Practice Problem 8.4

The circuit in Fig. 8.12 has reached steady state at $t = 0^-$. If the make-before-break switch moves to position b at $t = 0$, calculate $i(t)$ for $t > 0$.

Answer: $e^{-2.5t}(10\cos 1.6583t - 15.076\sin 1.6583t)$ A.

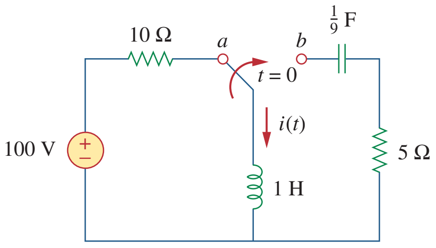

Figure 8.12
For Practice Prob. 8.4.

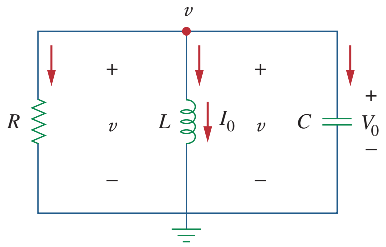

Figure 8.13
A source-free parallel *RLC* circuit.

8.4 The Source-Free Parallel *RLC* Circuit

Parallel *RLC* circuits find many practical applications, notably in communications networks and filter designs.

Consider the parallel *RLC* circuit shown in Fig. 8.13. Assume initial inductor current I_0 and initial capacitor voltage V_0,

$$i(0) = I_0 = \frac{1}{L}\int_{\infty}^{0} v(t)\,dt \tag{8.27a}$$

$$v(0) = V_0 \tag{8.27b}$$

Since the three elements are in parallel, they have the same voltage v across them. According to passive sign convention, the current is entering each element; that is, the current through each element is leaving the top node. Thus, applying KCL at the top node gives

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau + C\frac{dv}{dt} = 0 \tag{8.28}$$

Taking the derivative with respect to t and dividing by C results in

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \tag{8.29}$$

We obtain the characteristic equation by replacing the first derivative by s and the second derivative by s^2. By following the same reasoning used in establishing Eqs. (8.4) through (8.8), the characteristic equation is obtained as

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \tag{8.30}$$

The roots of the characteristic equation are

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

or

$$\boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}} \tag{8.31}$$

where

$$\boxed{\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}} \tag{8.32}$$

The names of these terms remain the same as in the preceding section, as they play the same role in the solution. Again, there are three possible solutions, depending on whether $\alpha > \omega_0$, $\alpha = \omega_0$, or $\alpha < \omega_0$. Let us consider these cases separately.

Overdamped Case ($\alpha > \omega_0$)

From Eq. (8.32), $\alpha > \omega_0$ when $L > 4R^2C$. The roots of the characteristic equation are real and negative. The response is

$$\boxed{v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \tag{8.33}$$

Critically Damped Case ($\alpha = \omega_0$)

For $\alpha = \omega_0$, $L = 4R^2C$. The roots are real and equal so that the response is

$$\boxed{v(t) = (A_1 + A_2 t)e^{-\alpha t}} \tag{8.34}$$

Underdamped Case ($\alpha < \omega_0$)

When $\alpha < \omega_0$, $L < 4R^2C$. In this case the roots are complex and may be expressed as

$$s_{1,2} = -\alpha \pm j\omega_d \tag{8.35}$$

where

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \tag{8.36}$$

The response is

$$\boxed{v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)} \tag{8.37}$$

The constants A_1 and A_2 in each case can be determined from the initial conditions. We need $v(0)$ and $dv(0)/dt$. The first term is known from Eq. (8.27b). We find the second term by combining Eqs. (8.27) and (8.28), as

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

or

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC} \tag{8.38}$$

The voltage waveforms are similar to those shown in Fig. 8.9 and will depend on whether the circuit is overdamped, underdamped, or critically damped.

Having found the capacitor voltage $v(t)$ for the parallel *RLC* circuit as shown above, we can readily obtain other circuit quantities such as individual element currents. For example, the resistor current is $i_R = v/R$ and the capacitor voltage is $v_C = C\, dv/dt$. We have selected the capacitor voltage $v(t)$ as the key variable to be determined first in order to take advantage of Eq. (8.1a). Notice that we first found the inductor current $i(t)$ for the *RLC* series circuit, whereas we first found the capacitor voltage $v(t)$ for the parallel *RLC* circuit.

Example 8.5

In the parallel circuit of Fig. 8.13, find $v(t)$ for $t > 0$, assuming $v(0) = 5$ V, $i(0) = 0$, $L = 1$ H, and $C = 10$ mF. Consider these cases: $R = 1.923\ \Omega$, $R = 5\ \Omega$, and $R = 6.25\ \Omega$.

Solution:

CASE 1 If $R = 1.923\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 1.923 \times 10 \times 10^{-3}} = 26$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10 \times 10^{-3}}} = 10$$

Since $\alpha > \omega_0$ in this case, the response is overdamped. The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2, -50$$

and the corresponding response is

$$v(t) = A_1 e^{-2t} + A_2 e^{-50t} \tag{8.5.1}$$

We now apply the initial conditions to get A_1 and A_2.

$$v(0) = 5 = A_1 + A_2 \tag{8.5.2}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{1.923 \times 10 \times 10^{-3}} = -260$$

But differentiating Eq. (8.5.1),

$$\frac{dv}{dt} = -2A_1 e^{-2t} - 50A_2 e^{-50t}$$

At $t = 0$,

$$-260 = -2A_1 - 50A_2 \tag{8.5.3}$$

From Eqs. (8.5.2) and (8.5.3), we obtain $A_1 = -0.2083$ and $A_2 = 5.208$. Substituting A_1 and A_2 in Eq. (8.5.1) yields

$$v(t) = -0.2083e^{-2t} + 5.208e^{-50t} \tag{8.5.4}$$

CASE 2 When $R = 5\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 5 \times 10 \times 10^{-3}} = 10$$

while $\omega_0 = 10$ remains the same. Since $\alpha = \omega_0 = 10$, the response is critically damped. Hence, $s_1 = s_2 = -10$, and

$$v(t) = (A_1 + A_2 t)e^{-10t} \tag{8.5.5}$$

To get A_1 and A_2, we apply the initial conditions

$$v(0) = 5 = A_1 \tag{8.5.6}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{5 \times 10 \times 10^{-3}} = -100$$

But differentiating Eq. (8.5.5),

$$\frac{dv}{dt} = (-10A_1 - 10A_2 t + A_2)e^{-10t}$$

At $t = 0$,

$$-100 = -10A_1 + A_2 \qquad \textbf{(8.5.7)}$$

From Eqs. (8.5.6) and (8.5.7), $A_1 = 5$ and $A_2 = -50$. Thus,

$$v(t) = (5 - 50t)e^{-10t} \text{ V} \qquad \textbf{(8.5.8)}$$

CASE 3 When $R = 6.25\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 6.25 \times 10 \times 10^{-3}} = 8$$

while $\omega_0 = 10$ remains the same. As $\alpha < \omega_0$ in this case, the response is underdamped. The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8 \pm j6$$

Hence,

$$v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t} \qquad \textbf{(8.5.9)}$$

We now obtain A_1 and A_2, as

$$v(0) = 5 = A_1 \qquad \textbf{(8.5.10)}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{6.25 \times 10 \times 10^{-3}} = -80$$

But differentiating Eq. (8.5.9),

$$\frac{dv}{dt} = (-8A_1 \cos 6t - 8A_2 \sin 6t - 6A_1 \sin 6t + 6A_2 \cos 6t)e^{-8t}$$

At $t = 0$,

$$-80 = -8A_1 + 6A_2 \qquad \textbf{(8.5.11)}$$

From Eqs. (8.5.10) and (8.5.11), $A_1 = 5$ and $A_2 = -6.667$. Thus,

$$v(t) = (5 \cos 6t - 6.667 \sin 6t)e^{-8t} \qquad \textbf{(8.5.12)}$$

Notice that by increasing the value of R, the degree of damping decreases and the responses differ. Figure 8.14 plots the three cases.

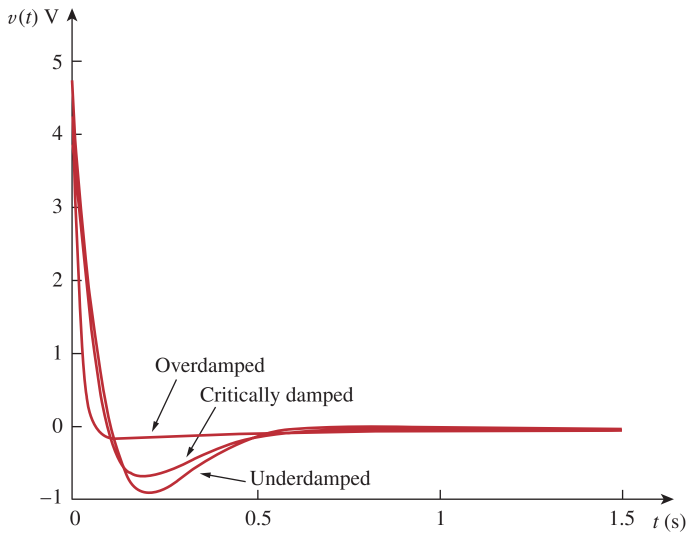

Figure 8.14
For Example 8.5: responses for three degrees of damping.

Practice Problem 8.5

In Fig. 8.13, let $R = 2\ \Omega$, $L = 0.4$ H, $C = 25$ mF, $v(0) = 0$, $i(0) = 50$ mA. Find $v(t)$ for $t > 0$.

Answer: $-2te^{-10t}\, u(t)$ V.

Example 8.6

Find $v(t)$ for $t > 0$ in the *RLC* circuit of Fig. 8.15.

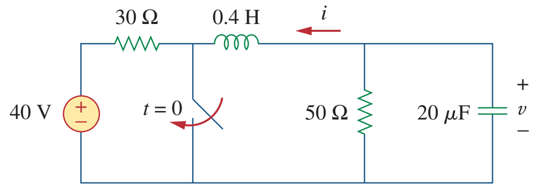

Figure 8.15
For Example 8.6.

Solution:
When $t < 0$, the switch is open; the inductor acts like a short circuit while the capacitor behaves like an open circuit. The initial voltage across the capacitor is the same as the voltage across the 50-Ω resistor; that is,

$$v(0) = \frac{50}{30 + 50}(40) = \frac{5}{8} \times 40 = 25\text{ V} \tag{8.6.1}$$

The initial current through the inductor is

$$i(0) = -\frac{40}{30 + 50} = -0.5\text{ A}$$

The direction of i is as indicated in Fig. 8.15 to conform with the direction of I_0 in Fig. 8.13, which is in agreement with the convention that current flows into the positive terminal of an inductor (see Fig. 6.23). We need to express this in terms of dv/dt, since we are looking for v.

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{25 - 50 \times 0.5}{50 \times 20 \times 10^{-6}} = 0 \tag{8.6.2}$$

When $t > 0$, the switch is closed. The voltage source along with the 30-Ω resistor is separated from the rest of the circuit. The parallel *RLC* circuit acts independently of the voltage source, as illustrated in Fig. 8.16. Next, we determine that the roots of the characteristic equation are

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 50 \times 20 \times 10^{-6}} = 500$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.4 \times 20 \times 10^{-6}}} = 354$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$
$$= -500 \pm \sqrt{250{,}000 - 124{,}997.6} = -500 \pm 354$$

or

$$s_1 = -854, \qquad s_2 = -146$$

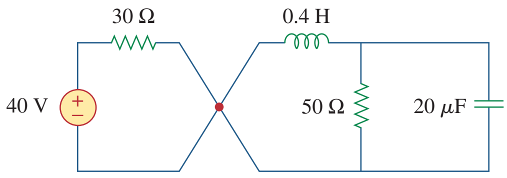

Figure 8.16
The circuit in Fig. 8.15 when $t > 0$. The parallel *RLC* circuit on the right-hand side acts independently of the circuit on the left-hand side of the junction.

Since $\alpha > \omega_0$, we have the overdamped response

$$v(t) = A_1 e^{-854t} + A_2 e^{-146t} \tag{8.6.3}$$

At $t = 0$, we impose the condition in Eq. (8.6.1),

$$v(0) = 25 = A_1 + A_2 \quad \Rightarrow \quad A_2 = 25 - A_1 \tag{8.6.4}$$

Taking the derivative of $v(t)$ in Eq. (8.6.3),

$$\frac{dv}{dt} = -854A_1 e^{-854t} - 146A_2 e^{-146t}$$

Imposing the condition in Eq. (8.6.2),

$$\frac{dv(0)}{dt} = 0 = -854A_1 - 146A_2$$

or

$$0 = 854A_1 + 146A_2 \tag{8.6.5}$$

Solving Eqs. (8.6.4) and (8.6.5) gives

$$A_1 = -5.156, \qquad A_2 = 30.16$$

Thus, the complete solution in Eq. (8.6.3) becomes

$$v(t) = -5.156e^{-854t} + 30.16e^{-146t} \text{ V}$$

Practice Problem 8.6

Refer to the circuit in Fig. 8.17. Find $v(t)$ for $t > 0$.

Answer: $150(e^{-10t} - e^{-2.5t})$ V.

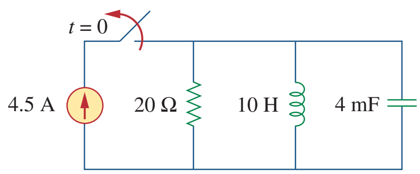

Figure 8.17
For Practice Prob. 8.6.

8.5 Step Response of a Series *RLC* Circuit

As we learned in the preceding chapter, the step response is obtained by the sudden application of a dc source. Consider the series *RLC* circuit shown in Fig. 8.18. Applying KVL around the loop for $t > 0$,

$$L\frac{di}{dt} + Ri + v = V_s \tag{8.39}$$

But

$$i = C\frac{dv}{dt}$$

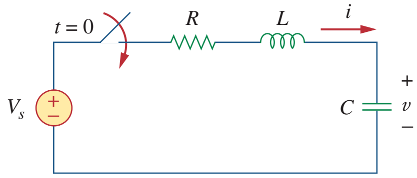

Figure 8.18
Step voltage applied to a series *RLC* circuit.