6.6 Function Approximation; Fourier Series

In this section we will show how orthogonal projections can be used to approximate certain types of functions by simpler functions. The ideas explained here have important applications in engineering and science. Calculus is required.

Best Approximations

All of the problems that we will study in this section will be special cases of the following general problem.

> **Approximation Problem** Given a function f that is continuous on an interval $[a, b]$, find the "best possible approximation" to f using only functions from a specified subspace W of $C[a, b]$.

Here are some examples of such problems:

(a) Find the best possible approximation to e^x over $[0, 1]$ by a polynomial of the form $a_0 + a_1x + a_2x^2$.

(b) Find the best possible approximation to $\sin \pi x$ over $[-1, 1]$ by a function of the form $a_0 + a_1e^x + a_2e^{2x} + a_3e^{3x}$.

(c) Find the best possible approximation to x over $[0, 2\pi]$ by a function of the form $a_0 + a_1 \sin x + a_2 \sin 2x + b_1 \cos x + b_2 \cos 2x$.

In the first example W is the subspace of $C[0, 1]$ spanned by 1, x, and x^2; in the second example W is the subspace of $C[-1, 1]$ spanned by 1, e^x, e^{2x}, and e^{3x}; and in the third example W is the subspace of $C[0, 2\pi]$ spanned by 1, $\sin x$, $\sin 2x$, $\cos x$, and $\cos 2x$.

Measurements of Error

To solve approximation problems of the preceding types, we first need to make the phrase "best approximation over $[a, b]$" mathematically precise. To do this we will need some way of quantifying the error that results when one continuous function is approximated by another over an interval $[a, b]$. If we were to approximate $f(x)$ by $g(x)$, and if we were concerned only with the error in that approximation at a *single point* x_0, then it would be natural to define the error to be

$$\text{error} = |f(x_0) - g(x_0)|$$

▲ Figure 6.6.1 The deviation between f and g at x_0.

sometimes called the ***deviation*** between f and g at x_0 (Figure 6.6.1). However, we are not concerned simply with measuring the error at a single point but rather with measuring it over the *entire* interval $[a, b]$. The difficulty is that an approximation may have small deviations in one part of the interval and large deviations in another. One possible way of accounting for this is to integrate the deviation $|f(x) - g(x)|$ over the interval $[a, b]$ and define the error over the interval to be

$$\text{error} = \int_a^b |f(x) - g(x)|\, dx \tag{1}$$

▲ Figure 6.6.2 The area between the graphs of f and g over $[a, b]$ measures the error in approximating f by g over $[a, b]$.

Geometrically, (1) is the area between the graphs of $f(x)$ and $g(x)$ over the interval $[a, b]$ (Figure 6.6.2)—the greater the area, the greater the overall error.

Although (1) is natural and appealing geometrically, most mathematicians and scientists generally favor the following alternative measure of error, called the ***mean square error***:

$$\text{mean square error} = \int_a^b [f(x) - g(x)]^2\, dx$$

Mean square error emphasizes the effect of larger errors because of the squaring and has the added advantage that it allows us to bring to bear the theory of inner product spaces. To see how, suppose that **f** is a continuous function on $[a, b]$ that we want to approximate by a function **g** from a subspace W of $C[a, b]$, and suppose that $C[a, b]$ is given the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\, dx$$

It follows that

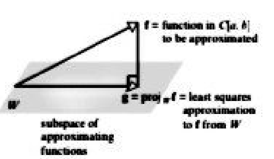
$$\|\mathbf{f} - \mathbf{g}\|^2 = \langle \mathbf{f} - \mathbf{g}, \mathbf{f} - \mathbf{g} \rangle = \int_a^b [f(x) - g(x)]^2\, dx = \text{mean square error}$$

so minimizing the mean square error is the same as minimizing $\|\mathbf{f} - \mathbf{g}\|^2$. Thus, the approximation problem posed informally at the beginning of this section can be restated more precisely as follows.

Least Squares Approximation

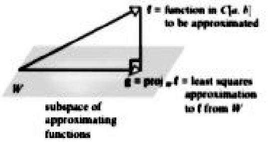
> **Least Squares Approximation Problem** Let **f** be a function that is continuous on an interval $[a, b]$, let $C[a, b]$ have the inner product
>
> $$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\, dx$$
>
> and let W be a finite-dimensional subspace of $C[a, b]$. Find a function **g** in W that minimizes
>
> $$\|\mathbf{f} - \mathbf{g}\|^2 = \int_a^b [f(x) - g(x)]^2\, dx$$

Since $\|\mathbf{f} - \mathbf{g}\|^2$ and $\|\mathbf{f} - \mathbf{g}\|$ are minimized by the same function **g**, this problem is equivalent to looking for a function **g** in W that is closest to **f**. But we know from Theorem 6.4.1 that $\mathbf{g} = \text{proj}_W \mathbf{f}$ is such a function (Figure 6.6.3). Thus, we have the following result.



► Figure 6.6.3

Since $\|\mathbf{f}-\mathbf{g}\|^2$ and $\|\mathbf{f}-\mathbf{g}\|$ are minimized by the same function $\mathbf{g}$, this problem is equivalent to looking for a function $\mathbf{g}$ in W that is closest to $\mathbf{f}$. But we know from Theorem 6.4.1 that $\mathbf{g} = \text{proj}_W \mathbf{f}$ is such a function (Figure 6.6.3). Thus, we have the following result.



▶ Figure 6.6.3

THEOREM 6.6.1 *If $\mathbf{f}$ is a continuous function on $[a, b]$, and W is a finite-dimensional subspace of $C[a, b]$, then the function $\mathbf{g}$ in W that minimizes the mean square error*

$$\int_a^b [f(x) - g(x)]^2\,dx$$

is $\mathbf{g} = \text{proj}_W \mathbf{f}$, where the orthogonal projection is relative to the inner product

$$\langle \mathbf{f}, \mathbf{g}\rangle = \int_a^b f(x)g(x)\,dx$$

*The function $\mathbf{g} = \text{proj}_W \mathbf{f}$ is called the **least squares approximation** to $\mathbf{f}$ from W.*

Fourier Series A function of the form

$$\begin{aligned} T(x) = c_0 + c_1\cos x + c_2\cos 2x + \cdots + c_n\cos nx \\ + d_1\sin x + d_2\sin 2x + \cdots + d_n\sin nx \end{aligned} \tag{2}$$

is called a ***trigonometric polynomial***; if c_n and d_n are not both zero, then $T(x)$ is said to have ***order n***. For example,

$$T(x) = 2 + \cos x - 3\cos 2x + 7\sin 4x$$

is a trigonometric polynomial of order 4 with

$$c_0 = 2,\ \ c_1 = 1,\ \ c_2 = -3,\ \ c_3 = 0,\ \ c_4 = 0,\ \ d_1 = 0,\ \ d_2 = 0,\ \ d_3 = 0,\ \ d_4 = 7$$

It is evident from (2) that the trigonometric polynomials of order n or less are the various possible linear combinations of

$$1,\ \cos x,\ \cos 2x, \ldots,\ \cos nx,\quad \sin x,\ \sin 2x, \ldots,\ \sin nx \tag{3}$$

It can be shown that these $2n+1$ functions are linearly independent and thus form a basis for a $(2n+1)$-dimensional subspace of $C[a, b]$.

Let us now consider the problem of finding the least squares approximation of a continuous function $f(x)$ over the interval $[0, 2\pi]$ by a trigonometric polynomial of order n or less. As noted above, the least squares approximation to $\mathbf{f}$ from W is the orthogonal projection of $\mathbf{f}$ on W. To find this orthogonal projection, we must find an orthonormal basis $\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_{2n}$ for W, after which we can compute the orthogonal projection on W from the formula

$$\text{proj}_W \mathbf{f} = \langle \mathbf{f}, \mathbf{g}_0\rangle \mathbf{g}_0 + \langle \mathbf{f}, \mathbf{g}_1\rangle \mathbf{g}_1 + \cdots + \langle \mathbf{f}, \mathbf{g}_{2n}\rangle \mathbf{g}_{2n} \tag{4}$$

[see Theorem 6.3.4(*b*)]. An orthonormal basis for W can be obtained by applying the Gram–Schmidt process to the basis vectors in (3) using the inner product

$$\langle \mathbf{f}, \mathbf{g}\rangle = \int_0^{2\pi} f(x)g(x)\,dx$$

This yields the orthonormal basis

$$\begin{aligned} \mathbf{g}_0 = \frac{1}{\sqrt{2\pi}},\quad \mathbf{g}_1 = \frac{1}{\sqrt{\pi}}\cos x, \ldots,\quad \mathbf{g}_n = \frac{1}{\sqrt{\pi}}\cos nx, \\ \mathbf{g}_{n+1} = \frac{1}{\sqrt{\pi}}\sin x, \ldots,\quad \mathbf{g}_{2n} = \frac{1}{\sqrt{\pi}}\sin nx \end{aligned} \tag{5}$$

(see Exercise 6). If we introduce the notation

$$\begin{aligned} a_0 = \frac{2}{\sqrt{2\pi}}\langle \mathbf{f}, \mathbf{g}_0\rangle,\quad a_1 = \frac{1}{\sqrt{\pi}}\langle \mathbf{f}, \mathbf{g}_1\rangle, \ldots,\quad a_n = \frac{1}{\sqrt{\pi}}\langle \mathbf{f}, \mathbf{g}_n\rangle \\ b_1 = \frac{1}{\sqrt{\pi}}\langle \mathbf{f}, \mathbf{g}_{n+1}\rangle, \ldots,\quad b_n = \frac{1}{\sqrt{\pi}}\langle \mathbf{f}, \mathbf{g}_{2n}\rangle \end{aligned} \tag{6}$$

then on substituting (5) in (4), we obtain

$$\text{proj}_W \mathbf{f} = \frac{a_0}{2} + [a_1\cos x + \cdots + a_n\cos nx] + [b_1\sin x + \cdots + b_n\sin nx] \tag{7}$$

where

$$a_0 = \frac{2}{\sqrt{2\pi}}\langle \mathbf{f}, \mathbf{g}_0\rangle = \frac{2}{\sqrt{2\pi}}\int_0^{2\pi} f(x)\frac{1}{\sqrt{2\pi}}\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_1 = \frac{1}{\sqrt{\pi}}\langle \mathbf{f}, \mathbf{g}_1\rangle = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} f(x)\frac{1}{\sqrt{\pi}}\cos x\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$$

$$\vdots$$

$$a_n = \frac{1}{\sqrt{\pi}}\langle \mathbf{f}, \mathbf{g}_n\rangle = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} f(x)\frac{1}{\sqrt{\pi}}\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_1 = \frac{1}{\sqrt{\pi}}\langle \mathbf{f}, \mathbf{g}_{n+1}\rangle = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} f(x)\frac{1}{\sqrt{\pi}}\sin x\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx$$

$$\vdots$$

$$b_n = \frac{1}{\sqrt{\pi}}\langle \mathbf{f}, \mathbf{g}_{2n}\rangle = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} f(x)\frac{1}{\sqrt{\pi}}\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

In short,

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx,\quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx \tag{8}$$

The numbers $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$ are called the ***Fourier coefficients*** of $\mathbf{f}$.

▶ **EXAMPLE 1 Least Squares Approximations**

Find the least squares approximation of $f(x) = x$ on $[0, 2\pi]$ by

Solution (a)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = 2\pi \qquad (9a)$$

For $k = 1, 2, \ldots$, integration by parts yields (verify)

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos kx\,dx = 0 \qquad (9b)$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin kx\,dx = -\frac{2}{k} \qquad (9c)$$

Thus, the least squares approximation to x on $[0, 2\pi]$ by a trigonometric polynomial of order 2 or less is

$$x \approx \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x$$

or, from (9a), (9b), and (9c),

$$x \approx \pi - 2\sin x - \sin 2x$$

Solution (b) The least squares approximation to x on $[0, 2\pi]$ by a trigonometric polynomial of order n or less is

$$x \approx \frac{a_0}{2} + [a_1\cos x + \cdots + a_n\cos nx] + [b_1\sin x + \cdots + b_n\sin nx]$$

or, from (9a), (9b), and (9c),

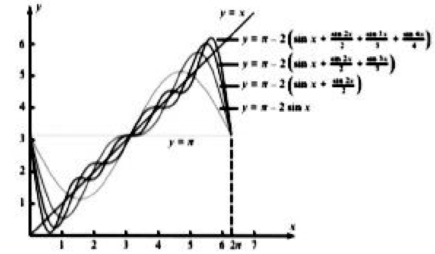
$$x \approx \pi - 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots + \frac{\sin nx}{n}\right)$$

The graphs of $y = x$ and some of these approximations are shown in Figure 6.6.4.

Jean Baptiste Fourier (1768–1830)

Historical Note Fourier was a French mathematician and physicist who discovered the Fourier series and related ideas while working on problems of heat diffusion. This discovery was one of the most influential in the history of mathematics; it is the cornerstone of many fields of mathematical research and a basic tool in many branches of engineering. Fourier, a political activist during the French revolution, spent time in jail for his defense of many victims during the Terror. He later became a favorite of Napoleon who made him a baron.

[Image: Hulton Archive/Getty Images]



▶ Figure 6.6.4

It is natural to expect that the mean square error will diminish as the number of terms in the least squares approximation

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)$$

increases. It can be proved that for functions f in $C[0, 2\pi]$, the mean square error approaches zero as $n \to +\infty$; this is denoted by writing

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx)$$

The right side of this equation is called the ***Fourier series*** for f over the interval $[0, 2\pi]$. Such series are of major importance in engineering, science, and mathematics. ◀

Exercise Set 6.6

1. Find the least squares approximation of $f(x) = 1 + x$ over the interval $[0, 2\pi]$ by
   (a) a trigonometric polynomial of order 2 or less.
   (b) a trigonometric polynomial of order n or less.

2. Find the least squares approximation of $f(x) = x^2$ over the interval $[0, 2\pi]$ by
   (a) a trigonometric polynomial of order 3 or less.
   (b) a trigonometric polynomial of order n or less.

3. (a) Find the least squares approximation of x over the interval $[0, 1]$ by a function of the form $a + be^x$.
   (b) Find the mean square error of the approximation.

4. (a) Find the least squares approximation of e^x over the interval $[0, 1]$ by a polynomial of the form $a_0 + a_1x$.
   (b) Find the mean square error of the approximation.

5. (a) Find the least squares approximation of $\sin \pi x$ over the interval $[-1, 1]$ by a polynomial of the form $a_0 + a_1x + a_2x^2$.
   (b) Find the mean square error of the approximation.

6. Use the Gram–Schmidt process to obtain the orthonormal basis (5) from the basis (3).

7. Carry out the integrations indicated in Formulas (9a), (9b), and (9c).

8. Find the Fourier series of $f(x) = \pi - x$ over the interval $[0, 2\pi]$.

9. Find the Fourier series of $f(x) = 1, 0 < x < \pi$ and $f(x) = 0, \pi \le x \le 2\pi$ over the interval $[0, 2\pi]$.

10. What is the Fourier series of $\sin(3x)$?

True-False Exercises

TF. In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

(a) If a function $\mathbf{f}$ in $C[a, b]$ is approximated by the function $\mathbf{g}$, then the mean square error is the same as the area between the graphs of $f(x)$ and $g(x)$ over the interval $[a, b]$.

(b) Given a finite-dimensional subspace W of $C[a, b]$, the function $\mathbf{g} = \text{proj}_W \mathbf{f}$ minimizes the mean square error.

(c) $\{1, \cos x, \sin x, \cos 2x, \sin 2x\}$ is an orthogonal subset of the vector space $C[0, 2\pi]$ with respect to the inner product $\langle \mathbf{f}, \mathbf{g}\rangle = \int_0^{2\pi} f(x)g(x)\,dx$.

(d) $\{1, \cos x, \sin x, \cos 2x, \sin 2x\}$ is an orthonormal subset of the vector space $C[0, 2\pi]$ with respect to the inner product $\langle \mathbf{f}, \mathbf{g}\rangle = \int_0^{2\pi} f(x)g(x)\,dx$.

(e) $\{1, \cos x, \sin x, \cos 2x, \sin 2x\}$ is a linearly independent subset of $C[0, 2\pi]$.

Chapter 6 Supplementary Exercises

1. Let R^4 have the Euclidean inner product.
   (a) Find a vector in R^4 that is orthogonal to $\mathbf{u}_1 = (1, 0, 0, 0)$ and $\mathbf{u}_4 = (0, 0, 0, 1)$ and makes equal angles with $\mathbf{u}_2 = (0, 1, 0, 0)$ and $\mathbf{u}_3 = (0, 0, 1, 0)$.
   (b) Find a vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$ of length 1 that is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_4$ above and such that the cosine of the angle between $\mathbf{x}$ and $\mathbf{u}_2$ is twice the cosine of the angle between $\mathbf{x}$ and $\mathbf{u}_3$.

2. Prove: If $\langle \mathbf{u}, \mathbf{v}\rangle$ is the Euclidean inner product on R^n, and if A is an $n \times n$ matrix, then
$$\langle \mathbf{u}, A\mathbf{v}\rangle = \langle A^T\mathbf{u}, \mathbf{v}\rangle$$
[*Hint:* Use the fact that $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v} = \mathbf{v}^T\mathbf{u}$.]

3. Let M_{22} have the inner product $\langle U, V\rangle = \text{tr}(U^TV) = \text{tr}(V^TU)$ that was defined in Example 6 of Section 6.1. Describe the

4. Let $A\mathbf{x} = \mathbf{0}$ be a system of m equations in n unknowns. Show that
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
is a solution of this system if and only if the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is orthogonal to every row vector of A with respect to the Euclidean inner product on R^n.

5. Use the Cauchy–Schwarz inequality to show that if $a_1, a_2, \ldots, a_n$ are positive real numbers, then
$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \ge n^2$$

the interval $[0, 2\pi]$ by

(a) a trigonometric polynomial of order 2 or less.

(b) a trigonometric polynomial of order n or less.

2. Find the least squares approximation of $f(x) = x^2$ over the interval $[0, 2\pi]$ by

(a) a trigonometric polynomial of order 3 or less.

(b) a trigonometric polynomial of order n or less.

3. (a) Find the least squares approximation of x over the interval $[0, 1]$ by a function of the form $a + be^x$.

(b) Find the mean square error of the approximation.

4. (a) Find the least squares approximation of e^x over the interval $[0, 1]$ by a polynomial of the form $a_0 + a_1x$.

(b) Find the mean square error of the approximation.

5. (a) Find the least squares approximation of $\sin \pi x$ over the interval $[-1, 1]$ by a polynomial of the form $a_0 + a_1x + a_2x^2$.

(b) Find the mean square error of the approximation.

6. Use the Gram–Schmidt process to obtain the orthonormal basis (5) from the basis (3).

7. Carry out the integrations indicated in Formulas (9a), (9b), and (9c).

8. ... $[0, 2\pi]$.

9. Find the Fourier series of $f(x) = 1, 0 < x < \pi$ and $f(x) = 0, \pi \le x \le 2\pi$ over the interval $[0, 2\pi]$.

10. What is the Fourier series of $\sin(3x)$?

True-False Exercises

TF. In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

(a) If a function $\mathbf{f}$ in $C[a, b]$ is approximated by the function $\mathbf{g}$, then the mean square error is the same as the area between the graphs of $f(x)$ and $g(x)$ over the interval $[a, b]$.

(b) Given a finite-dimensional subspace W of $C[a, b]$, the function $\mathbf{g} = \text{proj}_W \mathbf{f}$ minimizes the mean square error.

(c) $\{1, \cos x, \sin x, \cos 2x, \sin 2x\}$ is an orthogonal subset of the vector space $C[0, 2\pi]$ with respect to the inner product $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^{2\pi} f(x)g(x)\,dx$.

(d) $\{1, \cos x, \sin x, \cos 2x, \sin 2x\}$ is an orthonormal subset of the vector space $C[0, 2\pi]$ with respect to the inner product $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^{2\pi} f(x)g(x)\,dx$.

(e) $\{1, \cos x, \sin x, \cos 2x, \sin 2x\}$ is a linearly independent subset of $C[0, 2\pi]$.

Chapter 6 Supplementary Exercises

1. Let R^4 have the Euclidean inner product.

(a) Find a vector in R^4 that is orthogonal to $\mathbf{u}_1 = (1, 0, 0, 0)$ and $\mathbf{u}_4 = (0, 0, 0, 1)$ and makes equal angles with $\mathbf{u}_2 = (0, 1, 0, 0)$ and $\mathbf{u}_3 = (0, 0, 1, 0)$.

(b) Find a vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$ of length 1 that is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_4$ above and such that the cosine of the angle between $\mathbf{x}$ and $\mathbf{u}_2$ is twice the cosine of the angle between $\mathbf{x}$ and $\mathbf{u}_3$.

2. Prove: If $\langle \mathbf{u}, \mathbf{v} \rangle$ is the Euclidean inner product on R^n, and if A is an $n \times n$ matrix, then

$$\langle \mathbf{u}, A\mathbf{v} \rangle = \langle A^T\mathbf{u}, \mathbf{v} \rangle$$

[*Hint:* Use the fact that $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T\mathbf{u}$.]

3. Let M_{22} have the inner product $\langle U, V \rangle = \text{tr}(U^TV) = \text{tr}(V^TU)$ that was defined in Example 6 of Section 6.1. Describe the orthogonal complement of

(a) the subspace of all diagonal matrices.

(b) the subspace of symmetric matrices.

4. Let $A\mathbf{x} = \mathbf{0}$ be a system of m equations in n unknowns. Show that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is a solution of this system if and only if the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is orthogonal to every row vector of A with respect to the Euclidean inner product on R^n.

5. Use the Cauchy–Schwarz inequality to show that if $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \ge n^2$$

6. Show that if $\mathbf{x}$ and $\mathbf{y}$ are vectors in an inner product space and c is any scalar, then

$$\|c\mathbf{x} + \mathbf{y}\|^2 = c^2\|\mathbf{x}\|^2 + 2c\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2$$

7. Let R^3 have the Euclidean inner product. Find two vectors of length 1 that are orthogonal to all three of the vectors $\mathbf{u}_1 = (1, 1, -1)$, $\mathbf{u}_2 = (-2, -1, 2)$, and $\mathbf{u}_3 = (-1, 0, 1)$.

8. Find a weighted Euclidean inner product on R^n such that the vectors

$$\begin{aligned} \mathbf{v}_1 &= (1, 0, 0, \ldots, 0) \\ \mathbf{v}_2 &= (0, \sqrt{2}, 0, \ldots, 0) \\ \mathbf{v}_3 &= (0, 0, \sqrt{3}, \ldots, 0) \\ &\vdots \\ \mathbf{v}_n &= (0, 0, 0, \ldots, \sqrt{n}) \end{aligned}$$

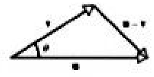
form an orthonormal set.

9. Is there a weighted Euclidean inner product on R^2 for which the vectors $(1, 2)$ and $(3, -1)$ form an orthonormal set? Justify your answer.

10. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space V, then $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ can be regarded as sides of a "triangle" in V (see the accompanying figure). Prove that the law of cosines holds for any such triangle; that is,

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.



◀ Figure Ex-10

11. (a) As shown in Figure 3.2.6, the vectors $(k, 0, 0)$, $(0, k, 0)$, and $(0, 0, k)$ form the edges of a cube in R^3 with diagonal (k, k, k). Similarly, the vectors

$$(k, 0, 0, \ldots, 0), \quad (0, k, 0, \ldots, 0), \ldots, \quad (0, 0, 0, \ldots, k)$$

can be regarded as edges of a "cube" in R^n with diagonal $(k, k, k, \ldots, k)$. Show that each of the above edges makes an angle of θ with the diagonal, where $\cos\theta = 1/\sqrt{n}$.

(b) (*Calculus required*) What happens to the angle θ in part (a) as the dimension of R^n approaches ∞?

12. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space.

(a) Prove that $\|\mathbf{u}\| = \|\mathbf{v}\|$ if and only if $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal.

(b) Give a geometric interpretation of this result in R^2 with the Euclidean inner product.

13. Let $\mathbf{u}$ be a vector in an inner product space V, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for V. Show that if α_i is the angle between $\mathbf{u}$ and $\mathbf{v}_i$, then

$$\cos^2\alpha_1 + \cos^2\alpha_2 + \cdots + \cos^2\alpha_n = 1$$

14. Prove: If $\langle \mathbf{u}, \mathbf{v} \rangle_1$ and $\langle \mathbf{u}, \mathbf{v} \rangle_2$ are two i... space V, then the quantity $\langle \mathbf{u}, \mathbf{v} \rangle$ ... inner product.

15. Prove Theorem 6.2.5.

16. Prove: If A has linearly independent ... is orthogonal to the column space of A, then the least squares solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{0}$.

17. Is there any value of s for which $x_1 = 1$ and $x_2 = 2$ is the least squares solution of the following linear system?

$$\begin{aligned} x_1 - x_2 &= 1 \\ 2x_1 + 3x_2 &= 1 \\ 4x_1 + 5x_2 &= s \end{aligned}$$

Explain your reasoning.

18. Show that if p and q are distinct positive integers, then the functions $f(x) = \sin px$ and $g(x) = \sin qx$ are orthogonal with respect to the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^{2\pi} f(x)g(x)\,dx$$

19. Show that if p and q are positive integers, then the functions $f(x) = \cos px$ and $g(x) = \sin qx$ are orthogonal with respect to the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^{2\pi} f(x)g(x)\,dx$$

20. Let W be the intersection of the planes

$$x + y + z = 0 \quad \text{and} \quad x - y + z = 0$$

in R^3. Find an equation for $W^\perp$.

21. Prove that if $ad - bc \neq 0$, then the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has a unique QR-decomposition $A = QR$, where

$$Q = \frac{1}{\sqrt{a^2 + c^2}} \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$$

$$R = \frac{1}{\sqrt{a^2 + c^2}} \begin{bmatrix} a^2 + c^2 & ab + cd \\ 0 & ad - bc \end{bmatrix}$$