Geodesic Problems

odesic is the curve of shortest length joining two points in space. Geodesic lems are similar to variational problems with constraints but sometimes be reduced to problems without constraint(s).

.1 General geodesic problem

eodesic connects two points in space or on a surface. Here we are concerned n geodesics on surfaces. The geodesic problem for a surface can be stated ollows:

en points $A(x_1,\ y_1,\ z_1)$ and $B(x_2,\ y_2,\ z_2)$ on a surface $z = z(x,\ y)$ find the arc of shortest length connecting A and B.

re we have to minimize

$$\ell = \int_A^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

$$= \int_{x_1}^{x_2} \sqrt{1 + y'^2 + (z_x + z_y\, y')^2}\, dx$$

here we have used the result

$$z = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy = (z_x + z_y y')\, dx$$

ubject to the conditions

$(x_1) = y_1,\ y(x_2) = y_2,\ z(x_1) = z_1,\ z(x_2) = z_2$

.6.2 Illustrative examples

We illustrate some well-known geodesic problems with examples.

Example 1

Find the curve of shortest length between two given points in a plane, using polar coordinates $(r,\ \theta)$.

Solution

The arc length ds is given by

$$\ldots = \sqrt{r'^2 + r^2}\, d\theta, \quad r' = \frac{dr}{d\theta}$$

us we have to minimize $\int_A^B ds = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + r'^2}\, d\theta$ subject to $r(\theta_1) =$ stant, and $r(\theta_2) =$ constant.

e $F(\theta, r, r') = \sqrt{r^2 + r'^2}$

e there is no explicit dependence on the independent variable θ, we use first integral of the E-L equation, (Beltrami's identity): $F - r'(\partial F/\partial r') =$ tant, which becomes

$$\sqrt{r^2 + r'^2} - r' \frac{r'}{\sqrt{r^2 + r'^2}} = c_1, \quad \text{a constant}$$

h on simplification reduces to $r^2 = c_1 \sqrt{r^2 + r'^2}$.

last equation gives

$$r'^2 = \frac{r^4 - c_1^2 r^2}{c_1^2} \quad \text{or} \quad \frac{dr}{d\theta} = \frac{r}{c_1} \sqrt{r^2 - c_1^2}$$

$$\int \frac{\pm c_1\, dr}{r\sqrt{r^2 - c_1^2}} = \theta + c_2 \quad \Rightarrow \quad \sec^{-1} \frac{r}{c_1} = \theta + c_2$$

efrom

$$\begin{aligned} c_1 &= r\cos(\theta + c_2) = r(\cos\theta \cos c_2 - \sin\theta \sin c_2) \\ &= (\cos c_2)\, x - (\sin c_2)\, y \end{aligned}$$

i is a linear equation in x, y of the form $\alpha x + \beta y + \gamma = 0$ and represents ight line.

nple 2

d the curve of shortest length (geodesic) on the surface of a sphere.

ion

and B be two points on the sphere S. Here the problem is to minimize egral

$^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

to the constraint $x^2 + y^2 + z^2 = a^2$ where a is the radius of the sphere. oblem can also be solved by using spherical polar coordinates (r, θ, ϕ). $\theta, \phi)$ and $(a, \theta + d\theta, \phi + d\phi)$ be the coordinates of two neighbouring on the curve through A and B and lying on the sphere. Then the above n is equivalent to minimizing the integral

I

or

He

Th

or

whic

or

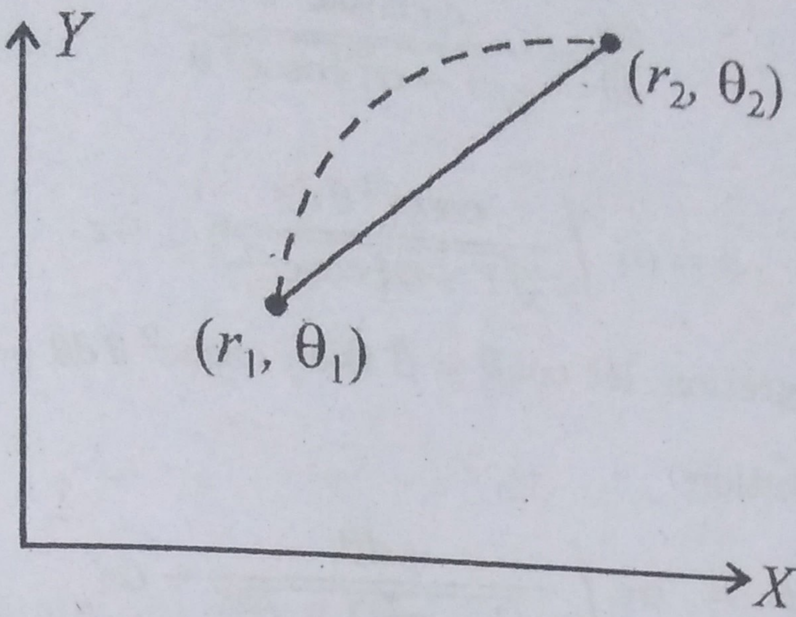

Figure 9.7: Points A and B on the surface of a sphere and the curve of shortest distance between them.

In case of a sphere $r = a,\ dr = 0$. Therefore the problem reduces to

$$\int_A^B \sqrt{a^2\,(d\theta)^2 + a^2 \sin^2\theta\,(d\phi)^2} \;\rightarrow\; \min$$

or

$$a \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2}\; d\theta \;\rightarrow\; \min$$

Here

$$F \equiv F(\theta; r; r') = \sqrt{1 + \sin^2\theta\,\phi'^2}, \quad \phi' = \frac{d\phi}{d\theta}$$

The required curve satisfies the E-L equation, (with $\phi' = d\phi/d\theta$)

$$\frac{\partial F}{\partial \phi} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \phi'}\right) = 0$$

or

$$0 - \frac{d}{d\theta}\left[\frac{1}{2}(1 + \sin^2\theta\phi'^2)^{-1/2}\; 2\sin^2\theta\;\phi'\right] = 0$$

which gives

$$\frac{\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\phi'^2}} = \alpha_1 \;\text{ or }\; \sin^4\theta\phi'^2 = \alpha_1^2(1 + \sin^2\theta\,\phi'^2)$$

or

$$\phi' = \frac{\alpha_1}{\sin\theta\sqrt{\sin^2\theta - \alpha_1^2}} \;\text{ or }\; \phi' = \frac{\alpha_1}{\sin^2\theta\sqrt{1 - \alpha_1^2\,\mathrm{cosec}^2\,\theta}}$$

$$\frac{d\phi}{d\theta} = \frac{\alpha_1 \operatorname{cosec}^2\theta}{\sqrt{1-\alpha_1^2 \operatorname{cosec}^2\theta}}$$

herefore

$$\phi = \alpha_1 \int \frac{\operatorname{cosec}^2\theta \, d\theta}{\sqrt{1-\alpha_1^2 \operatorname{cosec}^2\theta}} + \alpha_2$$

perform the integration, let $\cot\theta = \beta$ then $\operatorname{cosec}^2\theta \, d\theta = -d\beta$.

erefore on substitution

$$\begin{aligned}
\phi &= \alpha_1 \int \frac{-\,d\beta}{\sqrt{1-\alpha_1^2(1+\beta^2)}} + \alpha_2 \\
&= -\int \frac{\alpha_1 d\beta}{\sqrt{(1-\alpha_1^2)-\alpha_1^2\beta^2}} + \alpha_2 \\
&= -\int \frac{d\beta}{\sqrt{(1-\alpha_1^2)/\alpha_1^2-\beta^2}} + \alpha_2
\end{aligned}$$

ontinuing further

$$\begin{aligned}
\phi &= -\int \frac{d\beta}{\sqrt{\alpha_3^2-\beta^2}} + \alpha_2, \quad \alpha_3^2 = (1-\alpha_1^2)/\alpha_1^2 \\
&= \cos^{-1}\frac{\beta}{\alpha_3} + \alpha_2 = \cos^{-1}\frac{\cot\theta}{\alpha_3} + \alpha_2
\end{aligned}$$

$$\frac{\cot\theta}{\alpha_3} = \cos(\phi-\alpha_2) = \cos\alpha_2\cos\phi + \sin\alpha_2\sin\phi$$

n can also be written as

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \gamma_1\cos\phi + \gamma_2\sin\phi$$

γ_1 and γ_2 are new constants.

ally

$$a\cos\theta = a\,\gamma_1\sin\theta\cos\phi + a\,\gamma_2\sin\theta\sin\phi$$

ssing to the Cartesian coordinates $z = \gamma_1 x + \gamma_2 y$ which is the equation plane through the centre of the sphere. Hence the curve of shortest joining A and B is the arc of the great circle through A and B.

nple 3

he geodesic curve for the cylinder $x^2+y^2=a^2$.

ion

Chapter

In cylinc

We have

subject t

Here $F =$

which in

whose sol

On passin

The inters extremal c

Example

Find the s $B(2,\ 1,\ -$

Solution

Here we ha

ℓ

subject to

ylindrical coordinates $x = a\cos\theta,\ y = a\sin\theta,\ z = z.$

have to minimize

$$\begin{aligned} l &= \int_A^B ds = \int_A^B \sqrt{a^2\,(d\theta)^2 + dz^2} \\ &= \int_{\theta_1}^{\theta_2} \sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta \\ &= \int_{\theta_1}^{\theta_2} \sqrt{a^2 + z'^2}\, d\theta, \quad z' = \frac{dz}{d\theta} \end{aligned}$$

ect to no constraint, (where θ_1 and θ_2 correspond to A and B).

e $F = \sqrt{a^2 + z'^2}$. The E-L equation in this case is

$$\frac{\partial F}{\partial z} - \frac{d}{d\theta}\frac{\partial F}{\partial z'} = 0$$

ch in this case reduces to

$$0 - \frac{d}{d\theta}(2z') = 0 \quad \text{or} \quad \frac{d^2 z}{d\theta^2} = 0$$

se solution is given by $z = \alpha_1 + \alpha_2\theta$.

passing to Cartesian coordinates

$$z = \alpha_1 + \alpha_2 \tan^{-1}\frac{y}{x} \quad \text{or} \quad \tan\left(\frac{z - \alpha_1}{\alpha_2}\right) = \frac{y}{x}$$

intersection of this surface with the given cylinder gives the required emal curve.

mple 4

the shortest distance between the points $A(1,\ -1,\ 0)$ and $1,\ -1)$ in the plane $15x - 7y + z - 22 = 0$.

ution

e we have to minimize

$$\ell = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \int_{x_1=1}^{x_2=2} \sqrt{1 + y'^2 + z'^2}\, dx$$

ect to the constraint that the points A, B be on the plane

$$G \equiv 15x - 7y + z - 22 = 0$$

ere the auxiliary function is given by

$$H = F + \lambda G = \sqrt{1 + y'^2 + z'^2} + \lambda\,(15x - 7y + z - 22)$$

e corresponding E-L equations are

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \text{ and } \frac{\partial H}{\partial z} - \frac{d}{dx}\left(\frac{\partial H}{\partial z'}\right) = 0$$

$$-7\lambda - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0 \qquad (1)$$

d

$$\lambda - \frac{d}{dx}\left(\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0 \qquad (2)$$

e have to solve (1) and (2) using the condition

$$15x - 7y + z - 22 = 0 \qquad (3)$$

e endpoint conditions satisfied by the functions $y = y(x)$ and $z = z(x)$ are

$$y(1) = -1, \quad y(2) = 1, \quad z(1) = 0, \quad z(2) = -1 \qquad (4)$$

om (1) and (2), by eliminating λ

$$-\frac{d}{dx}\left[\frac{y' + 7z'}{\sqrt{1 + y'^2 + z'^2}}\right] = 0$$

ch gives

$$\frac{y' + 7z'}{\sqrt{1 + y'^2 + z'^2}} = c_1 \qquad (5)$$

m (3)

$$z' = 7y' - 15 \qquad (6)$$

stituting for z' from (6) into (5), we obtain

$$\frac{y' + 7(7y' - 15)}{\sqrt{1 + y'^2 + (7y' - 15)^2}} = c_1$$

$$50y' - 105 = c_1\left[1 + y'^2 + 49y'^2 - 210y' + 225\right]^{1/2}$$

$$25(10y' - 21)^2 = c_1^2\left[50y'^2 - 210y' + 226\right]$$

$$25(100y'^2 - 420y' + 441) = c_1^2(50y'^2 - 210y' + 226)$$

or

$$(2500 - 50c_1^2)\, y'^2 + (210c_1^2 - 10500)\, y' + (11025 - 226c_1^2) = 0$$

or

$$50(50 - c_1^2)\, y'^2 + 210(c_1^2 - 50)\cdot y' + (11025 - 226c_1^2) = 0$$

which is quadratic equation in y'. Since c_1 is arbitrary, we can always choose it so that the equation has real roots. Let α be one such root, then DE $y' = \alpha$ gives $y = \alpha x + \beta$.

Applying the given B.Cs., *viz.* $y(1) = -1,\ y(2) = 1,\ [z(1) = 0,\ z(2) = -1]$. we obtain $\alpha = 2,\ \beta = -3$. Therefore $y = 2x - 3$.

For z we have

$$z = 7y - 15x + 22 = 7(2x - 3) - 15x + 22 = -x + 1$$

The required least distance is

$$\ell = \int_1^2 \sqrt{1 + y'^2 + z'^2}\, dx = \int_1^2 \sqrt{1 + 4 + 1}\, dx = \sqrt{6}$$

Example 5

Find the shortest distance between the points $A(1, 0, -1)$ and $B(0, -1, 1)$ lying on the surface $x + y + z = 0$.

Solution

The problem is equivalent to

$$\ell = \int_0^1 \sqrt{1 + y'^2 + z'^2}\, dx \to \text{minimum}$$

subject to $y(0) = -1,\ z(0) = 1,\ y(1) = 0,\ z(1) = -1$.

Here

$$x + y + z = 0, \quad F = \sqrt{1 + y'^2 + z'^2}, \quad G = x + y + z$$

$$H = F + \lambda(x)\, G = \sqrt{1 + y'^2 + z'^2} + \lambda(x)\,(x + y + z)$$

From E-L equations for H, we have

$$\lambda - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0$$

and

$$\lambda - \frac{d}{dx}\left(\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0$$

$\int_{x_1}^{x_2}(py'^2 - qy^2)\,dx \to$ minimum, $\int_{x_1}^{x_2} r\,y^2\,dx = 1$, $y(x_1) = y_1$, $y(x_2) = y_2$.

solution of (b) above is $y = (\sqrt{3}/2)\ x$; find the corresponding minimum s of the integral and compare it with the values obtained by considering ation of $y = (\sqrt{3}/2)\ x$.

d the E-L equations of the two- dimensional isoperimetrical problem

$$\int\int_R F(x,\ y,\ u(x,\ y),\ u_x,\ u_y)\ dx\,dy \to \text{minimum}$$

$$\int\int_R G(x,\ y,\ u(x,\ y),\ u_x,\ u_y)\ dx\,dy = L$$

$u = u(x,\ y)$ satisfies the boundary condition $u(x,\ y) = u_0(x,\ y)$ on lary curve C of the region R.

d the curve joining the points (0, 0) and (1, 0) with the given length hat the y - coordinate of its centroid is minimum.

Applications to Mechanics

pplications of the Calculus of Variations in Mechanics are based on ving principle of least action and Hamilton's principle. These principles ted below.

Principle of least action

article move in an external field of force which is conservative. If the takes place in the interval of time from t_1 to t_2, where $t_2 > t_1$ then ual path traced by the particle is the one along which $I = \int_{t_1}^{t_2} L\ dt$ num. where L is the Lagrangian and for a conservative system $L =$ Energy - Potential Energy $= T - V$.

Hamilton's Principle

ng to this principle, the path of motion of a rigid body in the time $t_2 - t_1$ is such that the integral

$$A = \int_{t_1}^{t_2} Ldt$$

ationary value where L is the Lagrangian.

9.7.3 Illustrative Examples

Example 1

Find the equation of motion of a particle moving in a conservative field force described by the function $V(x, y, z)$.

Solution

Here

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

By the principle of least action, the equations of motion are given by

$\delta I = 0 = \int_{t_1}^{t_2} \delta L \, dt$

From this condition the following equations of motion follow

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial x}\right) = 0, \quad \frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial y}\right) = 0, \quad \frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial z}\right) = 0$$

The last three equations are called *Lagrangian equations of motion.*

Example 2 (Simple Harmonic Motion)

Use the principle of least action to obtain DE describing the vibrations of a simple harmonic oscillator.

Solution

By definition for a particle executing simple harmonic motion (S.H.M.) along the X - axis, the force F is given by $F = -k\,x$. where x denotes displacement from the centre of vibration (or the origin); the constant k which is positive is called the *spring constant*. The kinetic energy is given by $T = (1/2)\, m\, \dot{x}^2$.

For the potential energy, $F = -dV/dx$ and therefore P.E. is given by $V = \int k\,x\,dx = (1/2)\,k\,x^2$.

Therefore the Lagrangian L is given by

$L = T - V = (1/2)\, m\dot{x}^2 - (1/2)\, k\, x^2 \equiv L(x, \dot{x})$

The Euler-Lagrange equation

in this case reduces to

$$-kx - \frac{d}{dt}(m\dot{x}) = 0 \text{ or } m\ddot{x} + kx = 0$$

Example 3

The Lagrangian L for a system of n particles is a function of generalized coordinates q_i and generalized velocities $\dot{q}_i$. Show that minimization of the integral $I = \int_{t_1}^{t_2} L\, dt$ leads to the following equation of motion

$(d/dt)\ [\partial L/\partial \dot{q}_i] - (\partial L/\partial q_i) = 0$

Solution

Here $L = L(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots \dot{q}_n)$

Therefore

$$\delta I = \int_{t_1}^{t_2} \sum_{i=1}^{i=n} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt \tag{1}$$

Now consider

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i\, dt = \delta q_i \left. \frac{\partial L}{\partial \dot{q}_i} \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta q_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) dt$$

$$= - \int_{t_1}^{t_2} \delta q_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) dt$$

where we have used the fact that q_i are fixed at times t_1 and t_2.

So equation (1) becomes

$$\delta I = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial q_i'} \right) \delta q_i\, dt$$

For the extremal, $\delta I = 0$. Therefore we have

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial q_i'} \right) \right) \delta q_i\, dt = 0$$

Since δq_i is arbitrary, therefore by making an appeal to the fundamental lemma of calculus of variations, we have

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial q_i'} \right) = 0, \quad i = 1, 2, \cdots$$

Example 4

Obtain the equation of motion of a stretched string by invoking Hamilton's principle of least action.

Solution

et the string have uniform density ρ and length ℓ. Let it be initially stretched long the X - axis, and when plucked let it vibrate in the X Y- plane. Consider he K.E. of an element ds of the string. Then

$T = (1/2)\, dm\, v^2 = (1/2)\, \rho\, ds\, y_t^2$ and therefore $T = (1/2)\, \rho \int_0^\ell y_t^2\, ds$.

ere we have assumed that the deflection of the string is very small. There re the above expression for kinetic energy can be approximated as $T = $ /2) $\int_0^\ell y_t^2\, dx$

w the potential energy V of the string equals the increase in length due to nsion of the string, which in turn equals the work done by the tension in oducing extension $ds - dx$.

erefore we have

$$
\begin{aligned}
V &= \int_0^\ell (ds - dx)\, \tau \\
&= \tau \left\{ \int_0^\ell [1 + \frac{1}{2} y_x^2]\, dx - l \right\} \\
&= \tau \left\{ \ell + \frac{1}{2} \int_0^\ell y_x^2\, dx - \ell \right\} \\
&= \frac{\tau}{2} \int_0^\ell y_x^2\, dx
\end{aligned}
$$

$$
\begin{aligned}
L &= T - V = \frac{1}{2} \rho \int_0^\ell y_t^2\, dx - \frac{1}{2} \tau \int_0^\ell y_x^2\, dx \\
&= \frac{1}{2} \int_0^\ell (\rho y_t^2 - \tau y_x^2)\, dx
\end{aligned}
$$

milton's principle, the equation of motion of the string will correspond ionary value of

$$\int_{t_1}^{t_2} L\, dt = \frac{1}{2} \int_{t_1}^{t_2} \int_0^\ell [\rho y_t^2 - \tau y_x^2]\, dx\, dt$$

$= \rho y_t^2 - \tau y_x^2$.

extremal path

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) = 0 \qquad (1)$$

Now

The last rela

Also

Putting these

Example 5

Discuss the motion by m

Solution

We consider t is supposed t ton's principl membrane.

We choose X Y - plane. and are perpe figure 9.8). W B C D origina element dS in corresponding t, i.e. $z = z(x,$

To calculate p of the membra

T

where we have

From subsectio

In this problem

Now
$$\frac{\partial \mathcal{L}}{\partial y} = 0 \text{ and } \frac{\partial \mathcal{L}}{\partial y_x} = -2\tau\left(\frac{\partial y}{\partial x}\right)$$

The last relation gives
$$\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) = -2\tau\left(\frac{\partial^2 y}{\partial x^2}\right)$$

Also
$$\frac{\partial \mathcal{L}}{\partial y_t} = 2\rho\, y_t \implies \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) = 2\rho\, y_{tt}$$

Putting these values in (1), we obtain $y_{xx} - (\rho/\tau).y_{tt} = 0$

Example 5

Discuss the vibrations of a stretched membrane and obtain its equation of motion by making use of Hamilton's principle.

Solution

We consider the transverse vibrations of a membrane of arbitrary shape which is supposed to be perfectly flexible but inextensible. Before applying Hamilton's principle we calculate kinetic and potential energies of the vibrating membrane.

We choose the coordinate axes so that originally the membrane lies in the XY-plane. As in case of the string, we suppose that displacements are small and are perpendicular to the XY plane, i.e. in the direction of the Z-axis, (see Figure 9.8). When the membrane is vibrating a rectangular element of area $ABCD$ originally lying in the XY- plane is brought to the position of curved element dS in space centered around the point (x, y, z) at time t. Here the corresponding z coordinate denotes the displacement of the membrane at time t, i.e. $z = z(x, y, t)$.

To calculate potential and kinetic energies we analyse the motion of an element of the membrane. If T denotes the total kinetic energy, then
$$T = \frac{1}{2}\int (dm)\, z_t^2 = \frac{1}{2}\int (\rho\, dS)\, z_t^2 = \frac{1}{2}\rho \int z_t^2\, dx\, dy$$
where we have taken $dS = dx\, dy$ in the first approximation.

From subsection 9.4.3, we know that
$$dS = \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy$$
In this problem z_x and z_y are negligible because the vibrations are small.

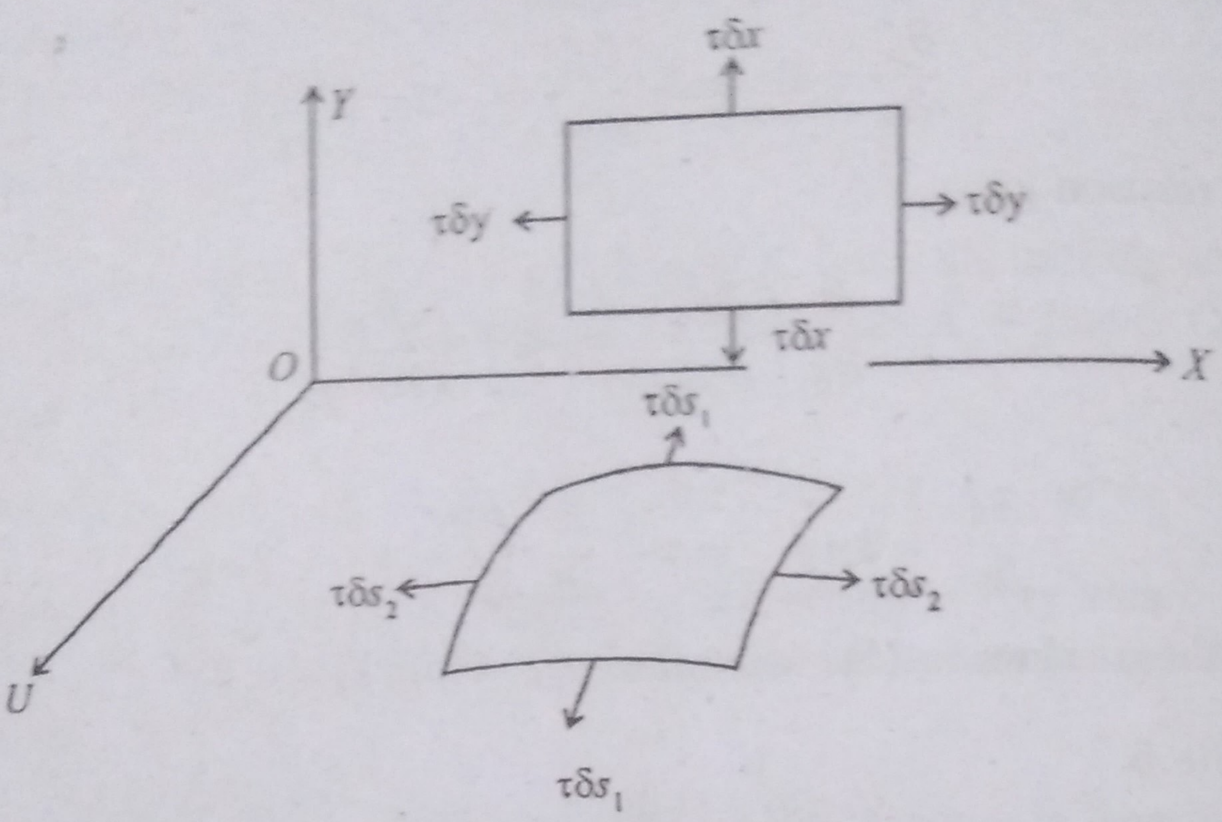

Figure 9.8: Vibrations of a stretched membrane.

calculate V, the total potential energy of the vibrating membrane, we have calculate the work done in bringing membrane from its original position XY-plane to its current position. First we calculate the work done on rectangular element $ABCD$ of area $dx\,dy$ in bringing it to the position of nent δS, (see figure 9.6).

notice that if τ is the tension per unit length in the membrane, then two ces each of magnitude $\tau\,dx$ are pulling the sides AD and BC, whereas two ces each of magnitude $\tau\,dy$ are pulling the sides AB and CD of the element CD.

membering that the work done on these sides is respectively $(ds_1 - dx)\,\tau\,dy$ d $(ds_2 - dy)\,\tau\,dx$, work done for the element of area d S will be

$$\begin{aligned} dV &= \tau\,(ds_1 - dx)\,dy + \tau\,(ds_2 - dy)\,dx \\ &= \tau(ds_1\,dy + ds_2\,dx - 2dx\,dy) \end{aligned}$$

w

$$\begin{aligned} ds_1\,dy &= \sqrt{(dx)^2 + (dz)^2}\,dy = \sqrt{1 + z_x^2}\,dx\,dy \\ &\doteq (1 + \frac{1}{2}\,z_x^2)\,dx\,dy \\ ds_2\,dx &= \sqrt{(dy)^2 + (dz)^2}\,dx = \sqrt{1 + z_y^2}\,dx\,dy \\ &\doteq (1 + \frac{1}{2}z_y^2)\,dx\,dy \end{aligned}$$

here we have used the fact th

being small. Therefore we can write

$$dV = \tau\left[\left(1+\frac{1}{2}z_x^2\right)dx\,dy + \left(1+\frac{1}{2}z_y^2\right)dx\,dy - 2dx\,dy\right]$$
$$\doteq \frac{1}{2}\tau\left(z_x^2+z_y^2\right)dx\,dy$$

Therefore total potential energy is given by

$$V = \tau\int_S \left(z_x^2+z_y^2\right)dx\,dy$$

Hence the Lagrangian is given by

$$L = T - V = \frac{1}{2}\int_S \left[\rho\, z_t^2 - \tau\left(z_x^2+z_y^2\right)\right]dx\,dy$$

By Hamilton's principle the condition $\delta\int_{t_1}^{t_2} L\,dt = 0$ leads to the equation of motion

$$\int_{t_1}^{t_2}\int_S \left(\rho\, z_t^2 - \tau\, z_x^2 - \tau\, z_y^2\right)dx\,dy\,dt = 0$$

Here

$$F(t,\ x,\ y;\ z,\ z_x,\ z_y,\ z_t) = \rho\, z_t^2 - \tau\, z_x^2 - \tau\, z_y^2$$

The Euler-Lagrange equation, called *Lagrangian equation* of motion, is given by

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial z_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 0$$

or on substitution

$$0 - \frac{\partial}{\partial t}(\rho\, z_t) - \frac{\partial}{\partial x}(-\tau\, z_x) - \frac{\partial}{\partial y}(-\tau\, z_y) = 0$$

or $z_{xx} + z_{yy} = (\rho/\tau)\, z_{tt}$

or $z_{xx} + z_{yy} = (1/c^2)\, z_{tt}$, where $c = \sqrt{\tau/\rho}$ is the velocity of the elastic waves.

9.7.4 Exercises

1. A light inextensible rope hangs vertically from a fixed support. It passes round a light pulley which supports a mass M, then goes up over a second pulley which is fixed. A second mass m is suspended at the end of the rope. Calculate the Lagrangian of the system and the obtain its equation of motion.

(**Hint**: If the origin is chosen at the point of suspension of mass M, and Z, z denote the vertical coordinates of the two masses measured downwards, then

Iint: The unconstrained Lagrangian is found to be

$$= (m_1/2)\,(\dot{x}_1^2 + \dot{y}_1^2 - 2gy_1)] + (m_2/2)\,(\dot{x}_2^2 + \dot{y}_2^2 - 2gy_2)$$

ext we impose the constraints.).

Discuss the motion of a particle moving on the surface of r
cylindrical coordinates $z = f(r)$.

Iint: The unconstrained Lagrangian is found to be

$$= (m_1/2)\,(\dot{x}_1^2 + \dot{y}_1^2 - 2gy_1)] + (m_2/2)\,(\dot{x}_2^2 + \dot{y}_2^2 - 2gy_2).$$

.8 Applications to Mechanics

he applications of the Calculus of Variations in Mechanics
nploying principle of least action and Hamilton's principle. Th
e stated below.

8.1 Principle of least action

et a particle move in an external field of force which is conser
otion takes place in the interval of time from t_1 to t_2, where
e actual path traced by the particle is the one along which

$$I = \int_{t_1}^{t_2} L\,dt$$

minimum. where L is the Lagrangian and for a conservative s

$$L = \text{Kinetic Energy} - \text{Potential Energy} = T -$$

.8.2 Hamilton's Principle

ccording to this principle, the path of motion of a rigid body in the time terval $t_2 - t_1$ is such that the integral

$$A = \int_{t_1}^{t_2} L dt$$

as a stationary value where L is the Lagrangian.

.8.3 Illustrative Examples

Example 1

ind the equation of motion of a particle moving in a conservative field of orce described by the function $V(x, y, z)$.

Solution

Here

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

By the principle of least action, the equations of motion are given by

$$\delta I = 0 = \int_{t_1}^{t_2} \delta L \, dt$$

From this condition the following equations of motion follow

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0, \quad \frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0, \quad \frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0$$

The last three equations are called *Lagrangian equations of motion.*

Example 2 (Simple Harmonic Motion)

Use the principle of least action to obtain DE describing the vibrations of a simple harmonic oscillator.

Solution

By definition for a particle executing simple harmonic motion (S.H.M.) along the X - axis, the force F is given by $F = -kx$. where x denotes splacement from the centre of vibration (or the origin); the constant k w positive, is called the *spring constant.* The kinetic energy is given by

$$T = \frac{1}{2} m \dot{x}^2$$

nd since $F = -dV/dx$, the potential energy is given by

$$V = \int k\,x\,dx = \frac{1}{2} k\,x^2$$

herefore the Lagrangian L is given by

$$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k\,x^2 \equiv L(x,\ \dot{x})$$

herefore the Lagrangian equation of is given by

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

hich in this case reduces to

$$-kx - \frac{d}{dt}(m\dot{x}) = 0 \quad \text{or} \quad m\,\ddot{x} + k\,x = 0$$

xample 3

he Lagrangian L for a system of n particles is a function of generalis oordinates q_i and generalised velocities q_i'. Show that minimization of th tegral $I = \int_{t_1}^{t_2} L\,dt$ leads to the following equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

lution

re

$$L = L(q_1,\ q_2,\ \cdots,q_n,\ q_1',\ q_2',\ \cdots q_n')$$

ow

$$\delta I = \int_{t_1}^{t_2} \sum_{i=1}^{i=n} \left(\frac{\partial L}{\partial q_i}\,\delta q_i + \frac{\partial L}{\partial q_i'}\,\delta q_i'\right) dt$$

ow consider

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial q_i'}\,\delta q_i'\,dt = \delta q_i \ \frac{\partial L}{\ }\Big]^{t_2} \ \int^{t_2} \ \frac{d}{\ }\left(\frac{\partial L}{\ }\right) dt$$

where we have

So equation (1)

For the extrem

Since δq_i is arbi of calculus of v

$$= -\int_{t_1}^{t_2} \delta\, q_i \frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) dt$$

e we have used the fact that q_i are fixed at times t_1 and t_2.

quation (1) becomes

$$\delta I = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial q_i'}\right] \delta q_i \, dt$$

the extremal, $\delta I = 0$. Therefore we have

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}(\frac{\partial L}{\partial q_i'})\right] \delta q_i \, dt = 0$$

ce δq_i is arbitrary, therefore by making an appeal to the fundamental lemma alculus of variations, we have

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right) = 0, \quad i = 1, 2, \cdots$$

Example 4

Obtain the equation of motion of a stretched string by invoking Ha
principle of least action.

Solution

Let the string have uniform density ρ and length ℓ. Let it be initially st
along the X - axis, and when plucked let it vibrate in the $X\,Y$- plane. Co
the K.E. of an element ds of the string. Then

$$dT = \frac{1}{2}\,dm\,v^2 = \frac{1}{2}\,\rho\,ds\,y_t^2$$

and therefore

$$T = \frac{1}{2}\rho\int_0^\ell y_t^2\,ds$$

Here we have assumed that the deflection of the string is very small. Then
the above expression for kinetic energy can be approximated as

$$T = \frac{\rho}{2}\int_0^\ell y_t^2\,dx$$

Now the potential energy V of the string equals the increase in length du
tension of the string, which in turn equals the work done by the tensio
producing extension $ds - dx$.

Therefore we have

$$\begin{aligned} V &= \int_0^\ell (ds - dx)\,\tau \\ &= \tau\left\{\int_0^\ell [1 + \frac{1}{2}\,y_x^2]\,dx - \ell\right\} \\ &= \tau\left\{\ell + \frac{1}{2}\int_0^\ell \left(\frac{dy}{dx}\right)^2 dx - \ell\right\} \\ &= \frac{\tau}{2}\int_0^\ell \left(\frac{dy}{dx}\right)^2 dx \end{aligned}$$

Hence

$$\begin{aligned} L = T - V &= \frac{1}{2}\rho\int_0^\ell y_t^2\,dx - \frac{1}{2}\tau\int_0^\ell y_x^2\,dx \\ &= \frac{1}{2}\int_0^\ell (\rho\,y_t^2 - \tau\,y_x^2)\,dx \end{aligned}$$

By Hamilton's principle, the equation of motion of the string will correspond to stationary value of

$$\int_{t_1}^{t_2} L\, dt = \frac{1}{2}\int_{t_1}^{t_2}\int_0^{\ell} \left[\rho\, y_t^2 - \tau\, y_x^2\right]\, dx\, dt$$

Let $\mathcal{L} = \rho\, y_t^2 - \tau\, y_x^2$.

For the extremal path

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) = 0 \tag{1}$$

Now

$$\frac{\partial \mathcal{L}}{\partial y} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial y_x} = -2\tau\left(\frac{\partial y}{\partial x}\right)$$

The last relation gives

$$\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) = -2\tau\left(\frac{\partial^2 y}{\partial x^2}\right)$$

Also

$$\frac{\partial \mathcal{L}}{\partial y_t} = 2\rho\, y_t \implies \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) = 2\rho\, y_{tt}$$

Putting these values in (1), we obtain

$$\frac{\partial^2 y}{\partial x^2} - \frac{\rho}{\tau}\frac{\partial^2 y}{\partial t^2} = 0$$

Example 5

Discuss the vibrations of a stretched membrane and obtain its equation of motion by making use of Hamilton's principle.

Solution

We consider the transverse vibrations of a membrane of arbitrary shape which is supposed to be perfectly flexible but inextensible. Before applying Hamilton's principle we calculate kinetic and potential energies of the vibrating membrane.

We choose the coordinate axes so that originally the membrane lies in the XY- plane. As in case of the string, we suppose that displacements