega}} \tag{9.28}$$

Integrating a sinusoid is equivalent to dividing its corresponding phasor by $j\omega$.

Equation (9.27) allows the replacement of a derivative with respect to time with multiplication of $j\omega$ in the phasor domain, whereas Eq. (9.28) allows the replacement of an integral with respect to time with division by $j\omega$ in the phasor domain. Equations (9.27) and (9.28) are useful in finding the steady-state solution, which does not require knowing the initial values of the variable involved. This is one of the important applications of phasors.

Besides time differentiation and integration, another important use of phasors is found in summing sinusoids of the same frequency. This is best illustrated with an example, and Example 9.6 provides one.

Adding sinusoids of the same frequency is equivalent to adding their corresponding phasors.

The differences between $v(t)$ and $\mathbf{V}$ should be emphasized:

1. $v(t)$ is the *instantaneous or time domain* representation, while $\mathbf{V}$ is the *frequency or phasor domain* representation.
2. $v(t)$ is time dependent, while $\mathbf{V}$ is not. (This fact is often forgotten by students.)
3. $v(t)$ is always real with no complex term, while $\mathbf{V}$ is generally complex.

Finally, we should bear in mind that phasor analysis applies only when frequency is constant; it applies in manipulating two or more sinusoidal signals only if they are of the same frequency.

Example 9.3

Evaluate these complex numbers:

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

Solution:

(a) Using polar to rectangular transformation,

$$40\angle 50^\circ = 40(\cos 50^\circ + j\sin 50^\circ) = 25.71 + j30.64$$

$$20\angle -30^\circ = 20[\cos(-30^\circ) + j\sin(-30^\circ)] = 17.32 - j10$$

Adding them up gives

$$40\angle 50^\circ + 20\angle -30^\circ = 43.03 + j20.64 = 47.72\angle 25.63^\circ$$

Taking the square root of this,

$$(40\angle 50^\circ + 20\angle -30^\circ)^{1/2} = 6.91\angle 12.81^\circ$$

(b) Using polar-rectangular transformation, addition, multiplication, and division,

$$\frac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*} = \frac{8.66 - j5 + (3 - j4)}{(2 + j4)(3 + j5)}$$

$$= \frac{11.66 - j9}{-14 + j22} = \frac{14.73\angle -37.66^\circ}{26.08\angle 122.47^\circ}$$

$$= 0.565\angle -160.13^\circ$$

Practice Problem 9.3

Evaluate the following complex numbers:

(a) $[(5 + j2)(-1 + j4) - 5\angle 60^\circ]^*$

(b) $\dfrac{10 + j5 + 3\angle 40^\circ}{-3 + j4} + 10\angle 30^\circ + j5$

Answer: (a) $-15.5 - j13.67$, (b) $8.293 + j7.2$.

Example 9.4

Transform these sinusoids to phasors:

(a) $i = 6\cos(50t - 40^\circ)$ A
(b) $v = -4\sin(30t + 50^\circ)$ V

Solution:

(a) $i = 6\cos(50t - 40^\circ)$ has the phasor

$$\mathbf{I} = 6\angle -40^\circ \text{ A}$$

(b) Since $-\sin A = \cos(A + 90^\circ)$,

$$v = -4\sin(30t + 50^\circ) = 4\cos(30t + 50^\circ + 90^\circ)$$

$$= 4\cos(30t + 140^\circ) \text{ V}$$

The phasor form of v is

$$\mathbf{V} = 4\angle 140^\circ \text{ V}$$

Practice Problem 9.4

Express these sinusoids as phasors:

(a) $v = 7\cos(2t + 40^\circ)$ V
(b) $i = -4\sin(10t + 10^\circ)$ A

Answer: (a) $\mathbf{V} = 7\angle 40^\circ$ V, (b) $\mathbf{I} = 4\angle 100^\circ$ A.

Example 9.5

Find the sinusoids represented by these phasors:

(a) $\mathbf{I} = -3 + j4$ A
(b) $\mathbf{V} = j8e^{-j20^\circ}$ V

Solution:

(a) $\mathbf{I} = -3 + j4 = 5\underline{/126.87^\circ}$. Transforming this to the time domain gives

$$i(t) = 5\cos(\omega t + 126.87^\circ)\text{ A}$$

(b) Since $j = 1\underline{/90^\circ}$,

$$\begin{aligned}\mathbf{V} = j8\underline{/-20^\circ} &= (1\underline{/90^\circ})(8\underline{/-20^\circ})\\ &= 8\underline{/90^\circ - 20^\circ} = 8\underline{/70^\circ}\text{ V}\end{aligned}$$

Converting this to the time domain gives

$$v(t) = 8\cos(\omega t + 70^\circ)\text{ V}$$

Practice Problem 9.5

Find the sinusoids corresponding to these phasors:

(a) $\mathbf{V} = -25\underline{/40^\circ}$ V
(b) $\mathbf{I} = j(12 - j5)$ A

Answer: (a) $v(t) = 25\cos(\omega t - 140^\circ)$ V or $25\cos(\omega t + 220^\circ)$ V, (b) $i(t) = 13\cos(\omega t + 67.38^\circ)$ A.

Example 9.6

Given $i_1(t) = 4\cos(\omega t + 30^\circ)$ A and $i_2(t) = 5\sin(\omega t - 20^\circ)$ A, find their sum.

Solution:

Here is an important use of phasors—for summing sinusoids of the same frequency. Current $i_1(t)$ is in the standard form. Its phasor is

$$\mathbf{I}_1 = 4\underline{/30^\circ}$$

We need to express $i_2(t)$ in cosine form. The rule for converting sine to cosine is to subtract 90°. Hence,

$$i_2 = 5\cos(\omega t - 20^\circ - 90^\circ) = 5\cos(\omega t - 110^\circ)$$

and its phasor is

$$\mathbf{I}_2 = 5\underline{/-110^\circ}$$

If we let $i = i_1 + i_2$, then

$$\begin{aligned}\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 &= 4\underline{/30^\circ} + 5\underline{/-110^\circ}\\ &= 3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698\\ &= 3.218\underline{/-56.97^\circ}\text{ A}\end{aligned}$$

Transforming this to the time domain, we get

$$i(t) = 3.218\cos(\omega t - 56.97^\circ)\text{ A}$$

Of course, we can find $i_1 + i_2$ using Eq. (9.9), but that is the hard way.

Practice Problem 9.6

If $v_1 = -10\sin(\omega t - 30^\circ)$ V and $v_2 = 20\cos(\omega t + 45^\circ)$ V, find $v = v_1 + v_2$.

Answer: $v(t) = 29.77\cos(\omega t + 49.98^\circ)$ V.

Example 9.7

Using the phasor approach, determine the current $i(t)$ in a circuit described by the integrodifferential equation

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75^\circ)$$

Solution:

We transform each term in the equation from time domain to phasor domain. Keeping Eqs. (9.27) and (9.28) in mind, we obtain the phasor form of the given equation as

$$4\mathbf{I} + \frac{8\mathbf{I}}{j\omega} - 3j\omega\mathbf{I} = 50\underline{/75^\circ}$$

But $\omega = 2$, so

$$\mathbf{I}(4 - j4 - j6) = 50\underline{/75^\circ}$$

$$\mathbf{I} = \frac{50\underline{/75^\circ}}{4 - j10} = \frac{50\underline{/75^\circ}}{10.77\underline{/-68.2^\circ}} = 4.642\underline{/143.2^\circ}\text{ A}$$

Converting this to the time domain,

$$i(t) = 4.642\cos(2t + 143.2^\circ)\text{ A}$$

Keep in mind that this is only the steady-state solution, and it does not require knowing the initial values.

Practice Problem 9.7

Find the voltage $v(t)$ in a circuit described by the integrodifferential equation

$$2\frac{dv}{dt} + 5v + 10\int v\,dt = 50\cos(5t - 30^\circ)$$

using the phasor approach.

Answer: $v(t) = 5.3\cos(5t - 88^\circ)$ V.

9.4 Phasor Relationships for Circuit Elements

Now that we know how to represent a voltage or current in the phasor or frequency domain, one may legitimately ask how we apply this to circuits involving the passive elements R, L, and C. What we need to do is to transform the voltage-current relationship from the time domain to the frequency domain for each element. Again, we will assume the passive sign convention.

We begin with the resistor. If the current through a resistor R is $i = I_m \cos(\omega t + \phi)$, the voltage across it is given by Ohm's law as

$$v = iR = RI_m \cos(\omega t + \phi) \tag{9.29}$$

The phasor form of this voltage is

$$\mathbf{V} = RI_m \underline{/\phi} \tag{9.30}$$

But the phasor representation of the current is $\mathbf{I} = I_m \underline{/\phi}$. Hence,

$$\mathbf{V} = R\mathbf{I} \tag{9.31}$$

showing that the voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain. Figure 9.9 illustrates the voltage-current relations of a resistor. We should note from Eq. (9.31) that voltage and current are in phase, as illustrated in the phasor diagram in Fig. 9.10.

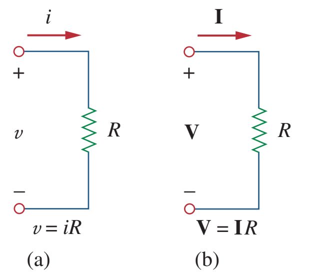


Figure 9.9
Voltage-current relations for a resistor in the: (a) time domain, (b) frequency domain.

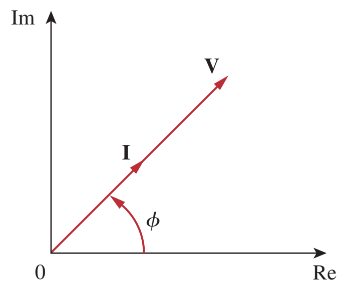


Figure 9.10
Phasor diagram for the resistor.

For the inductor L, assume the current through it is $i = I_m \cos(\omega t + \phi)$. The voltage across the inductor is

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) \tag{9.32}$$

Recall from Eq. (9.10) that $-\sin A = \cos(A + 90°)$. We can write the voltage as

$$v = \omega L I_m \cos(\omega t + \phi + 90°) \tag{9.33}$$

which transforms to the phasor

$$\mathbf{V} = \omega L I_m e^{j(\phi + 90°)} = \omega L I_m e^{j\phi} e^{j90°} = \omega L I_m \underline{/\phi + 90°} \tag{9.34}$$

But $I_m \underline{/\phi} = \mathbf{I}$, and from Eq. (9.19), $e^{j90°} = j$. Thus,

$$\mathbf{V} = j\omega L\mathbf{I} \tag{9.35}$$

showing that the voltage has a magnitude of $\omega L I_m$ and a phase of $\phi + 90°$. The voltage and current are 90° out of phase. Specifically, the current lags the voltage by 90°. Figure 9.11 shows the voltage-current relations for the inductor. Figure 9.12 shows the phasor diagram.

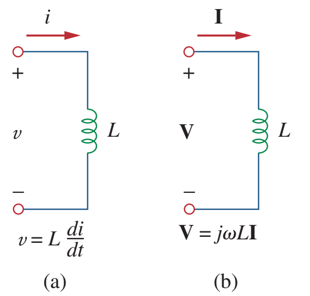


Figure 9.11
Voltage-current relations for an inductor in the: (a) time domain, (b) frequency domain.

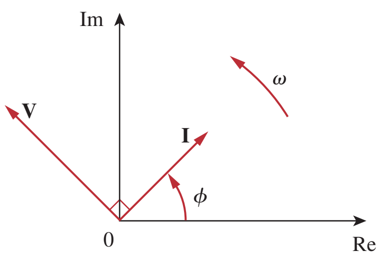


Figure 9.12
Phasor diagram for the inductor; **I** lags **V**.

Although it is equally correct to say that the inductor voltage leads the current by 90°, convention gives the current phase relative to the voltage.

For the capacitor C, assume the voltage across it is $v = V_m \cos(\omega t + \phi)$. The current through the capacitor is

$$i = C\frac{dv}{dt} \tag{9.36}$$

By following the same steps as we took for the inductor or by applying Eq. (9.27) on Eq. (9.36), we obtain

$$\mathbf{I} = j\omega C\mathbf{V} \quad \Rightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \tag{9.37}$$

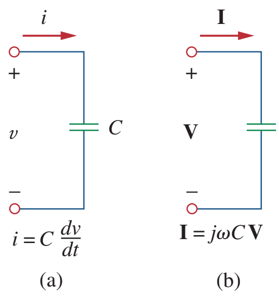


Figure 9.13
Voltage-current relations for a capacitor in the: (a) time domain, (b) frequency domain.

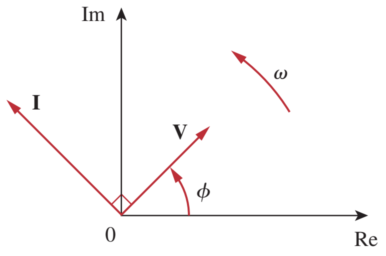


Figure 9.14
Phasor diagram for the capacitor; **I** leads **V**.

showing that the current and voltage are 90° out of phase. To be specific, the current leads the voltage by 90°. Figure 9.13 shows the voltage-current relations for the capacitor; Fig. 9.14 gives the phasor diagram. Table 9.2 summarizes the time domain and phasor domain representations of the circuit elements.

TABLE 9.2

Summary of voltage-current relationships.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

Example 9.8

The voltage $v = 12\cos(60t + 45^\circ)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

Solution:
For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and $\mathbf{V} = 12\underline{/45^\circ}$ V. Hence,

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\underline{/45^\circ}}{j60 \times 0.1} = \frac{12\underline{/45^\circ}}{6\underline{/90^\circ}} = 2\underline{/-45^\circ} \text{ A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45^\circ) \text{ A}$$

Practice Problem 9.8

If voltage $v = 10\cos(100t + 30^\circ)$ is applied to a 50 μF capacitor, calculate the current through the capacitor.

Answer: $50\cos(100t + 120^\circ)$ mA.

9.5 Impedance and Admittance

In the preceding section, we obtained the voltage-current relations for the three passive elements as

$$\mathbf{V} = R\mathbf{I}, \qquad \mathbf{V} = j\omega L\mathbf{I}, \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \tag{9.38}$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor current as

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C} \tag{9.39}$$

From these three expressions, we obtain Ohm's law in phasor form for any type of element as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{Z}\mathbf{I} \tag{9.40}$$

where $\mathbf{Z}$ is a frequency-dependent quantity known as *impedance*, measured in ohms.

The **impedance** $\mathbf{Z}$ of a circuit is the ratio of the phasor voltage $\mathbf{V}$ to the phasor current $\mathbf{I}$, measured in ohms (Ω).

The impedance represents the opposition that the circuit exhibits to the flow of sinusoidal current. Although the impedance is the ratio of two phasors, it is not a phasor, because it does not correspond to a sinusoidally varying quantity.

The impedances of resistors, inductors, and capacitors can be readily obtained from Eq. (9.39). Table 9.3 summarizes their impedances. From the table we notice that $\mathbf{Z}_L = j\omega L$ and $\mathbf{Z}_C = -j/\omega C$. Consider two extreme cases of angular frequency. When $\omega = 0$ (i.e., for dc sources), $\mathbf{Z}_L = 0$ and $\mathbf{Z}_C \to \infty$, confirming what we already know—that the inductor acts like a short circuit, while the capacitor acts like an open circuit. When $\omega \to \infty$ (i.e., for high frequencies), $\mathbf{Z}_L \to \infty$ and $\mathbf{Z}_C = 0$, indicating that the inductor is an open circuit to high frequencies, while the capacitor is a short circuit. Figure 9.15 illustrates this.

As a complex quantity, the impedence may be expressed in rectangular form as

$$\mathbf{Z} = R + jX \tag{9.41}$$

where $R = \text{Re}\,\mathbf{Z}$ is the *resistance* and $X = \text{Im}\,\mathbf{Z}$ is the *reactance*. The reactance X may be positive or negative. We say that the impedance is inductive when X is positive or capacitive when X is negative. Thus, impedance $\mathbf{Z} = R + jX$ is said to be *inductive* or lagging since current lags voltage, while impedance $\mathbf{Z} = R - jX$ is capacitive or leading because current leads voltage. The impedance, resistance, and reactance are all measured in ohms. The impedance may also be expressed in polar form as

$$\mathbf{Z} = |\mathbf{Z}| \underline{/\theta} \tag{9.42}$$

TABLE 9.3

Impedances and admittances of passive elements.

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \dfrac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \dfrac{1}{j\omega L}$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

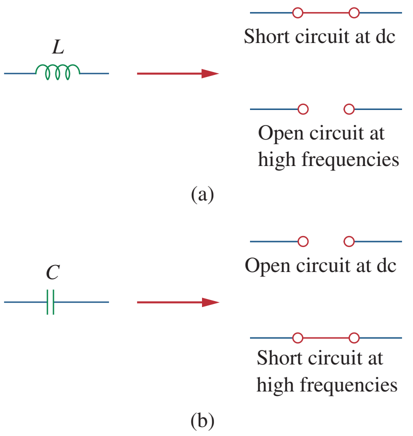


Figure 9.15
Equivalent circuits at dc and high frequencies: (a) inductor, (b) capacitor.

Comparing Eqs. (9.41) and (9.42), we infer that

$$\mathbf{Z} = R + jX = |\mathbf{Z}|\angle\theta \tag{9.43}$$

where

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{X}{R} \tag{9.44}$$

and

$$R = |\mathbf{Z}|\cos\theta, \qquad X = |\mathbf{Z}|\sin\theta \tag{9.45}$$

It is sometimes convenient to work with the reciprocal of impedance, known as *admittance*.

> The **admittance Y** is the reciprocal of impedance, measured in siemens (S).

The admittance **Y** of an element (or a circuit) is the ratio of the phasor current through it to the phasor voltage across it, or

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}} \tag{9.46}$$

The admittances of resistors, inductors, and capacitors can be obtained from Eq. (9.39). They are also summarized in Table 9.3.

As a complex quantity, we may write **Y** as

$$\mathbf{Y} = G + jB \tag{9.47}$$

where $G = \text{Re}\,\mathbf{Y}$ is called the *conductance* and $B = \text{Im}\,\mathbf{Y}$ is called the *susceptance*. Admittance, conductance, and susceptance are all expressed in the unit of siemens (or mhos). From Eqs. (9.41) and (9.47),

$$G + jB = \frac{1}{R + jX} \tag{9.48}$$

By rationalization,

$$G + jB = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2} \tag{9.49}$$

Equating the real and imaginary parts gives

$$G = \frac{R}{R^2 + X^2}, \qquad B = -\frac{X}{R^2 + X^2} \tag{9.50}$$

showing that $G \neq 1/R$ as it is in resistive circuits. Of course, if $X = 0$, then $G = 1/R$.

Example 9.9

Find $v(t)$ and $i(t)$ in the circuit shown in Fig. 9.16.

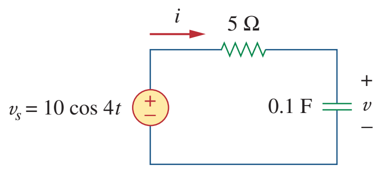


Figure 9.16 For Example 9.9.

Solution:
From the voltage source 10 cos 4t, $\omega = 4$,

$$\mathbf{V}_s = 10\underline{/0^\circ} \text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$\begin{aligned}\mathbf{I} &= \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\underline{/0^\circ}}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2} \\ &= 1.6 + j0.8 = 1.789\underline{/26.57^\circ} \text{ A}\end{aligned} \tag{9.9.1}$$

The voltage across the capacitor is

$$\begin{aligned}\mathbf{V} = \mathbf{I}\mathbf{Z}_C &= \frac{\mathbf{I}}{j\omega C} = \frac{1.789\underline{/26.57^\circ}}{j4 \times 0.1} \\ &= \frac{1.789\underline{/26.57^\circ}}{0.4\underline{/90^\circ}} = 4.47\underline{/-63.43^\circ} \text{ V}\end{aligned} \tag{9.9.2}$$

Converting **I** and **V** in Eqs. (9.9.1) and (9.9.2) to the time domain, we get

$$i(t) = 1.789 \cos(4t + 26.57^\circ) \text{ A}$$
$$v(t) = 4.47 \cos(4t - 63.43^\circ) \text{ V}$$

Notice that $i(t)$ leads $v(t)$ by 90° as expected.

Practice Problem 9.9

Refer to Fig. 9.17. Determine $v(t)$ and $i(t)$.

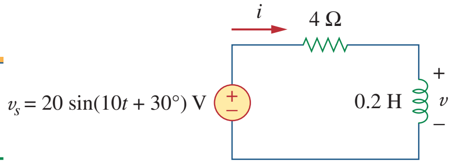


Figure 9.17 For Practice Prob. 9.9.

Answer: 8.944 sin(10t + 93.43°) V, 4.472 sin(10t + 3.43°) A.

9.6 †Kirchhoff's Laws in the Frequency Domain

We cannot do circuit analysis in the frequency domain without Kirchhoff's current and voltage laws. Therefore, we need to express them in the frequency domain.

For KVL, let $v_1, v_2, \ldots, v_n$ be the voltages around a closed loop. Then

$$v_1 + v_2 + \cdots + v_n = 0 \tag{9.51}$$

In the sinusoidal steady state, each voltage may be written in cosine form, so that Eq. (9.51) becomes

$$\begin{aligned}V_{m1} \cos(\omega t + \theta_1) + V_{m2} \cos(\omega t + \theta_2) \\ + \cdots + V_{mn} \cos(\omega t + \theta_n) = 0\end{aligned} \tag{9.52}$$