Bivariate Normal Distribution

The bivariate normal distribution is the statistical distribution with probability density function

$$P(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{z}{2(1-\rho^2)}\right], \quad (1)$$

where

$$z \equiv \frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}, \quad (2)$$

and

$$\rho \equiv \operatorname{cor}(x_1, x_2) = \frac{V_{12}}{\sigma_1\sigma_2}$$

is the correlation of x_1 and x_2 (Kenney and Keeping 1951, pp. 92 and 202-205; Whittaker and Robinson 1967, p. 329) and V_{12} is the covariance.

The marginal probabilities are then

$$P(x_1) = \int_{-\infty}^{\infty} P(x_1, x_2)\, d x_2 \quad (4)$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}} e^{-(x_1-\mu_1)^2/(2\sigma_1^2)} \quad (5)$$

and

$$P(x_2) = \int_{-\infty}^{\infty} P(x_1, x_2)\, d x_1 \quad (6)$$

$$= \frac{1}{\sigma_2\sqrt{2\pi}} e^{-(x_2-\mu_2)^2/(2\sigma_2^2)} \quad (7)$$

(Kenney and Keeping 1951, p. 202).

Let Z_1 and Z_2 be two independent normal variates with means $\mu_i = 0$ and $\sigma_i^2 = 1$ for $i = 1$, 2. Then the variables a_1 and a_2 defined below are normal bivariates with unit variance and correlation coefficient ρ:

$$a_1 = \sqrt{\frac{1+\rho}{2}}\, z_1 + \sqrt{\frac{1-\rho}{2}}\, z_2 \quad (8)$$

$$a_2 = \sqrt{\frac{1+\rho}{2}}\, z_1 - \sqrt{\frac{1-\rho}{2}}\, z_2. \tag{9}$$

To derive the bivariate normal probability function, let X_1 and X_2 be normally and independently distributed variates with mean 0 and variance 1, then define

$$Y_1 \equiv \mu_1 + \sigma_{11} X_1 + \sigma_{12} X_2 \tag{10}$$
$$Y_2 \equiv \mu_2 + \sigma_{21} X_1 + \sigma_{22} X_2 \tag{11}$$

(Kenney and Keeping 1951, p. 92). The variates Y_1 and Y_2 are then themselves normally distributed with means μ_1 and μ_2, variances

$$\sigma_1^2 \equiv \sigma_{11}^2 + \sigma_{12}^2 \tag{12}$$
$$\sigma_2^2 \equiv \sigma_{21}^2 + \sigma_{22}^2, \tag{13}$$

and covariance

$$V_{12} \equiv \sigma_{11}\,\sigma_{21} + \sigma_{12}\,\sigma_{22}. \tag{14}$$

The covariance matrix is defined by

$$V_{ij} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\,\sigma_2 \\ \rho\sigma_1\,\sigma_2 & \sigma_2^2 \end{bmatrix}, \tag{15}$$

where

$$\rho \equiv \frac{V_{12}}{\sigma_1\,\sigma_2} = \frac{\sigma_{11}\,\sigma_{21} + \sigma_{12}\,\sigma_{22}}{\sigma_1\,\sigma_2}. \tag{16}$$

Now, the joint probability density function for x_1 and x_2 is

$$f(x_1, x_2)\, d\,x_1\, d\,x_2 = \frac{1}{2\pi}\, e^{-\left(x_1^2 + x_2^2\right)/2}\, d\,x_1\, d\,x_2, \tag{17}$$

So, we proceed as

$$\begin{bmatrix} y_1 - \mu_1 \\ y_2 - \mu_2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{18}$$

As long as

$$\begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} \neq 0, \tag{19}$$

this can be inverted to give

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} y_1 - \mu_1 \\ y_2 - \mu_2 \end{bmatrix} \tag{20}$$

$$= \frac{1}{\sigma_{11}\,\sigma_{22} - \sigma_{12}\,\sigma_{21}} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{bmatrix} \begin{bmatrix} y_1 - \mu_1 \\ y_2 - \mu_2 \end{bmatrix}. \tag{21}$$

Therefore,

$$x_1^2 + x_2^2 = \frac{[\sigma_{22}\,(y_1 - \mu_1) - \sigma_{12}\,(y_2 - \mu_2)]^2}{(\sigma_{11}\,\sigma_{22} - \sigma_{12}\,\sigma_{21})^2} + \frac{[-\sigma_{21}\,(y_1 - \mu_1) + \sigma_{11}\,(y_2 - \mu_2)]^2}{(\sigma_{11}\,\sigma_{22} - \sigma_{12}\,\sigma_{21})^2}, \tag{22}$$

and expanding the numerator of (22) gives

$$\sigma_{22}^2\,(y_1 - \mu_1)^2 - 2\,\sigma_{12}\,\sigma_{22}\,(y_1 - \mu_1)\,(y_2 - \mu_2) + \sigma_{12}^2\,(y_2 - \mu_2)^2 + \sigma_{21}^2\,(y_1 - \mu_1)^2 - 2\,\sigma_{11}\,\sigma_{21}\,(y_1 - \mu_1)\,(y_2 - \mu_2) + \sigma_{11}^2\,(y_2 - \mu_2)^2, \tag{23}$$

so

$$\begin{aligned} &\left(x_1^2 + x_2^2\right)(\sigma_{11}\,\sigma_{22} - \sigma_{12}\,\sigma_{21})^2 \\ &= (y_1 - \mu_1)^2 \left(\sigma_{21}^2 + \sigma_{22}^2\right) - \\ &\quad 2\,(y_1 - \mu_1)\,(y_2 - \mu_2)\,(\sigma_{11}\,\sigma_{21} + \sigma_{12}\,\sigma_{22}) + (y_2 - \mu_2)^2 \left(\sigma_{11}^2 + \sigma_{12}^2\right) \\ &= \sigma_2^2\,(y_1 - \mu_1)^2 - 2\,(y_1 - \mu_1)\,(y_2 - \mu_2)\,(\rho\sigma_1\,\sigma_2) + \sigma_1^2\,(y_2 - \mu_2)^2 \\ &= \sigma_1^2\,\sigma_2^2 \left[\frac{(y_1 - \mu_1)^2}{\sigma_1^2} - \frac{2\,\rho\,(y_1 - \mu_1)\,(y_2 - \mu_2)}{\sigma_1\,\sigma_2} + \frac{(y_2 - \mu_2)^2}{\sigma_2^2}\right]. \end{aligned} \tag{24}$$

Now, the denominator of (24) is

$$\sigma_{11}^2\,\sigma_{21}^2 + \sigma_{11}^2\,\sigma_{22}^2 + \sigma_{12}^2\,\sigma_{21}^2 + \sigma_{12}^2\,\sigma_{22}^2 - \sigma_{11}^2\,\sigma_{21}^2 - 2\,\sigma_{11}\,\sigma_{12}\,\sigma_{21}\,\sigma_{22} - \sigma_{12}^2\,\sigma_{22}^2 = (\sigma_{11}\,\sigma_{22} - \sigma_{12}\,\sigma_{21})^2, \tag{25}$$

so

$$\frac{1}{1 - \rho^2} = \frac{1}{1 - \frac{V_{12}{}^2}{\sigma_1^2\,\sigma_2^2}} \tag{26}$$

$$= \frac{\sigma_1^2\,\sigma_2^2}{\sigma_1^2\,\sigma_2^2 - V_{12}{}^2} \tag{27}$$

$$= \frac{\sigma_1^2\,\sigma_2^2}{\left(\sigma_{11}^2 + \sigma_{12}^2\right)\left(\sigma_{21}^2 + \sigma_{22}^2\right) - (\sigma_{11}\,\sigma_{21} + \sigma_{12}\,\sigma_{22})^2}. \tag{28}$$

can be written simply as

$$\frac{1}{1-\rho^2} = \frac{\sigma_1^2\,\sigma_2^2}{(\sigma_{11}\,\sigma_{22} - \sigma_{12}\,\sigma_{21})^2}, \tag{29}$$

and

$$x_1^2 + x_2^2 = \frac{1}{1-\rho^2}\left[\frac{(y_1-\mu_1)^2}{\sigma_1^2} - \frac{2\,\rho\,(y_1-\mu_1)\,(y_2-\mu_2)}{\sigma_1\,\sigma_2} + \frac{(y_2-\mu_2)^2}{\sigma_2^2}\right]. \tag{30}$$

Solving for x_1 and x_2 and defining

$$\rho' \equiv \frac{\sigma_1\,\sigma_2\,\sqrt{1-\rho^2}}{\sigma_{11}\,\sigma_{22} - \sigma_{12}\,\sigma_{21}} \tag{31}$$

gives

$$x_1 = \frac{\sigma_{22}\,(y_1-\mu_1) - \sigma_{12}\,(y_2-\mu_2)}{\rho'} \tag{32}$$

$$x_2 = \frac{-\sigma_{21}\,(y_1-\mu_1) + \sigma_{11}\,(y_2-\mu_2)}{\rho'}. \tag{33}$$

But the Jacobian is

$$J\left(\frac{x_1,\,x_2}{y_1,\,y_2}\right) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{\sigma_{22}}{\rho'} & -\frac{\sigma_{12}}{\rho'} \\ -\frac{\sigma_{21}}{\rho'} & \frac{\sigma_{11}}{\rho'} \end{vmatrix} \tag{34}$$

$$= \frac{1}{\rho'^2}\,(\sigma_{11}\,\sigma_{22} - \sigma_{12}\,\sigma_{21}) \tag{35}$$

$$= \frac{1}{\rho'} = \frac{1}{\sigma_1\,\sigma_2\,\sqrt{1-\rho^2}}, \tag{36}$$

so

$$d\,x_1\,d\,x_2 = \frac{d\,y_1\,d\,y_2}{\sigma_1\,\sigma_2\,\sqrt{1-\rho^2}} \tag{37}$$

and

$$\frac{1}{2\,\pi}\,e^{-(x_1^2+x_2^2)/2}\,d\,x_1\,d\,x_2 = \frac{1}{2\,\pi\sigma_1\,\sigma_2\,\sqrt{1-\rho^2}}\,\exp\left[-\frac{z}{2\,(1-\rho^2)}\right]d\,y_1\,d\,y_2, \tag{38}$$

where

$$z \equiv \frac{(y_1 - \mu_1)^2}{\sigma_1^2} - \frac{2\,\rho\,(y_1 - \mu_1)\,(y_2 - \mu_2)}{\sigma_1\,\sigma_2} + \frac{(y_2 - \mu_2)^2}{\sigma_2^2}. \tag{39}$$

Q.E.D.

The characteristic function of the bivariate normal distribution is given by

$$\phi(t_1, t_2) \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\,(t_1\,x_1 + t_2\,x_2)}\,P(x_1, x_2)\,d\,x_1\,d\,x_2 \tag{40}$$

$$= N \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\,(t_1\,x_1 + t_2\,x_2)} \exp\left[-\frac{z}{2\left(1 - \rho^2\right)}\right] d\,x_1\,d\,x_2, \tag{41}$$

where

$$z \equiv \left[\frac{(x_1 - \mu_1)^2}{\sigma_1^2} - \frac{2\,\rho\,(x_1 - \mu_1)\,(x_2 - \mu_2)}{\sigma_1\,\sigma_2} + \frac{(x_2 - \mu_2)^2}{\sigma_2^2}\right] \tag{42}$$

and

$$N \equiv \frac{1}{2\,\pi\sigma_1\,\sigma_2\,\sqrt{1 - \rho^2}}. \tag{43}$$

Now let

$$u \equiv x_1 - \mu_1 \tag{44}$$

$$w \equiv x_2 - \mu_2. \tag{45}$$

Then

$$\phi(t_1, t_2) = N' \int_{-\infty}^{\infty} \left(e^{i\,t_2\,w} \exp\left[-\frac{1}{2\left(1 - \rho^2\right)}\,\frac{w^2}{\sigma_2^2}\right]\right) \int_{-\infty}^{\infty} e^{v}\,e^{i\,t_1\,u}\,d\,u\,d\,w, \tag{46}$$

where

$$v \equiv -\frac{1}{2\left(1 - \rho^2\right)}\,\frac{1}{\sigma_1^2}\left[u^2 - \frac{2\,\rho\sigma_1\,w}{\sigma_2}\,u\right] \tag{47}$$

$$N' \equiv \frac{e^{i\,(t_1\,\mu_1 + t_2\,\mu_2)}}{2\,\pi\sigma_1\,\sigma_2\,\sqrt{1 - \rho^2}}. \tag{48}$$

Complete the square in the inner integral

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\frac{1}{\sigma_1^2}\left[u^2 - \frac{2\rho\sigma_1 w}{\sigma_2}u\right]\right\} e^{t_1 u}\, d u$$
$$= \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma_1^2(1-\rho^2)}\left[u - \frac{\rho_1 \sigma_1 w}{\sigma^2}\right]^2\right\}\left\{\frac{1}{2\sigma_1^2(1-\rho^2)}\left(\frac{\rho_1 \sigma_1 w}{\sigma_2}\right)^2\right\} e^{i t_1 u}\, d u. \quad (49)$$

Rearranging to bring the exponential depending on w outside the inner integral, letting

$$v \equiv u - \rho\frac{\sigma_1 w}{\sigma_2}, \quad (50)$$

and writing

$$e^{i t_1 u} = \cos(t_1 u) + i \sin(t_1 u) \quad (51)$$

gives

$$\phi(t_1, t_2) =$$
$$N' \int_{-\infty}^{\infty} e^{i t_2 w} \exp\left[-\frac{1}{2\sigma_2^2(1-\rho^2)} w^2\right] \exp\left[\frac{\rho^2}{2\sigma_2^2(1-\rho^2)} w^2\right] \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma_2^2(1-\rho^2)}\right.$$
$$\left. v^2\right]\left\{\cos\left[t_1\left(v + \frac{\rho\sigma_1 w}{\sigma_2}\right)\right] + i \sin\left[t_1\left(v + \frac{\rho\sigma_1 w}{\sigma_2}\right)\right]\right\} d v\, d w. \quad (52)$$

Expanding the term in braces gives

$$\left[\cos(t_1 v)\cos\left(\frac{\rho\sigma_1 w t_1}{\sigma_2}\right) - \sin(t_1 v)\sin\left(\frac{\rho\sigma_1 w}{\sigma_2 t_1}\right)\right] +$$
$$i\left[\sin(t_1 v)\cos\left(\frac{\rho\sigma_1 w}{\sigma_2 t_1}\right) + \cos(t_1 v)\sin\left(\frac{\rho\sigma_1 w t_1}{\sigma_2}\right)\right]$$
$$= \left[\cos\left(\frac{\rho\sigma_1 w t_1}{\sigma_2}\right) + i \sin\left(\frac{\rho\sigma_1 w t_1}{\sigma_2}\right)\right][\cos(t_1 v) + i \sin(t_1 v)] =$$
$$\exp\left(\frac{i\rho\sigma_1 w}{\sigma_2} t_1\right)[\cos(t_1 v) + i \sin(t_1 v)]. \quad (53)$$

But $e^{-a x^2} \sin(b x)$ is odd, so the integral over the sine term vanishes, and we are left with

$$\phi(t_1, t_2) = N' \int_{-\infty}^{\infty} e^{i t_2 w} \exp\left[-\frac{w^2}{2\sigma_2^2}\right] \exp\left[\frac{\rho^2 w^2}{2\sigma_2^2 (1-\rho^2)}\right]$$

$$\exp\left[\frac{i\rho\sigma_1 w t_1}{\sigma_2}\right] d w \int_{-\infty}^{\infty} \exp\left[-\frac{v^2}{2\sigma_1^2 (1-\rho^2)}\right] \cos(t_1 v)\, d v$$

$$= N' \int_{-\infty}^{\infty} \exp\left[i w \left(t_2 + t_1 \left(\rho \frac{\sigma_1}{\sigma_2}\right)\right)\right] \exp\left[-\frac{w^2}{2\sigma_2^2}\right] d w \tag{54}$$

$$\int_{-\infty}^{\infty} \exp\left[-\frac{v^2}{2\sigma_1^2 (1-\rho^2)}\right] \cos(t_1 v)\, d v.$$

Now evaluate the Gaussian integral

$$\int_{-\infty}^{\infty} e^{i k x} e^{-a x^2} \, d x = \int_{-\infty}^{\infty} e^{-a x^2} \cos(k x)\, d x \tag{55}$$

$$= \sqrt{\frac{\pi}{a}} \, e^{-k^2/4a} \tag{56}$$

to obtain the explicit form of the characteristic function,

$$\phi(t_1, t_2) = \frac{e^{i(t_1 \mu_1 + t_2 \mu_2)}}{2\pi\sigma_1 \sigma_2 \sqrt{1-\rho^2}} \left\{\sigma_2 \sqrt{2\pi} \exp\left[-\frac{1}{4}\left(t_2 + \rho \frac{\sigma_1}{\sigma_2} t_1\right)^2 2\sigma_2^2\right]\right\}$$

$$\left\{\sigma_1 \sqrt{2\pi(1-\rho^2)} \exp\left[-\tfrac{1}{4} t_1^2 \, 2\sigma_1^2 (1-\rho^2)\right]\right\} \tag{57}$$

$$= e^{i(t_1 \mu_1 + t_2 \mu_2)} \exp\left\{-\tfrac{1}{2}\left[t_2^2 \sigma_2^2 + 2\rho\sigma_1 \sigma_2 t_1 t_2 + \rho^2 \sigma_1^2 t_1^2 + (1-\rho^2)\sigma_1^2 t_1^2\right]\right\}$$

$$= \exp\left[i(t_1 \mu_1 + t_2 \mu_2) - \tfrac{1}{2}\left(\sigma_1^2 t_1^2 + 2\rho\sigma_1 \sigma_2 t_1 t_2 + \sigma_2^2 t_2^2\right)\right].$$

In the singular case that

$$\begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} = 0 \tag{58}$$

(Kenney and Keeping 1951, p. 94), it follows that

$$\sigma_{11} \sigma_{22} = \sigma_{12} \sigma_{21} \tag{59}$$

$$y_1 = \mu_1 + \sigma_{11} x_1 + \sigma_{12} x_2 \tag{60}$$

$$y_2 = \mu_2 + \frac{\sigma_{12} \sigma_{21}}{\sigma_{11}} x_2 \tag{61}$$

$$= \mu_2 + \frac{\sigma_{11} \sigma_{21} x_1 + \sigma_{12} \sigma_{21} x_2}{\sigma_{11}} \tag{62}$$

$$= \mu_2 + \frac{\sigma_{21}}{\sigma_{11}} (\sigma_{11} x_1 + \sigma_{12} x_2), \tag{63}$$

so

$$y_1 = \mu_1 + x_3 \quad (64)$$
$$y_2 = \mu_2 + \frac{\sigma_{21}}{\sigma_{11}} x_3, \quad (65)$$

where

$$x_3 = y_1 - \mu_1 \quad (66)$$
$$= \frac{\sigma_{11}}{\sigma_{21}} (y_2 - \mu_2). \quad (67)$$

The standardized bivariate normal distribution takes $\sigma_1 = \sigma_2 = 1$ and $\mu_1 = \mu_2 = 0$. The quadrant probability in this special case is then given analytically by

$$P(x_1 \le 0, x_2 \le 0) = P(x_1 \ge 0, x_2 \ge 0) \quad (68)$$
$$= \int_{-\infty}^{0} \int_{-\infty}^{0} P(x_1, x_2)\, d\,x_1\, d\,x_2 \quad (69)$$
$$= \frac{1}{4} + \frac{\sin^{-1} \rho}{2\,\pi} \quad (70)$$

(Rose and Smith 1996; Stuart and Ord 1998; Rose and Smith 2002, p. 231). Similarly,

$$P(x_1 \le 0, x_2 \ge 0) = P(x_1 \ge 0, x_2 \le 0) \quad (71)$$
$$= \int_{-\infty}^{0} \int_{0}^{\infty} P(x_1, x_2)\, d\,x_1\, d\,x_2 \quad (72)$$
$$= \frac{\cos^{-1} \rho}{2\,\pi}. \quad (73)$$