Properties of the Normal and Multivariate Normal Distributions

By Students of the Course, edited by Will Welch

September 28, 2014

"Normal" and "Gaussian" may be used interchangeably.

1 Univariate Normal (Gaussian) Distribution

Let Y be a random variable with mean (expectation) μ and variance $\sigma^2 > 0$. Y is also normal, and its distribution is denoted by $\mathsf{N}(\mu, \sigma^2)$.

In the following a and b denote constants, i.e., they are not random variables.

1. **Density.** The normal distribution $\mathsf{N}(\mu, \sigma^2)$ has density

$$f_Y(y \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\mu)^2\right) \quad (-\infty < y < \infty).$$

2. **Shape.** The distribution is unimodal and the mode equals the mean equals the median.

3. **Exponential family.** The distribution belongs to the exponential family.

4. **Moment generating function.** The $\mathsf{N}(\mu, \sigma^2)$ distribution has MGF $M(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$, where t is real-valued.

5. **Characteristic function.** The $\mathsf{N}(\mu, \sigma^2)$ distribution has CF $\psi(t) = \exp\left(i\mu t - \frac{1}{2}\sigma^2 t^2\right)$, where t is real-valued.

6. **Linear transformation.**

   (a) The distribution of $a + bY$ is $\mathsf{N}(a + \mu, b^2\sigma^2)$.

   (b) The distribution of $(Y - \mu)/\sigma$ is $\mathsf{N}(0, 1)$, the standard normal.

7. **Smoothness.** The density is infinitely differentiable.

2 Multivariate Normal (Gaussian) Distribution

We have a vector of n random variables, $\mathbf{Y} = (Y_1, \ldots, Y_n)^T$. Denote the mean (expectation) of Y_i by μ_i, and let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^T$ be the $n \times 1$ vector of means. Similarly, $\boldsymbol{\Sigma}$ is the $n \times n$ matrix of covariances. Furthermore, the random variables in $\mathbf{Y}$ have a joint multivariate normal distribution, denoted by $\mathsf{MN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. We will assume the distribution is not degenerate, i.e., $\boldsymbol{\Sigma}$ is full rank, invertible, and hence positive definite.

The vector $\mathbf{a}$ denotes a vector of constants, i.e., not random variables, in the following. Similarly, $\mathbf{B}$ is a matrix of constants.

1. **Joint density.** The multivariate normal distribution $\mathsf{MN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ has joint density

$$f_{\mathbf{Y}}(\mathbf{y} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{n/2}} \frac{1}{\det^{1/2}(\boldsymbol{\Sigma})} \exp\left(-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\right) \quad (\mathbf{y} \in \mathcal{R}^n).$$

2. **Shape.** The contours of the joint distribution are n-dimensional ellipsoids.

3. **Mean and covariance specify the distribution.** The $\mathsf{MN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ joint distribution is specified by $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ only.

4. **Moment generating function.** The $\mathsf{MN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution has MGF $M(\mathbf{t}) = \exp\left(\boldsymbol{\mu}^T\mathbf{t} + \frac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}\right)$, where $\mathbf{t}$ is a real $n \times 1$ vector.

5. **Characteristic function.** The $\mathsf{MN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution has CF $\psi(\mathbf{t}) = \exp\left(i\boldsymbol{\mu}^T\mathbf{t} - \frac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}\right)$, where $\mathbf{t}$ is a real $n \times 1$ vector.

6. **Linear combinations.**

   (a) Let $\mathbf{a}$ be $n \times 1$. $\mathbf{Y}$ is $\mathsf{MN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ if and only if any linear combination $\mathbf{a}^T\mathbf{Y}$ has a (univariate) normal distribution.

   (b) Let $\mathbf{a}$ be $n \times 1$. The distribution of $\mathbf{a}^T\mathbf{Y}$ is $\mathsf{N}\left(\mathbf{a}^T\mu, \mathbf{a}^T\boldsymbol{\Sigma}\mathbf{a}\right)$.

   (c) Let $\mathbf{a}$ be $m \times 1$ and $\mathbf{B}$ be $m \times n$. The distribution of the m random variables $\mathbf{a} + \mathbf{B}\mathbf{Y}$ is $\mathsf{MN}\left(\mathbf{a} + \mathbf{B}\mu, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^T\right)$.

   (d) Let $\mathbf{Z}$ be n independent standard normal random variables. Then $\mathbf{Y} = \boldsymbol{\mu} + \mathbf{L}\mathbf{Z}$, with $\mathbf{L}\mathbf{L}^T = \boldsymbol{\Sigma}$, has a $\mathsf{MN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution.

   (e) Again let $\mathbf{L}\mathbf{L}^T = \boldsymbol{\Sigma}$. Then $\mathbf{Z} = \mathbf{L}^{-1}(\mathbf{Y} - \boldsymbol{\mu})$ has a $\mathsf{MN}(\mathbf{0}, \mathbf{I})$ distribution.

7. **Independence.**

   (a) Y_i and Y_j are independent if and only if $\boldsymbol{\Sigma}_{ij} = 0$.

   (b) Pairwise independence of Y_i and Y_j for all $i \neq j$ implies complete independence.

3 Marginal and Conditional Multivariate Normal (Gaussian) Distributions

Let the $n \times 1$ vector $\mathbf{Y}$ with a $\mathsf{MN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution be partitioned as $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2)$, where $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are $m \times 1$ and $(n-m) \times 1$, respectively. Similarly, partition $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

1. **Marginal distribution.** The m-dimensional marginal distribution of $\mathbf{Y}_1$ is $\mathsf{MN}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$.

2. **Conditional distribution.** The m-dimensional distribution of $\mathbf{Y}_1$ conditional on $\mathbf{Y}_2$ is

$$\mathsf{MN}\left(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{Y}_2 - \boldsymbol{\mu}_2), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right).$$

3. **Independence.** $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are independent if and only if $\boldsymbol{\Sigma}_{12} = \mathbf{0}$.