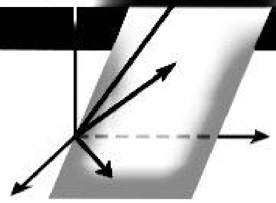
CHAPTER 1



# Systems of Linear Equations and Matrices

CHAPTER CONTENTS



INTRODUCTION

Information in science, business, and mathematics is often organized into rows and columns to form rectangular arrays called "matrices" (plural of "matrix"). Matrices often appear as tables of numerical data that arise from physical observations, but they occur in various mathematical contexts as well. For example, we will see in this chapter that all of the information required to solve a system of equations such as

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned}$$

is embodied in the matrix

$$\begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$$

and that the solution of the system can be obtained by performing appropriate operations on this matrix. This is particularly important in developing computer programs for solving systems of equations because computers are well suited for manipulating arrays of numerical information. However, matrices are not simply a notational tool for solving systems of equations; they can be viewed as mathematical objects in their own right, and there is a rich and important theory associated with them that has a multitude of practical applications. It is the study of matrices and related topics that forms the mathematical field that we call "linear algebra." In this chapter we will begin our study of matrices.





## 1.1 Introduction to Systems of Linear Equations

Systems of linear equations and their solutions constitute one of the major topics that we will study in this course. In this first section we will introduce some basic terminology and discuss a method for solving such systems.

*Linear Equations*

Recall that in two dimensions a line in a rectangular $xy$-coordinate system can be represented by an equation of the form

$$ax + by = c \quad (a, b \text{ not both } 0)$$

and in three dimensions a plane in a rectangular $xyz$-coordinate system can be represented by an equation of the form

$$ax + by + cz = d \quad (a, b, c \text{ not all } 0)$$

These are examples of "linear equations," the first being a linear equation in the variables $x$ and $y$ and the second a linear equation in the variables $x$, $y$, and $z$. More generally, we define a ***linear equation*** in the $n$ variables $x_1, x_2, \ldots, x_n$ to be one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where $a_1, a_2, \ldots, a_n$ and $b$ are constants, and the $a$'s are not all zero. In the special cases where $n = 2$ or $n = 3$, we will often use variables without subscripts and write linear equations as

$$a_1x + a_2y = b \quad (a_1, a_2 \text{ not both } 0) \tag{2}$$

$$a_1x + a_2y + a_3z = b \quad (a_1, a_2, a_3 \text{ not all } 0) \tag{3}$$

In the special case where $b = 0$, Equation (1) has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \tag{4}$$

which is called a ***homogeneous linear equation*** in the variables $x_1, x_2, \ldots, x_n$.

▶ **EXAMPLE 1 Linear Equations**

Observe that a linear equation does not involve any products or roots of variables. All

> The double subscripting on the coefficients $a_{ij}$ of the unknowns gives their location in the system—the first subscript indicates the equation in which the coefficient occurs, and the second indicates which unknown it multiplies. Thus, $a_{12}$ is in the first equation and multiplies $x_2$.

A general linear system of $m$ equations in the $n$ unknowns $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 &+ a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\vdots \\ a_{m1}x_1 &+ a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{aligned} \tag{7}$$

A ***solution*** of a linear system in $n$ unknowns $x_1, x_2, \ldots, x_n$ is a sequence of $n$ numbers $s_1, s_2, \ldots, s_n$ for which the substitution

$$x_1 = s_1, \quad x_2 = s_2, \ldots, \quad x_n = s_n$$

makes each equation a true statement. For example, the system in (5) has the solution

$$x = 1, \quad y = -2$$

and the system in (6) has the solution

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = -1$$

These solutions can be written more succinctly as

$$(1, -2) \quad \text{and} \quad (1, 2, -1)$$

in which the names of the variables are omitted. This notation allows us to interpret these solutions geometrically as points in two-dimensional and three-dimensional space. More generally, a solution

$$x_1 = s_1, \quad x_2 = s_2, \ldots, \quad x_n = s_n$$

of a linear system in $n$ unknowns can be written as

$$(s_1, s_2, \ldots, s_n)$$

which is called an ***ordered n-tuple***. With this notation it is understood that all variables appear in the same order in each equation. If $n = 2$, then the $n$-tuple is called an ***ordered pair***, and if $n = 3$, then it is called an ***ordered triple***.

*Linear Systems in Two and Three Unknowns*

Linear systems in two unknowns arise in connection with intersections of lines. For example, consider the linear system

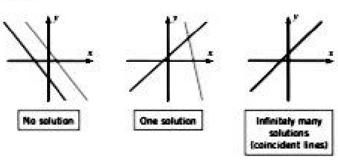
$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

in which the graphs of the equations are lines in the $xy$-plane. Each solution $(x, y)$ of this system corresponds to a point of intersection of the lines, so there are three possibilities (Figure 1.1.1):

1. The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.
2. The lines may intersect at only one point, in which case the system has exactly one solution.
3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.

In general, we say that a linear system is ***consistent*** if it has at least one solution and ***inconsistent*** if it has no solutions. Thus, a *consistent* linear systemof two equations in



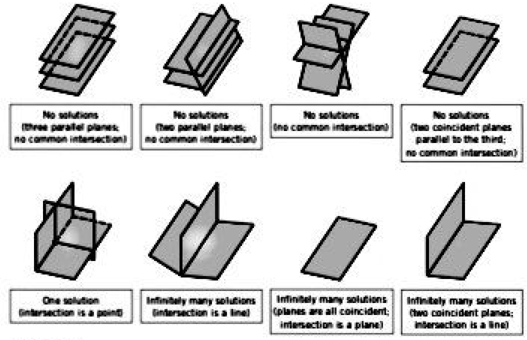



► Figure 1.1.1

two unknowns has either one solution or infinitely many solutions—there are no other possibilities. The same is true for a linear system of three equations in three unknowns

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

in which the graphs of the equations are planes. The solutions of the system, if any, correspond to points where all three planes intersect, so again we see that there are only three possibilities—no solutions, one solution, or infinitely many solutions (Figure 1.1.2).

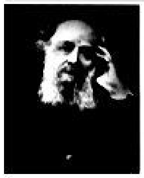



▲ Figure 1.1.2

We will prove later that our observations about the number of solutions of linear systems of two equations in two unknowns and linear systems of three equations in three unknowns actually hold for *all* linear systems. That is:

> ***Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.***

about the main diagonal



James Sylvester
(1814–1897)

Arthur Cayley
(1821–1895)

**Historical Note** The term *matrix* was first used by the English mathematician James Sylvester, who defined the term in 1850 to be an "oblong arrangement of terms." Sylvester communicated his work on matrices to a fellow English mathematician and lawyer named Arthur Cayley, who then introduced some of the basic operations on matrices in a book entitled *Memoir on the Theory of Matrices* that was published in 1858. As a matter of interest, Sylvester, who was Jewish, did not get his college degree because he refused to sign a required oath to the Church of England. He was appointed to a chair at the University of Virginia in the United States but resigned after swatting a student with a stick because he was reading a newspaper in class. Sylvester, thinking he had killed the student, fled back to England on the first available ship. Fortunately, the student was not dead, just in shock!

[*Images:* © Bettmann/CORBIS (*Sylvester*); *Photo Researchers/Getty Images* (*Cayley*)]



*Trace of a Matrix*

**DEFINITION 8** If $A$ is a square matrix, then the ***trace of*** $A$, denoted by $\text{tr}(A)$, is defined to be the sum of the entries on the main diagonal of $A$. The trace of $A$ is undefined if $A$ is not a square matrix.

**▶ EXAMPLE 12 Trace**

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\text{tr}(A) = a_{11} + a_{22} + a_{33} \qquad \text{tr}(B) = -1 + 5 + 7 + 0 = 11$$ ◀

In the exercises you will have some practice working with the transpose and trace operations.

## Exercise Set 1.3

▶ In Exercises 1–2, suppose that $A$, $B$, $C$, $D$, and $E$ are matrices with the following sizes:

| $A$ | $B$ | $C$ | $D$ | $E$ |
|---|---|---|---|---|
| $(4 \times 5)$ | $(4 \times 5)$ | $(5 \times 2)$ | $(4 \times 2)$ | $(5 \times 4)$ |

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix. ◀

**1.** (a) $BA$ (b) $AB^T$ (c) $AC + D$
(d) $E(AC)$ (e) $A - 3E^T$ (f) $E(5B + A)$

**2.** (a) $CD^T$ (b) $DC$ (c) $BC - 3D$
(d) $D^T(BE)$ (e) $B^TD + ED$ (f) $BA^T + D$

▶ In Exercises 3–6, use the following matrices to compute the indicated expression if it is defined.

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$ ◀

**3.** (a) $D + E$ (b) $D - E$ (c) $5A$
(d) $-7C$ (e) $2B - C$ (f) $4E - 2D$
(g) $-3(D + 2E)$ (h) $A - A$ (i) $\text{tr}(D)$
(j) $\text{tr}(D - 3E)$ (k) $4\,\text{tr}(7B)$ (l) $\text{tr}(A)$

**4.** (a) $2A^T + C$ (b) $D^T - E^T$ (c) $(D - E)^T$
(d) $B^T + 5C^T$ (e) $\frac{1}{2}C^T - \frac{1}{4}A$ (f) $B - B^T$
(g) $2E^T - 3D^T$ (h) $(2E^T - 3D^T)^T$ (i) $(CD)E$
(j) $C(BA)$ (k) $\text{tr}(DE^T)$ (l) $\text{tr}(BC)$

**5.** (a) $AB$ (b) $BA$ (c) $(3E)D$
(d) $(AB)C$ (e) $A(BC)$ (f) $CC^T$
(g) $(DA)^T$ (h) $(C^TB)A^T$ (i) $\text{tr}(DD^T)$
(j) $\text{tr}(4E^T - D)$ (k) $\text{tr}(C^TA^T + 2E^T)$ (l) $\text{tr}((EC^T)^TA)$

**6.** (a) $(2D^T - E)A$ (b) $(4B)C + 2B$
(c) $(-AC)^T + 5D^T$ (d) $(BA^T - 2C)^T$
(e) $B^T(CC^T - A^TA)$ (f) $D^TE^T - (ED)^T$

▶ In Exercises 7–8, use the following matrices and either the row method or the column method, as appropriate, to find the indicated row or column.

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$ ◀

**7.** (a) the first row of $AB$ (b) the third row of $AB$
(c) the second column of $AB$ (d) the first column of $BA$
(e) the third row of $AA$ (f) the third column of $AA$



**8.** (a) the first column of $AB$ (b) the third column of $BB$
(c) the second row of $BB$ (d) the first column of $AA$
(e) the third column of $AB$ (f) the first row of $BA$

▶ In Exercises 9–10, use matrices $A$ and $B$ from Exercises 7–8. ◀

**9.** (a) Express each column vector of $AA$ as a linear combination of the column vectors of $A$.
(b) Express each column vector of $BB$ as a linear combination of the column vectors of $B$.

**10.** (a) Express each column vector of $AB$ as a linear combination of the column vectors of $A$.
(b) Express each column vector of $BA$ as a linear combination of the column vectors of $B$.

▶ In each part of Exercises 11–12, find matrices $A$, $\mathbf{x}$, and $\mathbf{b}$ that express the given linear system as a single matrix equation $A\mathbf{x} = \mathbf{b}$, and write out this matrix equation. ◀

**11.** (a)
$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 &= 7 \\ 9x_1 - x_2 + x_3 &= -1 \\ x_1 + 5x_2 + 4x_3 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 4x_1 \qquad - 3x_3 + x_4 &= 1 \\ 5x_1 + x_2 \qquad - 8x_4 &= 3 \\ 2x_1 - 5x_2 + 9x_3 - x_4 &= 0 \\ 3x_2 - x_3 + 7x_4 &= 2 \end{aligned}$$

**12.** (a)
$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -3 \\ 2x_1 + x_2 \qquad &= 0 \\ -3x_2 + 4x_3 &= 1 \\ x_1 \qquad + x_3 &= 5 \end{aligned}$$

(b)
$$\begin{aligned} 3x_1 + 3x_2 + 3x_3 &= -3 \\ -x_1 - 5x_2 - 2x_3 &= 3 \\ -4x_2 + x_3 &= 0 \end{aligned}$$

▶ In Exercises 15–16, find all values of $k$, if any, that satisfy the equation. ◀

**15.** $$\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0$$

**16.** $$\begin{bmatrix} 2 & 2 & k \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ k \end{bmatrix} = 0$$

▶ In Exercises 17–20, use the column-row expansion of $AB$ to express this product as a sum of matrices. ◀

**17.** $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix}$

**18.** $A = \begin{bmatrix} 0 & -2 \\ 4 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 1 \\ -3 & 0 & 2 \end{bmatrix}$

**19.** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

**20.** $A = \begin{bmatrix} 0 & 4 & 2 \\ 1 & -2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 4 & 0 \\ 1 & -1 \end{bmatrix}$

**21.** For the linear system in Example 5 of Section 1.2, express the general solution that we obtained in that example as a linear combination of column vectors that contain only numerical entries. [*Suggestion:* Rewrite the general solution as a single column vector, then write that column vector as a sum of column vectors each of which contains at most one parameter, and then factor out the parameters.]

as guaranteed by Theorem 1.4.9. ◀

## Exercise Set 1.4

▶ In Exercises 1–2, verify that the following matrices and scalars satisfy the stated properties of Theorem 1.4.1.

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ 1 & -4 \end{bmatrix},$$

$$C = \begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix}, \quad a = 4, \quad b = -7 \quad ◀$$

1. (a) The associative law for matrix addition.
   (b) The associative law for matrix multiplication.
   (c) The left distributive law.
   (d) $(a + b)C = aC + bC$

2. (a) $a(BC) = (aB)C = B(aC)$
   (b) $A(B - C) = AB - AC$ (c) $(B + C)A = BA + CA$
   (d) $a(bC) = (ab)C$

▶ In Exercises 3–4, verify that the matrices and scalars in Exercise 1 satisfy the stated properties. ◀

3. (a) $(A^T)^T = A$ (b) $(AB)^T = B^TA^T$

4. (a) $(A + B)^T = A^T + B^T$ (b) $(aC)^T = aC^T$

▶ In Exercises 5–8, use Theorem 1.4.5 to compute the inverse of the matrix. ◀

5. $A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$ 6. $B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$



7. $C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ 8. $D = \begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}$

9. Find the inverse of

$$\begin{bmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^x - e^{-x}) \\ \frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \end{bmatrix}$$

10. Find the inverse of

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

▶ In Exercises 11–14, verify that the equations are valid for the matrices in Exercises 5–8. ◀

11. $(A^T)^{-1} = (A^{-1})^T$ 12. $(A^{-1})^{-1} = A$

13. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ 14. $(ABC)^T = C^TB^TA^T$

▶ In Exercises 15–18, use the given information to find $A$. ◀

15. $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$ 16. $(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$

17. $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$ 18. $A^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$

▶ In Exercises 19–20, compute the following using the given matrix $A$.

(a) $A^3$ (b) $A^{-3}$ (c) $A^2 - 2A + I$ ◀

19. $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ 20. $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$

▶ In Exercises 21–22, compute $p(A)$ for the given matrix $A$ and the following polynomials.

(a) $p(x) = x - 2$
(b) $p(x) = 2x^2 - x + 1$
(c) $p(x) = x^3 - 2x + 1$ ◀

21. $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ 22. $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$

▶ In Exercises 23–24, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad ◀$$

23. Find all values of $a$, $b$, $c$, and $d$ (if any) for which the matrices $A$ and $B$ commute.

24. Find all values of $a$, $b$, $c$, and $d$ (if any) for which the matrices $A$ and $C$ commute.

▶ In Exercises 25–28, use the method of Example 8 to find the unique solution of the given linear system. ◀

25. $3x_1 - 2x_2 = -1$
    $4x_1 + 5x_2 = 3$

26. $-x_1 + 5x_2 = 4$
    $-x_1 - 3x_2 = 1$

27. $6x_1 + x_2 = 0$
    $4x_1 - 3x_2 = -2$

28. $2x_1 - 2x_2 = 4$
    $x_1 + 4x_2 = 4$

▶ If a polynomial $p(x)$ can be factored as a product of lower degree polynomials, say

$$p(x) = p_1(x)p_2(x)$$

and if $A$ is a square matrix, then it can be proved that

$$p(A) = p_1(A)p_2(A)$$

In Exercises 29–30, verify this statement for the stated matrix $A$ and polynomials

$$p(x) = x^2 - 9, \quad p_1(x) = x + 3, \quad p_2(x) = x - 3 \quad ◀$$

29. The matrix $A$ in Exercise 21.

30. An arbitrary square matrix $A$.

31. (a) Give an example of two $2 \times 2$ matrices such that

$$(A + B)(A - B) \neq A^2 - B^2$$

   (b) State a valid formula for multiplying out

$$(A + B)(A - B)$$

   (c) What condition can you impose on $A$ and $B$ that will allow you to write $(A + B)(A - B) = A^2 - B^2$?

32. The numerical equation $a^2 = 1$ has exactly two solutions. Find at least eight solutions of the matrix equation $A^2 = I_3$. [*Hint:* Look for solutions in which all entries off the main diagonal are zero.]

33. (a) Show that if a square matrix $A$ satisfies the equation $A^2 + 2A + I = 0$, then $A$ must be invertible. What is the inverse?
   (b) Show that if $p(x)$ is a polynomial with a nonzero constant term, and if $A$ is a square matrix for which $p(A) = 0$, then $A$ is invertible.

34. Is it possible for $A^3$ to be an identity matrix without $A$ being invertible? Explain.

35. Can a matrix with a row of zeros or a column of zeros have an inverse? Explain.

36. Can a matrix with two identical rows or two identical columns have an inverse? Explain.



▶ In Exercises 37–38, determine whether $A$ is invertible, and if so, find the inverse. [*Hint:* Solve $AX = I$ for $X$ by equating corresponding entries on the two sides.] ◀

37. $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ 38. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

▶ In Exercises 39–40, simplify the expression assuming that $A$, $B$, $C$, and $D$ are invertible. ◀

39. $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$

40. $(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}$

41. Show that if $R$ is a $1 \times n$ matrix and $C$ is an $n \times 1$ matrix, then $RC = \text{tr}(CR)$.

42. If $A$ is a square matrix and $n$ is a positive integer, is it true that $(A^n)^T = (A^T)^n$? Justify your answer.

43. (a) Show that if $A$ is invertible and $AB = AC$, then $B = C$.
   (b) Explain why part (a) and Example 3 do not contradict one another.

44. Show that if $A$ is invertible and $k$ is any nonzero scalar, then $(kA)^n = k^nA^n$ for all integer values of $n$.

45. (a) Show that if $A$, $B$, and $A + B$ are invertible matrices with

49. Assuming that all matrices are $n \times n$ and invertible, solve for $D$.

$$C^TB^{-1}A^2BAC^{-1}DA^{-2}B^TC^{-2} = C^T$$

50. Assuming that all matrices are $n \times n$ and invertible, solve for $D$.

$$ABC^TDBA^TC = AB^T$$

### Working with Proofs

▶ In Exercises 51–58, prove the stated result. ◀

51. Theorem 1.4.1($a$) 52. Theorem 1.4.1($b$)

53. Theorem 1.4.1($f$) 54. Theorem 1.4.1($c$)

55. Theorem 1.4.2($c$) 56. Theorem 1.4.2($b$)

57. Theorem 1.4.8($d$) 58. Theorem 1.4.8($e$)

### True-False Exercises

TF. In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) Two $n \times n$ matrices, $A$ and $B$, are inverses of one another if and only if $AB = BA = 0$.

(b) For all square matrices $A$ and $B$ of the same size, it is true that $(A + B)^2 = A^2 + 2AB + B^2$.

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CHAPTER 2

# Determinants

CHAPTER CONTENTS



INTRODUCTION In this chapter we will study "determinants" or, more precisely, "determinant functions." Unlike real-valued functions, such as $f(x) = x^2$, that assign a real number to a real variable $x$, determinant functions assign a real number $f(A)$ to a matrix variable $A$. Although determinants first arose in the context of solving systems of linear equations, they are rarely used for that purpose in real-world applications. While they can be useful for solving very small linear systems (say two or three unknowns), our main interest in them stems from the fact that they link together various concepts in linear algebra and provide a useful formula for the inverse of a matrix.

## 2.1 Determinants by Cofactor Expansion

In this section we will define the notion of a "determinant." This will enable us to develop a specific formula for the inverse of an invertible matrix, whereas up to now we have had only a computational procedure for finding it. This, in turn, will eventually provide us with a formula for solutions of certain kinds of linear systems.

Recall from Theorem 1.4.5 that the $2 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$ and that the expression $ad - bc$ is called the ***determinant*** of the matrix $A$. Recall also that this determinant is denoted by writing

$$\det(A) = ad - bc \quad \text{or} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{1}$$

and that the inverse of $A$ can be expressed in terms of the determinant as

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

**WARNING** It is important to keep in mind that $\det(A)$ is a *number*, whereas $A$ is a *matrix*.

*Minors and Cofactors*

One of our main goals in this chapter is to obtain an analog of Formula (2) that is applicable to square matrices of *all orders*. For this purpose we will find it convenient to use subscripted entries when writing matrices or determinants. Thus, if we denote a $2 \times 2$ matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$





then the two equations in (1) take the form

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{3}$$

In situations where it is inconvenient to assign a name to the matrix, we can express this formula as

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{4}$$

There are various methods for defining determinants of higher-order square matrices. In this text, we will us an "inductive definition" by which we mean that the determinant of a square matrix of a given order will be defined in terms of determinants of square matrices of the next lower order. To start the process, let us define the determinant of a $1 \times 1$ matrix $[a_{11}]$ as

$$\det [a_{11}] = a_{11} \tag{5}$$

from which it follows that Formula (4) can be expressed as

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \det[a_{11}]\det[a_{22}] - \det[a_{12}]\det[a_{21}]$$

Now that we have established a starting point, we can define determinants of $3 \times 3$ matrices in terms of determinants of $2 \times 2$ matrices, then determinants of $4 \times 4$ matrices in terms of determinants of $3 \times 3$ matrices, and so forth, ad infinitum. The following terminology and notation will help to make this inductive process more efficient.

**DEFINITION 1** If $A$ is a square matrix, then the ***minor of entry*** $a_{ij}$ is denoted by $M_{ij}$ and is defined to be the determinant of the submatrix that remains after the $i$th row and $j$th column are deleted from $A$. The number $(-1)^{i+j}M_{ij}$ is denoted by $C_{ij}$ and is called the ***cofactor of entry*** $a_{ij}$.

▶ **EXAMPLE 1 Finding Minors and Cofactors**

Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The minor of entry $a_{11}$ is

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

**WARNING** We have followed the standard convention of using capital letters to denote minors and cofactors even though they are numbers, not matrices.

The cofactor of $a_{11}$ is

$$C_{11} = (-1)^{1+1}M_{11} = M_{11} = 16$$

Historical Note The term *determinant* was first introduced by the German mathematician Carl Friedrich Gauss in 1801 (see p. 15), who used them to "determine" properties of certain kinds of functions. Interestingly, the term *matrix* is derived from a Latin word for "womb" because it was viewed as a container of determinants.



Similarly, the minor of entry $a_{12}$ is

Similarly, the minor of entry $a_{32}$ is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

The cofactor of $a_{32}$ is

$$C_{32} = (-1)^{3+2}M_{32} = -M_{32} = -26 \blacktriangleleft$$

**Remark** Note that a minor $M_{ij}$ and its corresponding cofactor $C_{ij}$ are either the same or negatives of each other and that the relating sign $(-1)^{i+j}$ is either $+1$ or $-1$ in accordance with the pattern in the "checkerboard" array

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

For example,

$$C_{11} = M_{11}, \quad C_{21} = -M_{21}, \quad C_{22} = M_{22}$$

and so forth. Thus, it is never really necessary to calculate $(-1)^{i+j}$ to calculate $C_{ij}$—you can simply compute the minor $M_{ij}$ and then adjust the sign in accordance with the checkerboard pattern. Try this in Example 1.

**▶ EXAMPLE 2 Cofactor Expansions of a 2 × 2 Matrix**

The checkerboard pattern for a $2 \times 2$ matrix $A = [a_{ij}]$ is



$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

so that

$$C_{11} = M_{11} = a_{22} \qquad C_{12} = -M_{12} = -a_{21}$$
$$C_{21} = -M_{21} = -a_{12} \qquad C_{22} = M_{22} = a_{11}$$

We leave it for you to use Formula (3) to verify that $\det(A)$ can be expressed in terms of cofactors in the following four ways:

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}C_{11} + a_{12}C_{12} \\ &= a_{21}C_{21} + a_{22}C_{22} \\ &= a_{11}C_{11} + a_{21}C_{21} \\ &= a_{12}C_{12} + a_{22}C_{22} \end{aligned} \tag{6}$$

Each of the last four equations is called a *cofactor expansion* of $\det(A)$. In each cofactor expansion the entries and cofactors all come from the same row or same column of $A$.

**Historical Note** The term *minor* is apparently due to the English mathematician James Sylvester (see p. 35), who wrote the following in a paper published in 1850: "Now conceive any one line and any one column be struck out, we get...a square, one term less in breadth and depth than the original square; and by varying in every possible selection of the line and column excluded, we obtain, supposing the original square to consist of $n$ lines and $n$ columns, $n^2$ such minor squares, each of which will represent what I term a "First Minor Determinant" relative to the principal or complete determinant."



For example, in the first equation the entries and cofactors all come from the first row of $A$, in the second they all come from the second row of $A$, in the third they all come from the first column of $A$, and in the fourth they all come from the second column of $A$. ◀

*Definition of a General Determinant*

Formula (6) is a special case of the following general result, which we will state without proof.

**THEOREM 2.1.1** *If $A$ is an $n \times n$ matrix, then regardless of which row or column of $A$ is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.*

This result allows us to make the following definition.

**DEFINITION 2** If $A$ is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of $A$ by the corresponding cofactors and adding the resulting products is called the ***determinant of A***, and the sums themselves are called ***cofactor expansions of A***. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \tag{7}$$

[cofactor expansion along the $j$th column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \tag{8}$$

[cofactor expansion along the $i$th row]

**▶ EXAMPLE 3 Cofactor Expansion Along the First Row**

Find the determinant of the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

by cofactor expansion along the first row.

**Charles Lutwidge Dodgson (Lewis Carroll) (1832–1898)**

**Historical Note** Cofactor expansion is not the only method for expressing the determinant of a matrix in terms of determinants of lower order. For example, although it is not well known, the English mathematician Charles Dodgson, who was the author of *Alice's Adventures in Wonderland* and *Through the Looking Glass* under the pen name of Lewis Carroll, invented such a method, called condensation. That method has recently been resurrected from obscurity because of its suitability for parallel processing on computers.

[Image: Oscar G. Rejlander/ Time & Life Pictures/Getty Images]



*Solution*

*Solution*

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - (1)(-11) + 0 = -1$$

▶ **EXAMPLE 4 Cofactor Expansion Along the First Column**

Let $A$ be the matrix in Example 3, and evaluate $\det(A)$ by cofactor expansion along the first column of $A$.

> Note that in Example 4 we had to compute three cofactors, whereas in Example 3 only two were needed because the third was multiplied by zero. As a rule, the best strategy for cofactor expansion is to expand along a row or column with the most zeros.

*Solution*

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) - (-2)(-2) + 5(3) = -1$$

This agrees with the result obtained in Example 3.

▶ **EXAMPLE 5 Smart Choice of Row or Column**

If $A$ is the $4 \times 4$ matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

then to find $\det(A)$ it will be easiest to use cofactor expansion along the second column, since it has the most zeros:

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

For the $3 \times 3$ determinant, it will be easiest to use cofactor e[illegible] column, since it has the most zeros:

$$\det(A) = 1 \cdot -2 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -2(1 + 2)$$

$$= -6$$

▶ **EXAMPLE 6 Determinant of a Lower Triangular Matrix**

The following computation shows that the determinant of a $4 \times 4$ lower triangular matrix is the product of its diagonal entries. Each part of the computation uses a cofactor expansion along the first row.

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11}a_{22}\begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33}|a_{44}| = a_{11}a_{22}a_{33}a_{44}$$ ◀



The method illustrated in Example 6 can be easily adapted to prove the following general result.

> **THEOREM 2.1.2** *If $A$ is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is, $\det(A) = a_{11}a_{22}\cdots a_{nn}$.*

*A Useful Technique for Evaluating $2 \times 2$ and $3 \times 3$ Determinants*

Determinants of $2 \times 2$ and $3 \times 3$ matrices can be evaluated very efficiently using the pattern suggested in Figure 2.1.1.

▶ Figure 2.1.1

In the $2 \times 2$ case, the determinant can be computed by forming the product of the entries on the rightward arrow and subtracting the product of the entries on the leftward arrow. In the $3 \times 3$ case we first recopy the first and second columns as shown in the figure, after which we can compute the determinant by summing the products of the entries on the rightward arrows and subtracting the products on the leftward arrows. These procedures execute the computations

> **WARNING** The arrow technique works only for determinants of $2 \times 2$ and $3 \times 3$ matrices. It *does not* work for matrices of size $4 \times 4$ or higher.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

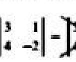
$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

which agrees with the cofactor expansions along the first row.

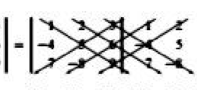
▶ **EXAMPLE 7 A Technique for Evaluating 2 × 2 and 3 × 3 Determinants**

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} =$$



$$= (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} =$$



$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$ ◀



## Exercise Set 2.1

▶ In Exercises 1–2, find all the minors and cofactors of the matrix $A$. ◀

17. $A = \begin{bmatrix} \lambda - 1 & 0 \\ \ldots & \ldots \end{bmatrix}$

18. $A = \begin{bmatrix} \lambda - 4 & 4 & 0 \\ -1 & \lambda & 0 \\ \ldots \end{bmatrix}$

## Exercise Set 2.1

In Exercises 1–2, find all the minors and cofactors of the matrix $A$.

1. $A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{bmatrix}$

3. Let

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

Find

(a) $M_{13}$ and $C_{13}$.
(b) $M_{23}$ and $C_{23}$.
(c) $M_{22}$ and $C_{22}$.
(d) $M_{21}$ and $C_{21}$.

4. Let

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}$$

Find

(a) $M_{32}$ and $C_{32}$.
(b) $M_{44}$ and $C_{44}$.
(c) $M_{41}$ and $C_{41}$.
(d) $M_{24}$ and $C_{24}$.

In Exercises 5–8, evaluate the determinant of the given matrix. If the matrix is invertible, use Equation (2) to find its inverse.

5. $\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$

7. $\begin{bmatrix} -5 & 7 \\ -7 & -2 \end{bmatrix}$

8. $\begin{bmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{bmatrix}$

In Exercises 9–14, use the arrow technique to evaluate the determinant.

9. $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix}$

10. $\begin{vmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{vmatrix}$

11. $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix}$

12. $\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix}$

13. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix}$

14. $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix}$

In Exercises 15–18, find all values of $\lambda$ for which $\det(A) = 0$.

15. $A = \begin{bmatrix} \lambda-2 & 1 \\ -5 & \lambda+4 \end{bmatrix}$

16. $A = \begin{bmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda-1 \end{bmatrix}$

17. $A = \begin{bmatrix} \lambda-1 & 0 \\ 2 & \lambda+1 \end{bmatrix}$

18. $A = \begin{bmatrix} \lambda-4 & 4 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda-5 \end{bmatrix}$

19. Evaluate the determinant in Exercise 13 by a cofactor expansion along

(a) the first row.
(b) the first column.
(c) the second row.
(d) the second column.
(e) the third row.
(f) the third column.

20. Evaluate the determinant in Exercise 12 by a cofactor expansion along

(a) the first row.
(b) the first column.
(c) the second row.
(d) the second column.
(e) the third row.
(f) the third column.

In Exercises 21–26, evaluate $\det(A)$ by a cofactor expansion along a row or column of your choice.

21. $A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$

22. $A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}$

23. $A = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$

24. [illegible]

25. $A = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$

26. $A = \begin{bmatrix} 4 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & -1 & 0 \\ 1 & 2 & 4 & 2 & 3 \\ 9 & 4 & 6 & 2 & 3 \\ 2 & 2 & 4 & 2 & 3 \end{bmatrix}$

In Exercises 27–32, evaluate the determinant of the given matrix by inspection.

27. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

28. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

29. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 1 & 2 & 3 & 8 \end{bmatrix}$

30. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$





31. $\begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

32. $\begin{bmatrix} -3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 40 & 10 & -1 & 0 \\ 100 & 200 & -23 & 3 \end{bmatrix}$

33. In each part, show that the value of the determinant is independent of $\theta$.

(a) $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$

(b) $\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$

34. Show that the matrices

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$$

commute if and only if

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$

35. By inspection, what is the relationship between the following determinants?

$$d_1 = \begin{vmatrix} a & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix} \quad \text{and} \quad d_2 = \begin{vmatrix} a+\lambda & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix}$$

36. Show that

$$\det(A) = \frac{1}{2}\begin{vmatrix} \operatorname{tr}(A) & 1 \\ \operatorname{tr}(A^2) & \operatorname{tr}(A) \end{vmatrix}$$

for every $2 \times 2$ matrix $A$.

37. What can you say about an $n$th-order determinant all of whose entries are 1? Explain.

38. What is the maximum number of zeros that a $3 \times 3$ matrix can have without having a zero determinant? Explain.

39. Explain why the determinant of a matrix with integer entries must be an integer.

### Working with Proofs

40. Prove that $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ are collinear points if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

41. Prove that the equation of the line through the distinct points $(a_1, b_1)$ and $(a_2, b_2)$ can be written as

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$$

42. Prove that if $A$ is upper triangular and $B_{ij}$ is the matrix that results when the $i$th row and $j$th column of $A$ are deleted, then $B_{ij}$ is upper triangular if $i < j$.

### True-False Exercises

TF. In parts (a)–(j) determine whether the statement is true or false, and justify your answer.

(a) The determinant of the $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad + bc$.

(b) Two square matrices that have the same determinant must have the same size.

(c) The minor $M_{ij}$ is the same as the cofactor $C_{ij}$ if $i + j$ is even.

(d) If $A$ is a $3 \times 3$ symmetric matrix, then $C_{ij} = C_{ji}$ for all $i$ and $j$.

(e) The number obtained by a cofactor expansion of a matrix $A$ is independent of the row or column chosen for the expansion.

(f) If $A$ is a square matrix whose minors are all zero, then $\det(A) = 0$.

(g) The determinant of a lower triangular matrix is the sum of the entries along the main diagonal.

(h) For every square matrix $A$ and every scalar $c$, it is true that $\det(cA) = c\det(A)$.

(i) For all square matrices $A$ and $B$, it is true that

$$\det(A + B) = \det(A) + \det(B)$$

(j) For every $2 \times 2$ matrix $A$ it is true that $\det(A^2) = (\det(A))^2$.

### Working with Technology

T1. (a) Use the determinant capability of your technology utility to find the determinant of the matrix

$$A = \begin{bmatrix} 4.2 & -1.3 & 1.1 & 6.0 \\ 0.0 & 0.0 & -3.2 & 3.4 \\ 4.5 & 1.3 & 0.0 & 14.8 \\ 4.7 & 1.0 & 3.4 & 2.3 \end{bmatrix}$$

(b) Compare the result obtained in part (a) to that obtained by a cofactor expansion along the second row of $A$.

T2. Let $A_n$ be the $n \times n$ matrix with 2's along the main diagonal, 1's along the diagonal lines immediately above and below the main diagonal, and zeros everywhere else. Make a conjecture about the relationship between $n$ and $\det(A_n)$.



2.2 Evaluating Determinants by Row Reduction

## 2.2 Evaluating Determinants by Row Reduction

In this section we will show how to evaluate a determinant by reducing the associated matrix to row echelon form. In general, this method requires less computation than cofactor expansion and hence is the method of choice for large matrices.

*A Basic Theorem*

We begin with a fundamental theorem that will lead us to an efficient procedure for evaluating the determinant of a square matrix of any size.

**THEOREM 2.2.1** *Let $A$ be a square matrix. If $A$ has a row of zeros or a column of zeros, then $\det(A) = 0$.*

*Proof* Since the determinant of $A$ can be found by a cofactor expansion along any row or column, we can use the row or column of zeros. Thus, if we let $C_1, C_2, \ldots, C_n$ denote the cofactors of $A$ along that row or column, then it follows from Formula (7) or (8) in Section 2.1 that

$$\det(A) = 0 \cdot C_1 + 0 \cdot C_2 + \cdots + 0 \cdot C_n = 0 \blacktriangleleft$$

The following useful theorem relates the determinant of a matrix and the determinant of its transpose.

> Because transposing a matrix changes its columns to rows and its rows to columns, almost every theorem about the rows of a determinant has a companion version about columns, and vice versa.

**THEOREM 2.2.2** *Let $A$ be a square matrix. Then $\det(A) = \det(A^T)$.*

*Proof* Since transposing a matrix changes its columns to rows and its rows to columns, the cofactor expansion of $A$ along any row is the same as the cofactor expansion of $A^T$ along the corresponding column. Thus, both have the same determinant. ◀

*Elementary Row Operations*

The next theorem shows how an elementary row operation on a square matrix affects the value of its determinant. In place of a formal proof we have provided a table to illustrate the ideas in the $3 \times 3$ case (see Table 1).



> The first panel of Table 1 shows that you can bring a common factor from any row (column) of a determinant through the determinant sign. This is a slightly different way of thinking about part (*a*) of Theorem 2.2.3.

Table 1

| Relationship | Operation |
|---|---|
| $\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ <br> $\det(B) = k\det(A)$ | In the matrix $B$ the first row of $A$ was multiplied by $k$. |
| $\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ <br> $\det(B) = -\det(A)$ | In the matrix $B$ the first and second rows of $A$ were interchanged. |
| $\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ <br> $\det(B) = \det(A)$ | In the matrix $B$ a multiple of the second row of $A$ was added to the first row. |



**THEOREM 2.2.3** *Let $A$ be an $n \times n$ matrix.*

*(a) If $B$ is the matrix that results when a single row or single column of $A$ is multiplied by a scalar $k$, then $\det(B) = k\det(A)$.*

*(b) If $B$ is the matrix that results when two rows or two columns of $A$ are interchanged, then $\det(B) = -\det(A)$.*

*(c) If $B$ is the matrix that results when a multiple of one row of $A$ is added to another or when a multiple of one column is added to another, then $\det(B) = \det(A)$.*

We will verify the first equation in Table 1 and leave the other two for you. To start, note that the determinants on the two sides of the equation differ only in the first row, so these determinants have the same cofactors, $C_{11}, C_{12}, C_{13}$, along that row (since those cofactors depend only on the entries in the *second* two rows). Thus, expanding the left side by cofactors along the first row yields

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13}$$
$$= k(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13})$$
$$= k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

*Elementary Matrices*

It will be useful to consider the special case of Theorem 2.2.3 in which $A = I_n$ is the $n \times n$ identity matrix and $E$ (rather than $B$) denotes the elementary matrix that results when the row operation is performed on $I_n$. In this special case Theorem 2.2.3 implies the following result.

**THEOREM 2.2.4** *Let $E$ be an $n \times n$ elementary matrix.*

*(a) If $E$ results from multiplying a row of $I_n$ by a nonzero number $k$, then $\det(E) = k$.*

*(b) If $E$ results from interchanging two rows of $I_n$, then $\det(E) = -1$.*

*(c) If $E$ results from adding a multiple of one row of $I_n$ to another, then $\det(E) = 1$.*

▶ EXAMPLE 1 **Determinants of Elementary Matrices**

> Observe that the determinant of an elementary matrix cannot be zero.

The following determinants of elementary matrices, which are evaluated by inspection, illustrate Theorem 2.2.4.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3, \quad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1, \quad \begin{vmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \blacktriangleleft$$

The second row of $I_4$ was multiplied by 3. The first and last rows of $I_4$ were interchanged. 7 times the last row of $I_4$ was added to the first row.

*Matrices with Proportional Rows or Columns*

If a square matrix $A$ has two proportional rows, then a row of zeros can be introduced by adding a suitable multiple of one of the rows to the other. Similarly for columns. But adding a multiple of one row or column to another does not change the determinant, so from Theorem 2.2.1, we must have $\det(A) = 0$. This proves the following theorem.



**THEOREM 2.2.5** *If $A$ is a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.*

**THEOREM 2.2.5** *If A is a square matrix with two proportional rows or two proportional columns, then* $\det(A) = 0$.

► **EXAMPLE 2 Proportional Rows or Columns**

Each of the following matrices has two proportional rows or columns; thus, each has a determinant of zero.

$$\begin{bmatrix} -1 & 4 \\ -2 & 8 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{bmatrix}$$ ◄

*Evaluating Determinants by Row Reduction*

We will now give a method for evaluating determinants that involves substantially less computation than cofactor expansion. The idea of the method is to reduce the given matrix to upper triangular form by elementary row operations, then compute the determinant of the upper triangular matrix (an easy computation), and then relate that determinant to that of the original matrix. Here is an example.

► **EXAMPLE 3 Using Row Reduction to Evaluate a Determinant**

Evaluate $\det(A)$ where

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

*Solution* We will reduce $A$ to row echelon form (which … apply Theorem 2.1.2.

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$ ← The first and second rows of $A$ were interchanged.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$ ← A common factor of 3 from the first row was taken through the determinant sign.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$ ← −2 times the first row was added to the third row.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$ ← −10 times the second row was added to the third row.

$$= (-3)(-55)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$ ← A common factor of −55 from the last row was taken through the determinant sign.

$$= (-3)(-55)(1) = 165$$

Even with today's fastest computers it would take millions of years to calculate a $25 \times 25$ determinant by cofactor expansion, so methods based on row reduction are often used for large determinants. For determinants of small size (such as those in this text), cofactor expansion is often a reasonable choice.



► **EXAMPLE 4 Using Column Operations to Evaluate a Determinant**

Compute the determinant of

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$$

*Solution* This determinant could be computed as above by using elementary row operations to reduce $A$ to row echelon form, but we can put $A$ in lower triangular form in one step by adding $-3$ times the first column to the fourth to obtain

$$\det(A) = \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{bmatrix} = (1)(7)(3)(-26) = -546$$ ◄

Example 4 points out that it is always wise to keep an eye open for column operations that can shorten computations.

Cofactor expansion and row or column operations can sometimes be used in combination to provide an effective method for evaluating determinants. The following example illustrates this idea.

► **EXAMPLE 5 Row Operations and Cofactor Expansion**

Evaluate $\det(A)$ where

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

*Solution* By adding suitable multiples of the second row to the remaining rows, we obtain

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$ ← Cofactor expansion along the first column

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$ ← We added the first row to the third row.

$$= -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix}$$ ← Cofactor expansion along the first column

$$= -18$$ ◄



Exercise Set 2.2

## Exercise Set 2.2

► In Exercises 1–4, verify that $\det(A) = \det(A^T)$.

1. $A = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} -6 & 1 \\ 2 & -2 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{bmatrix}$

4. $A = \begin{bmatrix} 4 & 2 & -1 \\ 0 & 2 & -3 \\ -1 & 1 & 5 \end{bmatrix}$

► In Exercises 5–8, find the determinant of the given elementary matrix by inspection.

5. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

► In Exercises 9–14, evaluate the determinant of the matrix by first reducing the matrix to row echelon form and then using some combination of row operations and cofactor expansion.

9. $\begin{bmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{bmatrix}$

► In Exercises 15–22, evaluate the determinant, given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

15. $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$

16. $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$

17. $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$

18. $\begin{vmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix}$

19. $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$

20. $\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix}$

21. $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$

22. $\begin{vmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{vmatrix}$

23. Use row reduction to show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

24. Verify the formulas in parts (a) and (b) and then make a conjecture about a general result of which these results are special cases.

(a) $\det\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = -a_{13}a_{22}a_{31}$

(b) $\det\begin{bmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{14}a_{23}a_{32}a_{41}$

► In Exercises 25–28, confirm the identities without evaluating the determinants directly.

25. $\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

26. $\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

27. $\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

28. $\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$



► In Exercises 29–30, show that $\det(A) = 0$ without directly evaluating the determinant.

29. $A = \begin{bmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{bmatrix}$

30. $A = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}$

► It can be proved that if a square matrix $M$ is partitioned into ***block triangular form*** as

$$M = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \quad \text{or} \quad M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

in which $A$ and $B$ are square, then $\det(M) = \det(A)\det(B)$. Use this result to compute the determinants of the matrices in Exercises 31 and 32.

31. $M = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 8 & 6 & -9 \\ 2 & 5 & 0 & 4 & 7 & 5 \\ -1 & 3 & 2 & 6 & 9 & -2 \\ \hline 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 8 & -4 \end{array}\right]$

32. $M = \left[\begin{array}{ccc|cc} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 \end{array}\right]$

33. Let $A$ be an $n \times n$ matrix, and let $B$ be the matrix that results when the rows of $A$ are written in reverse order. State a theorem that describes how $\det(A)$ and $\det(B)$ are related.

34. Find the determinant of the following matrix.

$$\begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$$

### True-False Exercises

**TF.** In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) If $A$ is a $4 \times 4$ matrix and $B$ is obtained from $A$ by interchanging the first two rows and then interchanging the last two rows, then $\det(B) = \det(A)$.

(b) If $A$ is a $3 \times 3$ matrix and $B$ is obtained from $A$ by multiplying the first column by 4 and multiplying the third column by $\frac{3}{4}$, then $\det(B) = 3\det(A)$.

(c) If $A$ is a $3 \times 3$ matrix and $B$ is obtained from $A$ by adding 5 times the first row to each of the second and third rows, then $\det(B) = 25\det(A)$.

(d) If $A$ is an $n \times n$ matrix and $B$ is obtained from $A$ by multiplying each row of $A$ by its row number, then

$$\det(B) = \frac{n(n+1)}{2}\det(A)$$

(e) If $A$ is a square matrix with two identical columns, then $\det(A) = 0$.

(f) If the sum of the second and fourth row vectors of a $6 \times 6$ matrix $A$ is equal to the last row vector, then $\det(A) = 0$.

### Working with Technology

**T1.** Find the determinant of

$$A = \begin{bmatrix} 4.2 & -1.3 & 1.1 & 6.0 \\ 0.0 & 0.0 & -3.2 & 3.4 \\ 4.5 & 1.3 & 0.0 & 14.8 \\ 4.7 & 1.0 & 3.4 & 2.3 \end{bmatrix}$$

by reducing the matrix to reduced row echelon form, and compare the result obtained in this way to that obtained in Exercise T1 of Section 2.1.

## 2.3 Properties of Determinants; Cramer's Rule

In this section we will develop some fundamental properties of matrices, and we will use these results to derive a formula for the inverse of an invertible matrix and formulas for the solutions of certain kinds of linear systems.

*Basic Properties of Determinants*

Suppose that $A$ and $B$ are $n \times n$ matrices and $k$ is any scalar. We begin by considering possible relationships among $\det(A)$, $\det(B)$, and

$$\det(kA), \quad \det(A+B), \quad \text{and} \quad \det(AB)$$

Since a common factor of any row of a matrix can be moved through the determinant sign, and since each of the $n$ rows in $kA$ has a common factor of $k$, it follows that

Since a common factor of any row of a matrix can be moved through the determinant sign, and since each of the $n$ rows in $kA$ has a common factor of $k$, it follows that



$$\det(kA) = k^n \det(A) \tag{1}$$

For example,

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Unfortunately, no simple relationship exists among $\det(A)$, $\det(B)$, and $\det(A+B)$. In particular, $\det(A+B)$ will usually *not* be equal to $\det(A)+\det(B)$. The following example illustrates this fact.

▶ EXAMPLE 1 $\det(A+B) \neq \det(A) + \det(B)$

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad A+B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

We have $\det(A) = 1$, $\det(B) = 8$, and $\det(A+B) = 23$; thus

$$\det(A+B) \neq \det(A) + \det(B) \quad ◀$$

In spite of the previous example, there is a useful relationship concerning sums of determinants that is applicable when the matrices involved are the same except for *one* row (column). For example, consider the following two matrices that differ only in the second row:



$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Calculating the determinants of $A$ and $B$, we obtain

$$\begin{aligned} \det(A) + \det(B) &= (a_{11}a_{22} - a_{12}a_{21}) + (a_{11}b_{22} - a_{12}b_{21}) \\ &= a_{11}(a_{22} + b_{22}) - a_{12}(a_{21} + b_{21}) \\ &= \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \end{aligned}$$

Thus

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \det \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix} = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

This is a special case of the following general result.

**THEOREM 2.3.1** *Let $A$, $B$, and $C$ be $n \times n$ matrices that differ only in a single row, say the $r$th, and assume that the $r$th row of $C$ can be obtained by adding corresponding entries in the $r$th rows of $A$ and $B$. Then*

$$\det(C) = \det(A) + \det(B)$$

*The same result holds for columns.*

▶ EXAMPLE 2 **Sums of Determinants**

We leave it to you to confirm the following equality by evaluating the determinants.

$$\det \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} = \det \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} + \det \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix} \quad ◀$$



*Determinant of a Matrix Product*

Considering the complexity of the formulas for determinants and matrix multiplication, it would seem unlikely that a simple relationship should exist between them. This is what makes the simplicity of our next result so surprising. We will show that if $A$ and $B$ are square matrices of the same size, then

$$\det(AB) = \det(A)\det(B) \tag{2}$$

The proof of this theorem is fairly intricate, so we will have to develop some preliminary results first. We begin with the special case of (2) in which $A$ is an elementary matrix. Because this special case is only a prelude to (2), we call it a lemma.

**LEMMA 2.3.2** *If $B$ is an $n \times n$ matrix and $E$ is an $n \times n$ elementary matrix, then*

$$\det(EB) = \det(E)\det(B)$$

*Proof* We will consider three cases, each in accordance with the row operation that produces the matrix $E$.

*Case 1* If $E$ results from multiplying a row of $I_n$ by $k$, then by Theorem 1.5.1, $EB$ results from $B$ by multiplying the corresponding row by $k$; so from Theorem 2.2.3(*a*) we have

$$\det(EB) = k\det(B)$$

But from Theorem 2.2.4(*a*) we have $\det(E) = k$, so

$$\det(EB) = \det(E)\det(B)$$

*Cases 2 and 3* The proofs of the cases where $E$ results from interchanging two rows of $I_n$ or from adding a multiple of one row to another follow the same pattern as Case 1 and are left as exercises. ◀

**Remark** It follows by repeated applications of Lemma 2.3.2 that if $B$ is an $n \times n$ matrix and $E_1, E_2, \ldots, E_r$ are $n \times n$ elementary matrices, then

$$\det(E_1E_2 \cdots E_rB) = \det(E_1)\det(E_2) \cdots \det(E_r)\det(B) \tag{3}$$

*Determinant Test for Invertibility*

Our next theorem provides an important criterion for determining whether a matrix is invertible. It also takes us a step closer to establishing Formula (2).

**THEOREM 2.3.3** *A square matrix $A$ is invertible if and only if $\det(A) \neq 0$.*

*Proof* Let $R$ be the reduced row echelon form of $A$. As a preliminary step, we will show that $\det(A)$ and $\det(R)$ are both zero or both nonzero: Let $E_1, E_2, \ldots, E_r$ be the elementary matrices that correspond to the elementary row operations that produce $R$ from $A$. Thus

$$R = E_r \cdots E_2E_1A$$

and from (3),

$$\det(R) = \det(E_r) \cdots \det(E_2)\det(E_1)\det(A) \tag{4}$$

We pointed out in the margin note that accompanies Theorem 2.2.4 that the determinant of an elementary matrix is nonzero. Thus, it follows from Formula (4) that $\det(A)$ and $\det(R)$ are either both zero or both nonzero, which sets the stage for the main part of the proof. If we assume first that $A$ is invertible, then it follows from Theorem 1.6.4 that

It follows from Theorems 2.3.3 and 2.2.5 that a square matrix with two proportional rows or two proportional columns is not invertible.

$R = I$ and hence that $\det(R) = 1\ (\neq 0)$. This, in turn, implies that $\det(A) \neq 0$, which is what we wanted to show.

Conversely, assume that $\det(A) \neq 0$. It follows from this that $\det(R) \neq 0$, which tells us that $R$ cannot have a row of zeros. Thus, it follows from Theorem 1.4.3 that $R = I$ and hence that $A$ is invertible by Theorem 1.6.4. ◀

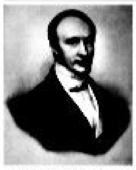
▶ EXAMPLE 3 **Determinant Test for Invertibility**

Since the first and third rows of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$$

are proportional, $\det(A) = 0$. Thus $A$ is not invertible. ◀

We are now ready for the main result concerning products of matrices.



**Augustin Louis Cauchy (1789–1857)**

Historical Note In 1815 the great French mathematician Augustin Cauchy published a landmark paper in which he gave the first systematic and modern treatment of determinants. It was in that paper that Theorem 2.3.4 was stated and proved in full generality for the first time. Special cases of the theorem had been stated and proved earlier, but it was Cauchy who made the final jump.

[*Image: © Bettmann/CORBIS*]

**THEOREM 2.3.4** *If A and B are square matrices of the same size, then*

$$\det(AB) = \det(A)\det(B)$$

*Proof* We divide the proof into two cases that depend on whether or not $A$ is invertible. If the matrix $A$ is not invertible, then by Theorem 1.6.5 neither is the product $AB$. Thus, from Theorem 2.3.3, we have $\det(AB) = 0$ and $\det(A) = 0$, so it follows that $\det(AB) = \det(A)\det(B)$.

Now assume that $A$ is invertible. By Theorem 1.6.4, the matrix $A$ is expressible as a product of elementary matrices, say

$$A = E_1E_2\cdots E_r$$

so

$$AB = E_1E_2\cdots E_rB$$

Applying (3) to this equation yields

$$\det(AB) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B)$$

and applying (3) again yields

$$\det(AB) = \det(E_1E_2\cdots E_r)\det(B)$$

which, from (5), can be written as $\det(AB) = \det(A)\det(B)$. ◀

▶ EXAMPLE 4 **Verifying that $\det(AB) = \det(A)\det(B)$**

Consider the matrices

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}, \quad AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}$$

We leave it for you to verify that

$$\det(A) = 1, \quad \det(B) = -23, \quad \text{and} \quad \det(AB) = -23$$

Thus $\det(AB) = \det(A)\det(B)$, as guaranteed by Theorem 2.3.4. ◀

The following theorem gives a useful relationship between the determinant of an invertible matrix and the determinant of its inverse.



**THEOREM 2.3.5** *If A is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

*Proof* Since $A^{-1}A = I$, it follows that $\det(A^{-1}A) = \det(I)$. Therefore, we must have $\det(A^{-1})\det(A) = 1$. Since $\det(A) \neq 0$, the proof can be completed by dividing through by $\det(A)$. ◀

*Adjoint of a Matrix*

In a cofactor expansion we compute $\det(A)$ by multiplying the entries in a row or column by their cofactors and adding the resulting products. It turns out that if one multiplies the entries in any row by the corresponding cofactors from a *different* row, the sum of these products is always zero. (This result also holds for columns.) Although we omit the general proof, the next example illustrates this fact.

▶ EXAMPLE 5 **Entries and Cofactors from Different Rows**

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

We leave it for you to verify that the cofactors of $A$ are

$$\begin{matrix} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{matrix}$$

so, for example, the cofactor expansion of $\det(A)$ along the first row is

$$\det(A) = 3C_{11} + 2C_{12} + (-1)C_{13} = 36 + 12 + 16 = 64$$

and along the first column is

$$\det(A) = 3C_{11} + C_{21} + 2C_{31} = 36 + 4 + 24 = 64$$

Suppose, however, we multiply the entries in the first row by the corresponding cofactors from the *second row* and add the resulting products. The result is

$$3C_{21} + 2C_{22} + (-1)C_{23} = 12 + 4 - 16 = 0$$

Or suppose we multiply the entries in the first column by the corresponding cofactors from the *second column* and add the resulting products. The result is again zero since

$$3C_{12} + 1C_{22} + 2C_{32} = 18 + 2 - 20 = 0 \;◀$$

**Leonard Eugene Dickson (1874–1954)**

Historical Note The use of the term *adjoint* for the transpose of the matrix of cofactors appears to have been introduced by the American mathematician L. E. Dickson in a research paper that he published in 1902.

[*Image: Courtesy of the American Mathematical Society www.ams.org*]

**DEFINITION 1** If $A$ is any $n \times n$ matrix and $C_{ij}$ is the cofactor of $a_{ij}$, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the ***matrix of cofactors from A***. The transpose of this matrix is called the ***adjoint of A*** and is denoted by $\text{adj}(A)$.

▶ **EXAMPLE 6 Adjoint of a 3 × 3 Matrix**

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

As noted in Example 5, the cofactors of $A$ are

$$\begin{matrix} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{matrix}$$

so the matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

and the adjoint of $A$ is

$$\text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} \blacktriangleleft$$

In Theorem 1.4.5 we gave a formula for the inverse of a 2 × 2 invertible matrix. Our next theorem extends that result to $n \times n$ invertible matrices.

It follows from Theorems 2.3.5 and 2.1.2 that if $A$ is an invertible triangular matrix, then

$$\det(A^{-1}) = \frac{1}{a_{11}}\frac{1}{a_{22}}\cdots\frac{1}{a_{nn}}$$

Moreover, by using the adjoint formula it is possible to show that

$$\frac{1}{a_{11}}, \frac{1}{a_{22}}, \ldots, \frac{1}{a_{nn}}$$

are actually the successive diagonal entries of $A^{-1}$ (compare $A$ and $A^{-1}$ in Example 3 of Section 1.7).

**THEOREM 2.3.6 Inverse of a Matrix Using Its Adjoint**

*If $A$ is an invertible matrix, then*

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$$



*Proof* We show first that

$$A\,\text{adj}(A) = \det(A)I$$

Consider the product

$$A\,\text{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}$$

The entry in the $i$th row and $j$th column of the product $A\,\text{adj}(A)$ is

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} \tag{7}$$

(see the shaded lines above).

If $i = j$, then (7) is the cofactor expansion of $\det(A)$ along the $i$th row of $A$ (Theorem 2.1.1), and if $i \neq j$, then the $a$'s and the cofactors come from different rows of $A$, so the value of (7) is zero (as illustrated in Example 5). Therefore,



$$A\,\text{adj}(A) = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix} = \det(A)I \tag{8}$$

Since $A$ is invertible, $\det(A) \neq 0$. Therefore, Equation (8) can be rewritten as

$$\frac{1}{\det(A)}[A\,\text{adj}(A)] = I \quad \text{or} \quad A\left[\frac{1}{\det(A)}\text{adj}(A)\right] = I$$

Multiplying both sides on the left by $A^{-1}$ yields

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) \blacktriangleleft$$

▶ **EXAMPLE 7 Using the Adjoint to Find an Inverse Matrix**

Use Formula (6) to find the inverse of the matrix $A$ in Example 6.

*Solution* We showed in Example 5 that $\det(A) = 64$. Thus,

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & -\frac{10}{64} \\ -\frac{16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix} \blacktriangleleft$$

*Cramer's Rule*

Our next theorem uses the formula for the inverse of an invertible matrix to produce a formula, called ***Cramer's rule***, for the solution of a linear system $A\mathbf{x} = \mathbf{b}$ of $n$ equations in $n$ unknowns in the case where the coefficient matrix $A$ is invertible (or, equivalently, when $\det(A) \neq 0$).

**Gabriel Cramer (1704–1752)**

**Historical Note** Variations of Cramer's rule were fairly well known before the Swiss mathematician discussed it in work he published in 1750. It was Cramer's superior notation that popularized the method and led mathematicians to attach his name to it.

[Image: Science Source/Photo Researchers]

**THEOREM 2.3.7 Cramer's Rule**

*If $A\mathbf{x} = \mathbf{b}$ is a system of $n$ linear equations in $n$ unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

*where $A_j$ is the matrix obtained by replacing the entries in the $j$th column of $A$ by the entries in the matrix*

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

*Proof* If $\det(A) \neq 0$, then $A$ is invertible, and by Theorem 1.6.2, $\mathbf{x} = A^{-1}\mathbf{b}$ is the unique solution of $A\mathbf{x} = \mathbf{b}$. Therefore, by Theorem 2.3.6 we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}\text{adj}(A)\mathbf{b} = \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

## Exercise Set 2.3

In Exercises 1–4, verify that $\det(kA) = k^n \det(A)$.

1. $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$; $k = 2$

2. $A = \begin{bmatrix} 2 & 2 \\ 5 & -2 \end{bmatrix}$; $k = -4$

3. $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}$; $k = -2$

4. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}$; $k = 3$

In Exercises 5–6, verify that $\det(AB) = \det(BA)$ and determine whether the equality $\det(A + B) = \det(A) + \det(B)$ holds.

5. $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} -1 & 8 & 2 \\ 1 & 0 & -1 \\ -2 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & -4 \\ 1 & 1 & 3 \\ 0 & 3 & -1 \end{bmatrix}$

In Exercises 7–14, use determinants to decide whether the given matrix is invertible.

7. $A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

8. $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$

9. $A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$

10. $A = \begin{bmatrix} -3 & 0 & 1 \\ 5 & 0 & 6 \\ 8 & 0 & 3 \end{bmatrix}$

11. $A = \begin{bmatrix} 4 & 2 & 8 \\ -2 & 1 & -4 \\ 3 & 1 & 6 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 0 & -1 \\ 9 & -1 & 4 \\ 8 & 9 & -1 \end{bmatrix}$

13. $A = \begin{bmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{bmatrix}$

14. $A = \begin{bmatrix} \sqrt{2} & -\sqrt{7} & 0 \\ 3\sqrt{2} & -3\sqrt{7} & 0 \\ 5 & -9 & 0 \end{bmatrix}$

In Exercises 15–18, find the values of $k$ for which the matrix $A$ is invertible.

15. $A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$

16. $A = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix}$

17. $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & 2 & 0 \\ k & 1 & k \\ 0 & 2 & 1 \end{bmatrix}$

In Exercises 19–23, decide whether the matrix is invertible, and if so, use the adjoint method to find its inverse.

19. $A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

20. $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$

21. $A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$

22. $A = \begin{bmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{bmatrix}$

23. $A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{bmatrix}$

In Exercises 24–29, solve by Cramer's rule, where it applies.

24. $\begin{aligned} 7x_1 - 2x_2 &= 3 \\ 3x_1 + x_2 &= 5 \end{aligned}$

25. $\begin{aligned} 4x + 5y \phantom{{}+2z} &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$

26. $\begin{aligned} x - 4y + z &= 6 \\ 4x - y + 2z &= -1 \\ 2x + 2y - 3z &= -20 \end{aligned}$

27. $\begin{aligned} x_1 - 3x_2 + x_3 &= 4 \\ 2x_1 - x_2 &= -2 \\ 4x_1 - 3x_3 &= 0 \end{aligned}$

28. $\begin{aligned} -x_1 - 4x_2 + 2x_3 + x_4 &= -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 &= 1[illegible] \\ -x_1 + x_2 + 3x_3 + x_4 &= [illegible] \\ x_1 - 2x_2 + x_3 - 4x_4 &= [illegible] \end{aligned}$



29. $\begin{aligned} 3x_1 - x_2 + x_3 &= 4 \\ -x_1 + 7x_2 - 2x_3 &= 1 \\ 2x_1 + 6x_2 - x_3 &= 5 \end{aligned}$

30. Show that the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is invertible for all values of $\theta$; then find $A^{-1}$ using Theorem 2.3.6.

31. Use Cramer's rule to solve for $y$ without solving for the unknowns $x$, $z$, and $w$.

$$\begin{aligned} 4x + y + z + w &= 6 \\ 3x + 7y - z + w &= 1 \\ 7x + 3y - 5z + 8w &= -3 \\ x + y + z + 2w &= 3 \end{aligned}$$

32. Let $A\mathbf{x} = \mathbf{b}$ be the system in Exercise 31.
   (a) Solve by Cramer's rule.
   (b) Solve by Gauss–Jordan elimination.
   (c) Which method involves fewer computations?



33. Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Assuming that $\det(A) = -7$, find

(a) $\det(3A)$ (b) $\det(A^{-1})$ (c) $\det(2A^{-1})$

(d) $\det((2A)^{-1})$ (e) $\det \begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix}$

34. In each part, find the determinant given that $A$ is a $4 \times 4$ matrix for which $\det(A) = -2$.

(a) $\det(-A)$ (b) $\det(A^{-1})$ (c) $\det(2A^T)$ (d) $\det(A^3)$

35. In each part, find the determinant given that $A$ is a $3 \times 3$ matrix for which $\det(A) = 7$.

(a) $\det(3A)$ (b) $\det(A^{-1})$

(c) $\det(2A^{-1})$ (d) $\det((2A)^{-1})$

### Working with Proofs

36. Prove that a square matrix $A$ is invertible if and only if $A^TA$ is invertible.

37. Prove that if $A$ is a square matrix, then $\det(A^TA) = \det(AA^T)$.

38. Let $A\mathbf{x} = \mathbf{b}$ be a system of $n$ linear equations in $n$ unknowns with integer coefficients and integer constants. Prove that if $\det(A) = 1$, the solution $\mathbf{x}$ has integer entries.

39. Prove that if $\det(A) = 1$ and all the entries in $A$ are integers, then all the entries in $A^{-1}$ are integers.

### True-False Exercises

TF. In parts (a)–(l) determine whether the statement is true or false, and justify your answer.

(a) If $A$ is a $3 \times 3$ matrix, then $\det(2A) = 2\det(A)$.

(b) If $A$ and $B$ are square matrices of the same size such that $\det(A) = \det(B)$, then $\det(A + B) = 2\det(A)$.

(c) If $A$ and $B$ are square matrices of the same size and $A$ is invertible, then

$$\det(A^{-1}BA) = \det(B)$$

(d) A square matrix $A$ is invertible if and only if $\det(A) = 0$.

(e) The matrix of cofactors of $A$ is precisely $[\operatorname{adj}(A)]^T$.

(f) For every $n \times n$ matrix $A$, we have

$$A \cdot \operatorname{adj}(A) = (\det(A))I_n$$

(g) If $A$ is a square matrix and the linear system $A\mathbf{x} = \mathbf{0}$ has multiple solutions for $\mathbf{x}$, then $\det(A) = 0$.

(h) If $A$ is an $n \times n$ matrix and there exists an $n \times 1$ matrix $\mathbf{b}$ such that the linear system $A\mathbf{x} = \mathbf{b}$ has no solutions, then the reduced row echelon form of $A$ cannot be $I_n$.

(i) If $E$ is an elementary matrix, then $E\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(j) If $A$ is an invertible matrix, then the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if the linear system $A^{-1}\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(k) If $A$ is invertible, then $\operatorname{adj}(A)$ must also be invertible.

(l) If $A$ has a row of zeros, then so does $\operatorname{adj}(A)$.

### Working with Technology

T1. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1+\epsilon \end{bmatrix}$$

in which $\epsilon > 0$. Since $\det(A) = \epsilon \neq 0$, it follows from Theorem 2.3.8 that $A$ is invertible. Compute $\det(A)$ for various small nonzero values of $\epsilon$ until you find a value that produces $\det(A) = 0$, thereby leading you to conclude erroneously that $A$ is not invertible. Discuss the cause of this.

T2. We know from Exercise 39 that if $A$ is a square matrix then $\det(A^TA) = \det(AA^T)$. By experimenting, make a conjecture as to whether this is true if $A$ is not square.

T3. The French mathematician Jacques Hadamard (1865–1963) proved that if $A$ is an $n \times n$ matrix each of whose entries satisfies the condition $|a_{ij}| \leq M$, then

$$|\det(A)| \leq \sqrt{n^n} M^n$$

(*Hadamard's inequality*). For the following matrix $A$, use this result to find an interval of possible values for $\det(A)$, and then use your technology utility to show that the value of $\det(A)$ falls within this interval.

$$A = \begin{bmatrix} 0.3 & -2.4 & -1.7 & 2.5 \\ 0.2 & -0.3 & -1.2 & 1.4 \\ 2.5 & 2.3 & 0.0 & 1.8 \\ 1.7 & 1.0 & -2.1 & 2.3 \end{bmatrix}$$