We can find $di_R(0^+)/dt$ although it is not required. Since $v_R = 5i_R$,

$$\frac{di_R(0^+)}{dt} = \frac{1}{5}\frac{dv_R(0^+)}{dt} = \frac{1}{5}\frac{2}{3} = \frac{2}{15}\text{ A/s}$$

(c) As $t \to \infty$, the circuit reaches steady state. We have the equivalent circuit in Fig. 8.6(a) except that the 3-A current source is now operative. By current division principle,

$$\begin{aligned} i_L(\infty) &= \frac{2}{2+4}3\text{ A} = 1\text{ A} \\ v_R(\infty) &= \frac{4}{2+4}3\text{ A} \times 2 = 4\text{ V}, \qquad v_C(\infty) = -20\text{ V} \end{aligned} \tag{8.2.12}$$

Practice Problem 8.2

For the circuit in Fig. 8.7, find: (a) $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$, (b) $di_L(0^+)/dt$, $dv_C(0^+)/dt$, $dv_R(0^+)/dt$, (c) $i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$.

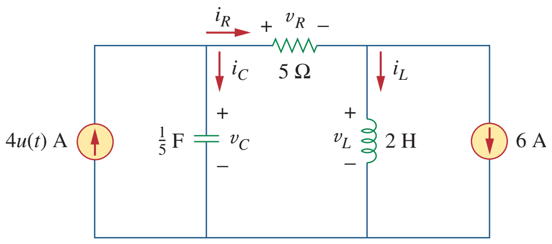

Figure 8.7
For Practice Prob. 8.2.

Answer: (a) −6 A, 0, 0, (b) 0, 20 V/s, 0, (c) −2 A, 20 V, 20 V.

8.3 The Source-Free Series *RLC* Circuit

An understanding of the natural response of the series *RLC* circuit is a necessary background for future studies in filter design and communications networks.

Consider the series *RLC* circuit shown in Fig. 8.8. The circuit is being excited by the energy initially stored in the capacitor and inductor. The energy is represented by the initial capacitor voltage V_0 and initial inductor current I_0. Thus, at $t = 0$,

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0 \tag{8.2a}$$

$$i(0) = I_0 \tag{8.2b}$$

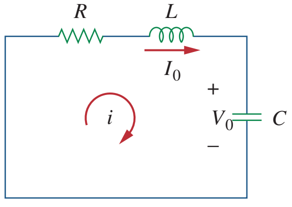

Figure 8.8
A source-free series *RLC* circuit.

Applying KVL around the loop in Fig. 8.8,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = 0 \tag{8.3}$$

To eliminate the integral, we differentiate with respect to t and rearrange terms. We get

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \tag{8.4}$$

This is a *second-order differential equation* and is the reason for calling the *RLC* circuits in this chapter second-order circuits. Our goal is to solve Eq. (8.4). To solve such a second-order differential equation requires that we have two initial conditions, such as the initial value of i and its first derivative or initial values of some i and v. The initial value of i is given in Eq. (8.2b). We get the initial value of the derivative of i from Eqs. (8.2a) and (8.3); that is,

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0$$

or

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) \tag{8.5}$$

With the two initial conditions in Eqs. (8.2b) and (8.5), we can now solve Eq. (8.4). Our experience in the preceding chapter on first-order circuits suggests that the solution is of exponential form. So we let

$$i = Ae^{st} \tag{8.6}$$

where A and s are constants to be determined. Substituting Eq. (8.6) into Eq. (8.4) and carrying out the necessary differentiations, we obtain

$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

or

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0 \tag{8.7}$$

Since $i = Ae^{st}$ is the assumed solution we are trying to find, only the expression in parentheses can be zero:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \tag{8.8}$$

See Appendix C.1 for the formula to find the roots of a quadratic equation.

This quadratic equation is known as the *characteristic equation* of the differential Eq. (8.4), since the roots of the equation dictate the character of i. The two roots of Eq. (8.8) are

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{8.9a}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{8.9b}$$

A more compact way of expressing the roots is

$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}} \tag{8.10}$$

where

$$\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}} \tag{8.11}$$

The roots s_1 and s_2 are called *natural frequencies*, measured in nepers per second (Np/s), because they are associated with the natural response of the circuit; ω_0 is known as the *resonant frequency* or strictly as the *undamped natural frequency*, expressed in radians per second (rad/s); and α is the *neper frequency* or the *damping factor*, expressed in nepers per second. In terms of α and ω_0, Eq. (8.8) can be written as

The *neper* (Np) is a dimensionless unit named after John Napier (1550–1617), a Scottish mathematician.

$$s^2 + 2\alpha s + \omega_0^2 = 0 \tag{8.8a}$$

The variables s and ω_0 are important quantities we will be discussing throughout the rest of the text.

The ratio α/ω_0 is known as the *damping ratio* ζ.

The two values of s in Eq. (8.10) indicate that there are two possible solutions for i, each of which is of the form of the assumed solution in Eq. (8.6); that is,

$$i_1 = A_1 e^{s_1 t}, \qquad i_2 = A_2 e^{s_2 t} \tag{8.12}$$

Since Eq. (8.4) is a linear equation, any linear combination of the two distinct solutions i_1 and i_2 is also a solution of Eq. (8.4). A complete or total solution of Eq. (8.4) would therefore require a linear combination of i_1 and i_2. Thus, the natural response of the series *RLC* circuit is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{8.13}$$

where the constants A_1 and A_2 are determined from the initial values $i(0)$ and $di(0)/dt$ in Eqs. (8.2b) and (8.5).

From Eq. (8.10), we can infer that there are three types of solutions:

1. If $\alpha > \omega_0$, we have the *overdamped* case.
2. If $\alpha = \omega_0$, we have the *critically damped* case.
3. If $\alpha < \omega_0$, we have the *underdamped* case.

The response is *overdamped* when the roots of the circuit's characteristic equation are unequal and real, *critically damped* when the roots are equal and real, and *underdamped* when the roots are complex.

We will consider each of these cases separately.

Overdamped Case ($\alpha > \omega_0$)

From Eqs. (8.9) and (8.10), $\alpha > \omega_0$ implies $C > 4L/R^2$. When this happens, both roots s_1 and s_2 are negative and real. The response is

$$\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \tag{8.14}$$

which decays and approaches zero as t increases. Figure 8.9(a) illustrates a typical overdamped response.

Critically Damped Case ($\alpha = \omega_0$)

When $\alpha = \omega_0$, $C = 4L/R^2$ and

$$s_1 = s_2 = -\alpha = -\frac{R}{2L} \tag{8.15}$$

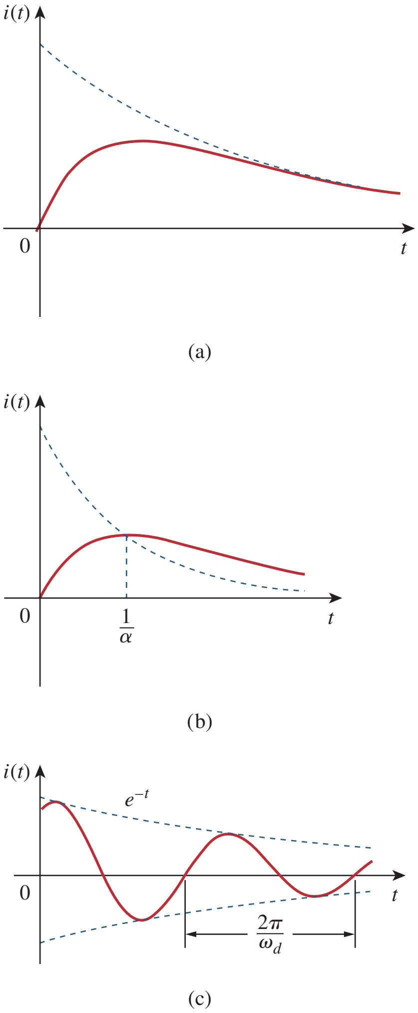

Figure 8.9
(a) Overdamped response, (b) critically damped response, (c) underdamped response.

For this case, Eq. (8.13) yields

$$i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

where $A_3 = A_1 + A_2$. This cannot be the solution, because the two initial conditions cannot be satisfied with the single constant A_3. What then could be wrong? Our assumption of an exponential solution is incorrect for the special case of critical damping. Let us go back to Eq. (8.4). When $\alpha = \omega_0 = R/2L$, Eq. (8.4) becomes

$$\frac{d^2 i}{dt^2} + 2\alpha\frac{di}{dt} + \alpha^2 i = 0$$

or

$$\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0 \qquad \textbf{(8.16)}$$

If we let

$$f = \frac{di}{dt} + \alpha i \qquad \textbf{(8.17)}$$

then Eq. (8.16) becomes

$$\frac{df}{dt} + \alpha f = 0$$

which is a first-order differential equation with solution $f = A_1 e^{-\alpha t}$, where A_1 is a constant. Equation (8.17) then becomes

$$\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t}$$

or

$$e^{\alpha t}\frac{di}{dt} + e^{\alpha t}\alpha i = A_1 \qquad \textbf{(8.18)}$$

This can be written as

$$\frac{d}{dt}(e^{\alpha t} i) = A_1 \qquad \textbf{(8.19)}$$

Integrating both sides yields

$$e^{\alpha t} i = A_1 t + A_2$$

or

$$i = (A_1 t + A_2)e^{-\alpha t} \qquad \textbf{(8.20)}$$

where A_2 is another constant. Hence, the natural response of the critically damped circuit is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term, or

$$\boxed{i(t) = (A_2 + A_1 t)e^{-\alpha t}} \qquad \textbf{(8.21)}$$

A typical critically damped response is shown in Fig. 8.9(b). In fact, Fig. 8.9(b) is a sketch of $i(t) = te^{-\alpha t}$, which reaches a maximum value of e^{-1}/α at $t = 1/\alpha$, one time constant, and then decays all the way to zero.

Underdamped Case ($\alpha < \omega_0$)

For $\alpha < \omega_0$, $C < 4L/R^2$. The roots may be written as

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d \tag{8.22a}$$

$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d \tag{8.22b}$$

where $j = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called the *damping frequency*. Both ω_0 and ω_d are natural frequencies because they help determine the natural response; while ω_0 is often called the *undamped natural frequency*, ω_d is called the *damped natural frequency*. The natural response is

$$\begin{aligned} i(t) &= A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \\ &= e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}) \end{aligned} \tag{8.23}$$

Using Euler's identities,

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta \tag{8.24}$$

we get

$$\begin{aligned} i(t) &= e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)] \\ &= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t] \end{aligned} \tag{8.25}$$

Replacing constants $(A_1 + A_2)$ and $j(A_1 - A_2)$ with constants B_1 and B_2, we write

$$\boxed{i(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)} \tag{8.26}$$

With the presence of sine and cosine functions, it is clear that the natural response for this case is exponentially damped and oscillatory in nature. The response has a time constant of $1/\alpha$ and a period of $T = 2\pi/\omega_d$. Figure 8.9(c) depicts a typical underdamped response. [Figure 8.9 assumes for each case that $i(0) = 0$.]

Once the inductor current $i(t)$ is found for the *RLC* series circuit as shown above, other circuit quantities such as individual element voltages can easily be found. For example, the resistor voltage is $v_R = Ri$, and the inductor voltage is $v_L = L\,di/dt$. The inductor current $i(t)$ is selected as the key variable to be determined first in order to take advantage of Eq. (8.1b).

We conclude this section by noting the following interesting, peculiar properties of an *RLC* network:

1. The behavior of such a network is captured by the idea of *damping*, which is the gradual loss of the initial stored energy, as evidenced by the continuous decrease in the amplitude of the response. The damping effect is due to the presence of resistance R. The damping factor α determines the rate at which the response is damped. If $R = 0$, then $\alpha = 0$, and we have an LC circuit with $1/\sqrt{LC}$ as the undamped natural frequency. Since $\alpha < \omega_0$ in this case, the response is not only undamped but also oscillatory. The circuit is said to be *loss-less*, because the dissipating or damping element (R) is absent. By adjusting the value of R, the response may be made undamped, overdamped, critically damped, or underdamped.
2. Oscillatory response is possible due to the presence of the two types of storage elements. Having both L and C allows the flow of

$R = 0$ produces a perfectly sinusoidal response. This response cannot be practically accomplished with L and C because of the inherent losses in them. See Figs 6.8 and 6.26. An electronic device called an *oscillator* can produce a perfectly sinusoidal response.

Examples 8.5 and 8.7 demonstrate the effect of varying R.

The response of a second-order circuit with two storage elements of the same type, as in Fig. 8.1(c) and (d), cannot be oscillatory.

energy back and forth between the two. The damped oscillation exhibited by the underdamped response is known as *ringing*. It stems from the ability of the storage elements L and C to transfer energy back and forth between them.

3. Observe from Fig. 8.9 that the waveforms of the responses differ. In general, it is difficult to tell from the waveforms the difference between the overdamped and critically damped responses. The critically damped case is the borderline between the underdamped and overdamped cases and it decays the fastest. With the same initial conditions, the overdamped case has the longest settling time, because it takes the longest time to dissipate the initial stored energy. If we desire the response that approaches the final value most rapidly without oscillation or ringing, the critically damped circuit is the right choice.

What this means in most practical circuits is that we seek an overdamped circuit that is as close as possible to a critically damped circuit.

Example 8.3

In Fig. 8.8, $R = 40\ \Omega$, $L = 4$ H, and $C = 1/4$ F. Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped, or critically damped?

Solution:

We first calculate

$$\alpha = \frac{R}{2L} = \frac{40}{2(4)} = 5, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times \frac{1}{4}}} = 1$$

The roots are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -5 \pm \sqrt{25 - 1}$$

or

$$s_1 = -0.101, \qquad s_2 = -9.899$$

Since $\alpha > \omega_0$, we conclude that the response is overdamped. This is also evident from the fact that the roots are real and negative.

Practice Problem 8.3

If $R = 10\ \Omega$, $L = 5$ H, and $C = 2$ mF in Fig. 8.8, find α, ω_0, s_1, and s_2. What type of natural response will the circuit have?

Answer: 1, 10, $-1 \pm j9.95$, underdamped.

Example 8.4

Find $i(t)$ in the circuit of Fig. 8.10. Assume that the circuit has reached steady state at $t = 0^-$.

Solution:

For $t < 0$, the switch is closed. The capacitor acts like an open circuit while the inductor acts like a shunted circuit. The equivalent circuit is shown in Fig. 8.11(a). Thus, at $t = 0$,

$$i(0) = \frac{10}{4 + 6} = 1\text{ A}, \qquad v(0) = 6i(0) = 6\text{ V}$$

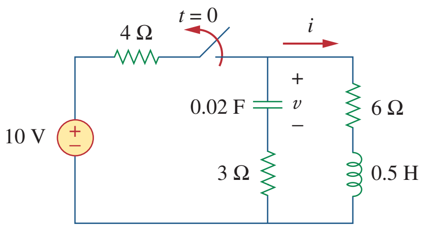

Figure 8.10
For Example 8.4.

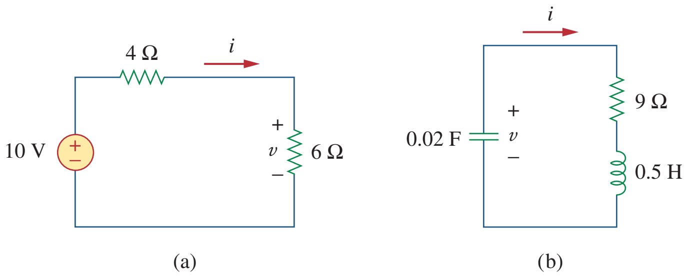

Figure 8.11
The circuit in Fig. 8.10: (a) for $t < 0$, (b) for $t > 0$.

where $i(0)$ is the initial current through the inductor and $v(0)$ is the initial voltage across the capacitor.

For $t > 0$, the switch is opened and the voltage source is disconnected. The equivalent circuit is shown in Fig. 8.11(b), which is a source-free series *RLC* circuit. Notice that the 3-Ω and 6-Ω resistors, which are in series in Fig. 8.10 when the switch is opened, have been combined to give $R = 9\ \Omega$ in Fig. 8.11(b). The roots are calculated as follows:

$$\alpha = \frac{R}{2L} = \frac{9}{2(\frac{1}{2})} = 9, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times \frac{1}{50}}} = 10$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -9 \pm \sqrt{81 - 100}$$

or

$$s_{1,2} = -9 \pm j4.359$$

Hence, the response is underdamped ($\alpha < \omega$); that is,

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) \tag{8.4.1}$$

We now obtain A_1 and A_2 using the initial conditions. At $t = 0$,

$$i(0) = 1 = A_1 \tag{8.4.2}$$

From Eq. (8.5),

$$\left.\frac{di}{dt}\right|_{t=0} = -\frac{1}{L}[Ri(0) + v(0)] = -2[9(1) - 6] = -6 \text{ A/s} \tag{8.4.3}$$

Note that $v(0) = V_0 = -6$ V is used, because the polarity of v in Fig. 8.11(b) is opposite that in Fig. 8.8. Taking the derivative of $i(t)$ in Eq. (8.4.1),

$$\begin{aligned}\frac{di}{dt} &= -9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)\\ &\quad + e^{-9t}(4.359)(-A_1 \sin 4.359t + A_2 \cos 4.359t)\end{aligned}$$

Imposing the condition in Eq. (8.4.3) at $t = 0$ gives

$$-6 = -9(A_1 + 0) + 4.359(-0 + A_2)$$

But $A_1 = 1$ from Eq. (8.4.2). Then

$$-6 = -9 + 4.359A_2 \qquad \Rightarrow \qquad A_2 = 0.6882$$

Substituting the values of A_1 and A_2 in Eq. (8.4.1) yields the complete solution as

$$i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359t) \text{ A}$$

Practice Problem 8.4

The circuit in Fig. 8.12 has reached steady state at $t = 0^-$. If the make-before-break switch moves to position b at $t = 0$, calculate $i(t)$ for $t > 0$.

Answer: $e^{-2.5t}(10 \cos 1.6583t - 15.076 \sin 1.6583t)$ A.

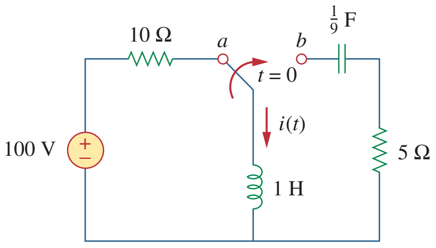

Figure 8.12
For Practice Prob. 8.4.

8.4 The Source-Free Parallel *RLC* Circuit

Parallel *RLC* circuits find many practical applications, notably in communications networks and filter designs.

Consider the parallel *RLC* circuit shown in Fig. 8.13. Assume initial inductor current I_0 and initial capacitor voltage V_0,

$$i(0) = I_0 = \frac{1}{L}\int_{\infty}^{0} v(t)\,dt \tag{8.27a}$$

$$v(0) = V_0 \tag{8.27b}$$

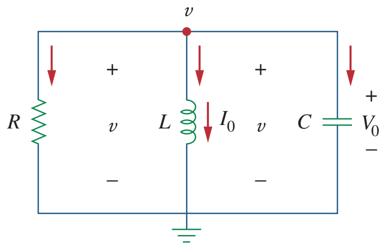

Figure 8.13
A source-free parallel *RLC* circuit.

Since the three elements are in parallel, they have the same voltage v across them. According to passive sign convention, the current is entering each element; that is, the current through each element is leaving the top node. Thus, applying KCL at the top node gives

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau + C\frac{dv}{dt} = 0 \tag{8.28}$$

Taking the derivative with respect to t and dividing by C results in

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \tag{8.29}$$

We obtain the characteristic equation by replacing the first derivative by s and the second derivative by s^2. By following the same reasoning used in establishing Eqs. (8.4) through (8.8), the characteristic equation is obtained as

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \tag{8.30}$$

The roots of the characteristic equation are

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

or

$$\boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}} \tag{8.31}$$

where

$$\boxed{\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}} \tag{8.32}$$