Three-Level Factorial Design

We have discussed the two-level series of factorial and fractional factorial designs in previous lectures, that are widely used in industrial research and development. Some extensions and variations of these designs are occasionally of interest, such as the designs for cases where all the factors are present at three levels.

The three-level design is written as a 3^k factorial design. It means that k factors are considered, each at 3 levels. These are usually reffered to as low, intermediate and high levels. These levels are numerically expressed as 0, 1 and 2. One could have considered the digits −1, 0 and +1 but this may be confusing with respect to the 2-level design since 0 is reserved for center points. Therefore we will use the 0, 1, 2 scheme.

The reason that the three-Level designs were proposed is to model possible curvature in the response function and to handle the case of nominal (Qualitative) factors at 3 levels. A third level for a continuous (Quantitative) factor facilitates investigation of a quadratic relationship between the response and each of the factors.

However the 3^k design is not the most efficient way to model a quadratic relationship. The "Response surface designs"* are superior alternatives which we will discussed in coming Lectures.

⇒ The 3^2 Design:

The simplest design in the 3^k system is the 3^2 design, which has two factors, each at three levels.

A			B		
Low	medium	High	Low	medium	high
0	1	2	0	1	2

→ Possible combinations $= 3^2 = 9$

$(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)$

$1, b, b^2, a, ab, ab^2, a^2, a^2b, a^2b^2$.

we can arrange it

$1, a, a^2, b, b^2, ab, a^2b, ab^2, a^2b^2$

→ Factor effect :

$$3^2 - 1 = 8$$

$A, A^2, B, B^2, AB, A^2B, AB^2, A^2B^2$

Linear Quadratic

There are eight degrees of freedom between these treatment combinations. The main effects of A and B each have two degrees of freedom and the AB interaction has four degrees of freedom. If there are r replicaticates, there will be $3^2(r) - 1$ total degrees of freedom and $3^2(r-1)$ degrees of freedom for error.

→ The 3^3 Design:

Now suppose there are three factors (A, B and C) under study and that each factor is at three levels arranged in a factorial experiment.

	A			B			C	
Low	medium	high	Low	medium	high	Low	medium	high
0	1	2	0	1	2	0	1	2

Possible combinations $= 3^3 = 27$

(0,0,0), (0,0,1), (0,0,2), (0,1,0), (0,1,1),
(0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,0,0),
(1,0,1), (1,0,2), (1,1,0), (1,1,1), (1,1,2),
(1,2,0), (1,2,1), (1,2,2), (2,0,0), (2,0,1),
(2,0,2), (2,1,0), (2,1,1), (2,1,2), (2,2,0),
(2,2,1), (2,2,2).

$1, c, c^2, b, bc, bc^2, b^2, b^2c, b^2c^2, a, ac,$
$ac^2, ab, abc, abc^2, ab^2, ab^2c, ab^2c^2, a^2,$
$a^2c, a^2c^2, a^2b, a^2bc, a^2bc^2, a^2b^2, a^2b^2c, a^2b^2c^2.$

we can arrange it.

$1, (a, a^2), (b, b^2), \underline{ab, a^2b, ab^2, a^2b^2}, (c, c^2),$
$\underline{ac, a^2c, ac^2, a^2c^2}, \underline{bc, b^2c, bc^2, b^2c^2}, \underline{abc,}$
$\underline{a^2bc, ab^2c, abc^2, a^2b^2c, ab^2c^2, a^2bc^2,}$
$\underline{a^2b^2c^2}.$

⇒ Factor effect: $3^3 - 1 = 26$

$\underline{A, A^2}, \underline{B, B^2}, \underline{AB, A^2B, AB^2, A^2B^2}, \underline{C, C^2},$
$\underline{AC, A^2C, AC^2, A^2C^2}, \underline{BC, B^2C, BC^2, B^2C^2},$

$\underline{ABC, A^2BC, AB^2C, ABC^2, A^2B^2C, AB^2C^2, A^2BC^2, A^2B^2C^2}$

there are 26 degree of freedom between these treatment combinations. Each main effect has two degrees of freedom, each two-factor interaction has four degrees of freedom and the three-factor interaction has eight degree of freedom. If there are "r" replicates, there are $3^3(r)-1$ total degree of freedom and $3^3(r-1)$ degrees of freedom for error.

Confounding in the 3^k factorial Design

Even when a single replicate of the 3^k design is considered, the design requires so many runs that it is unlikely that all 3^k runs can be made under uniform conditions. Thus, confounding in blocks is often neccesary. The 3^k design may be confounded in 3^c incomplete blocks, where $c < k$. Thus, these design may be confounded in three blocks $(c = 1)$, nine blocks $(c = 2)$ and so on.

→ **The 3^k factorial Design in Three Blocks:**

The general procedure is to construct a defining contrast

$$L = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$$

where α_i represents the exponent on the ith factor in the effect to be confounded and x_i is the level of the ith factor in a particular treatment combination.

For the 3^k series, we have $\alpha_i = 0, 1$ or 2. and $x_i = 0$ (low level), 1 (intermediate) or 2 (high level).

For example, suppose we wish to construct a 3^2 factorial design in three blocks with AB^2.

Possible combinations: $3^2 = 9$

$1, a, a^2, b, b^2, ab, a^2b, ab^2, a^2b^2$

(0,0), (1,0), (2,0), (0,1), (0,2), (1,1), (2,1), (1,2), (2,2)

Confounding Term AB^2, So defining contrast will be.

$$L = x_1 + 2x_2$$

The value of L (mod 3) of each treatment combination may be found as follows:

(0,0) : $L = 0 + 2(0) = 0 = 0 \pmod 3$
(1,0) : $L = 1 + 2(0) = 1 = 1 \pmod 3$
(2,0) : $L = 2 + 2(0) = 2 = 2 \pmod 3$
(0,1) : $L = 0 + 2(1) = 2 = 2 \pmod 3$
(0,2) : $L = 0 + 2(2) = 4 = 1 \pmod 3$
(1,1) : $L = 1 + 2(1) = 3 = 0 \pmod 3$
(2,1) : $L = 2 + 2(1) = 4 = 1 \pmod 3$
(1,2) : $L = 1 + 2(2) = 5 = 2 \pmod 3$
(2,2) : $L = 2 + 2(2) = 6 = 0 \pmod 3$

Block 1	Block 2	Block 3
(0,0)	(1,0)	(2,0)
(1,1)	(0,2)	(0,1)
(2,2)	(2,1)	(1,2)

ANOVA table if complete confounding (one replication)

SoV	d.f	SS
Block	$P-1$	$\sum B_k^2/n_B - (T_{..})^2/p^n$
A	$P-1$	$[(x_1+0x_2=0)^2 + (x_1+0x_2=1)^2 + (x_1+0x_2=2)^2]/n_B - (T_{..})^2/p^n$
B	$P-1$	$[(0x_1+x_2=0)^2 + (0x_1+x_2=1)^2 + (0x_1+x_2=2)^2]/n_B - (T_{..})^2/p^n$
Error	By subtraction	By subtraction
Total	3^2-1	$\sum_i\sum_j\sum_k Y_{ijk}^2 - \frac{(T_{..})^2}{p^n}$

Example

Perform statistical analysis of the 3^2 design confounded with AB^2 in three blocks by using the following data, which come from the single replicate of the 3^2 design:

Block 1	Block 2	Block 3	
00 = 4	10 = -2	01 = 5	
11 = -4	21 = 1	12 = -5	
22 = 0	02 = 8	20 = 0	
Block Total 0	7	0	7 = Grand Total

$$SS\,Block = \frac{\sum B_k^2}{n_B} - \frac{(T_{..})^2}{p^n}$$

$$= \frac{(0)^2+(7)^2+(0)^2}{3} - \frac{(7)^2}{3^2}$$

$$= 16.33 - 5.44$$
$$= 10.89$$

$$SSA = \frac{[(x_1+0x_2=0)^2+(x_1+0x_2=1)^2+(x_1+0x_2=2)^2]}{n_B} - \frac{(T_{..})^2}{p^n}$$

$$= \frac{[(00+01+02)^2+(10+11+12)^2+(20+21+22)^2]}{n_B} - \frac{(T_{..})^2}{p^n}$$

$$= \frac{(4+5+8)^2+(-2-4-5)^2+(0+1+0)^2}{3} - \frac{(7)^2}{3^2}$$

$$= \frac{289+121+1}{3} - 5.44$$

$$= 131.56$$

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$$SS\,B = \frac{\left[(0x_1+x_2=0)^2+(0x_1+x_2=1)^2+(0x_1+x_2=2)^2\right]}{n_B} - \frac{(T_{..})^2}{p^n}$$

$$= \frac{\left[(00+10+20)^2+(01+11+21)^2+(02+12+22)^2\right]}{n_B} - \frac{(T_{..})^2}{p^n}$$

$$= \frac{(4-2+0)^2+(5+(-4)+1)^2+(8+(-5)+0)^2}{3} - \frac{(7)^2}{3^2}$$

$$= \frac{(2)^2+(2)^2+(3)^2}{3} - \frac{(7)^2}{3^2}$$

$$= 5.67 - 5.44$$
$$= 0.23$$

$$SS\,Total = \sum\sum\sum Y_{ijk}^2 - \frac{(T_{..})^2}{p^n}$$

$$= \left[(4)^2+(-2)^2+(5)^2+\cdots+(0)^2\right] - \frac{(7)^2}{3^2}$$

$$= 151 - 5.44$$
$$= 145.56$$

ANOVA Table:

SoV	d.f	SS	MS	F
Block	3-1=2	10.89	5.445	
A	3-1=2	131.56	65.78	45.52
B	3-1=2	0.23	0.11	0.08
Error	2	2.89	1.445	
Total	$3^2-1=8$	145.56		

A 3^3 factorial confounded in three blocks of nine runs each. The AB^2C^2 component of the three-factor interaction will be confounded with blocks.

Possible combination $= 3^3 = 27$

$1, a, a^2, b, b^2, ab, a^2b, ab^2, a^2b^2, c, c^2, ac,$
$a^2c, ac^2, a^2c^2, bc, b^2c, bc^2, b^2c^2, abc, a^2bc,$
$ab^2c, abc^2, a^2b^2c, ab^2c^2, a^2bc^2, a^2b^2c^2$

(000), (100), (200), (010), (020), (110), (210), (120), (220)
(001), (002), (101), (201), (102), (202), (011), (021), (012)
(022), (111), (211), (121), (112), (221), (122), (212), (222).

Confounded term : AB^2C^2.
So defining contrast will be:

$$L = x_1 + 2x_2 + 2x_3$$

The value of $L \pmod 3$ of each treatment combination may be found as follows:

$000: L = 0+2(0)+2(0) = 0 = 0 \pmod 3$
$100: L = 1+2(0)+2(0) = 1 = 1 \pmod 3$
$200: L = 2+2(0)+2(0) = 2 = 2 \pmod 3$
$010: L = 0+2(1)+2(0) = 2 = 2 \pmod 3$
$020: L = 0+2(2)+2(0) = 4 = 1 \pmod 3$
$110: L = 1+2(1)+2(0) = 3 = 0 \pmod 3$
$210: L = 2+2(1)+2(0) = 4 = 1 \pmod 3$
$120: L = 1+2(2)+2(0) = 5 = 2 \pmod 3$
$220: L = 2+2(2)+2(0) = 6 = 0 \pmod 3$ $\quad \begin{array}{r} 2 \\ 3\overline{)6} \\ \underline{6} \\ \boxed{0} \end{array}$
$001: L = 0+2(0)+2(1) = 2 = 2 \pmod 3$
$002: L = 0+2(0)+2(2) = 4 = 1 \pmod 3$
$101: L = 1+2(0)+2(1) = 3 = 0 \pmod 3$
$201: L = 2+2(0)+2(1) = 4 = 1 \pmod 3$
$102: L = 1+2(0)+2(2) = 5 = 2 \pmod 3$
$202: L = 2+2(0)+2(2) = 6 = 0 \pmod 3$

$011 : L = 0 + 2(1) + 2(1) = 4 = 1 \pmod 3$

$021 : L = 0 + 2(2) + 2(1) = 6 = 0 \pmod 3$

$012 : L = 0 + 2(1) + 2(2) = 6 = 0 \pmod 3$

$022 : L = 0 + 2(2) + 2(2) = 8 = 2 \pmod 3$

$111 : L = 1 + 2(1) + 2(1) = 5 = 2 \pmod 3$

$211 : L = 2 + 2(1) + 2(1) = 6 = 0 \pmod 3$

$121 : L = 1 + 2(2) + 2(1) = 7 = 1 \pmod 3$

$112 : L = 1 + 2(1) + 2(2) = 7 = 1 \pmod 3$

$221 : L = 2 + 2(2) + 2(1) = 8 = 2 \pmod 3$

$122 : L = 1 + 2(2) + 2(2) = 9 = 0 \pmod 3$

$212 : L = 2 + 2(1) + 2(2) = 8 = 2 \pmod 3$

$222 : L = 2 + 2(2) + 2(2) = 10 = 1 \pmod 3$

Block 1	Block 2	Block 3
000	100	200
110	020	010
220	210	120
101	002	001
202	201	102
021	011	022
012	121	111
211	112	221
122	222	212