$F'_y - (d/dx)\, F''_y =$ constant, if the integrand does not depend on y

and the first integral

$F - y'\left(F_{y'} - (d/dx)\, F_{y''} - y'' F_{y''}\right) =$ constant, if the integrand does not depen on x

14. Find the extremals of the functional $I[y] = \int_0^1 (y'^2 + x^2)\, dx$ subject t conditions $y(0) = 0,\ y(1) = 0$ and the side condition $\int_0^1 y^2 dx = 2$.

(Ans. $x/a + y/b = 1$ with $ab = 2A$).

(Hint $A = \int_0^a y dx$, Area $= \int_0^a 2\pi y\, ds \rightarrow$ minimum).

15. Find the curve that extremizes the functional $I[y] = \int_0^1 (360\, x\, y^2 - y''^2)$ subject to $y(0) = 0,\ y'(0) = 1,\ y(1) = 0,\ y'(1) = 5/2$.

9.4 Euler-Lagrange Equation for two and three Independent Variables

9.4.1 Euler-Lagrange equation for two independent variable

Let $F = F(x, y, u, u_x, u_y)$ where x and y are independent variables. In th case the problem is to determine the surface which extremizes the functiona

$$I[u(x,\ y)] = \int_R\!\!\int F(x,\ y,\ u,\ u_x,\ u_y)\, dx\, dy$$

The relevant theorem is given below.

Theorem

The extremal (surface) of the functional

$$I[u(x,\ y)] = \int_R\!\!\int F(x,\ y,\ u,\ u_x,\ u_y)\, dx\, dy$$

where u has different values inside the region R but is prescribed on the boun ary of the region R. Moreover u is supposed to possess continuous parti derivatives upto second order, is given by the PDE

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$$

Proof

$I[u(x, y)] = \int_R \int$

We consider the variation δI of I in (9.4.1) corresponding to variatio
Using the shortened procedure discussed above (subsection 9.2.3), w

$$\delta I = \int_R \int \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u_x} \delta u_x + \frac{\partial F}{\partial u_y} \delta u_y \right) dx\, dy$$

Now consider

$$\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \delta u \right) = \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) \delta u + \frac{\partial F}{\partial u_x} \delta u_x$$

Therefore

$$\frac{\partial F}{\partial u_x} \delta u_x = \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \delta u \right) - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) \delta u$$

Similarly

$$\frac{\partial F}{\partial u_y} \delta u_y = \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \delta u \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \delta u$$

Substituting in (9.4.2),

$$\begin{aligned} \delta I &= \int_R \int \left[\frac{\partial F}{\partial u} \delta u + \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \delta u \right) - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) \delta u \right. \\ &+ \left. \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \delta u \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \delta u \right] dx\, dy \end{aligned}$$

or on rearranging terms

$$\begin{aligned} \delta I &= \int_R \int \left[\frac{\partial F}{\partial u} \delta u - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) \delta u - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \delta u \right] dx\, dy \\ &+ \int_R \int \left[\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \delta u \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \delta u \right) \right] dx\, dy \\ &= I_1 + I_2 \end{aligned}$$

According to Green's theorem, (see appendix A)

$$\int_R \int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_C (P\, dx + Q\, dy)$$

Using this theorem, we can prove that the second integral above vanishes.
have

$$I_2 = \int_C \left(\frac{\partial F}{\partial u_x} dy - \frac{\partial F}{\partial u_y} dx \right) \delta u$$

$$\delta I = \int\int_V\int \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right] \delta u \, dx \, d$$

r the extremal $\delta I = 0$, which implies that

$$\int\int_V\int \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right] \delta u \, dx \, dy \, dz$$

ence by fundamental lemma of the calculus of variations,

$$\left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right] \delta u = 0$$

ince $\delta u \neq 0$ in the region, therefore

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) = 0$$

.4.3 Plateau's problem

To find the surface of minimal area which is bounded by a given closed This problem is called problem of *minimal surface* or *Plateau's* probl is named after the Belgian physicist Plateau (1801–1883) who was the study such problems systematically. To formulate this problem we not for a surface $z = z(x, y)$ which is described parametrically by the para (u, v), arc element of any curve on it is given by

$$(ds)^2 = E\,(du)^2 + 2F\,du\,dv + G\,(dv)^2$$

where E, F and G are fundamental quantities of the surface given by

$$E = \frac{\partial \mathbf{r}}{\partial u}\cdot\frac{\partial \mathbf{r}}{\partial u}, \quad F = \frac{\partial \mathbf{r}}{\partial u}\cdot\frac{\partial \mathbf{r}}{\partial v}, \quad G = \frac{\partial \mathbf{r}}{\partial v}\cdot\frac{\partial \mathbf{r}}{\partial v}$$

Since u, v are parametric coordinates for points on the surface $z = z$ therefore $x = x(u, v)$, $y = y(u, v)$.

We take (x, y) as parameters i.e. $u = x$, $v = y$, then

$$E = \left(\frac{\partial \mathbf{r}}{\partial x}\right)^2 = \left(\frac{\partial x}{\partial x}\mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k}\right)^2 = 1 + z_x^2$$

3. Find E-L equations for

$$I[u] = \iiint_R \int \frac{1}{2}(u_t^2 - u_x^2 - u_y^2 - u_z^2 - m^2u^2)\, dx\, dy\, dz\, dt, \quad m = \text{constant}$$

4. Obtain the PDE satisfied by the extremal surface $z = z(x, y)$ for the variational problems

(a) $\displaystyle\iint_D [(z_x)^2 - (z_y)^2]\, dx\, dy \to$ minimum
with $z = z_0$ on the boundary of the region D.

(b) $\displaystyle\iint_D [(z_{xx})^2 + (z_{yy})^2 + 2(z_{xy})^2 - 2z f(x, y)]\, dx\, dy \to$ minimum
with $z = z_0$ on the boundary of the region D.

5. Generalize the three dimensional problem discussed in section 9.5 tc n dimensions. Find E-L equation for the functional

$$I[u] = \int \cdots \int \sum_{i=1}^{n} u_{x_i}^2\, dx_1 dx_2 \cdots dx_n$$

6. Find the E-L equation for the functional

$$I[y]' = \iint_R \int \sqrt{1 + u_x^2 + u_y^2 + u_z^2}\, dx\, dy\, dz$$

7. Write the appropriate generalization of the E-L equation for the functional

$$I[u] = \iint_R F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy})\, dx\, dy$$

9.5 Constrained Extrema

These problems are also called *variational problems with constraints* or *variational problems with side conditions*. A subclass of these problems is called *isoperimetrical problems*.

There are certain variational problems in which we have to find stationary values of a functional of the form with constraints

$$G_j(x,\ y_1,\ \cdots y_n) = \text{constant}, \quad j = 1,\ 2, \cdots m$$

satisfy the Euler-Lagrange equations corresponding to the functional

$$\begin{aligned} H[y_1, y_2, \cdots, y_n] &= \int_{x_1}^{x_2} \left[F(x, y_1, y_2, \cdots, y_n) + \sum_{i=1}^{m} \lambda_i(x) G_i(x) \right] dx \\ &= \int_{x_1}^{x_2} H(x,\ y_1, \cdots y_n)\ dx \end{aligned}$$

where $\lambda_i(x)$ are suitably chosen multipliers.

It is clear that the Euler-Lagrange equation in this case will be

$$\frac{\partial H}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial H}{\partial y_i'}\right) = 0, \quad i = 1,\ 2, \quad \cdots, n \tag{9.7.1}$$

The curves $y_i = y_i(x),\ i = 1, 2, \cdots, n$ be determined from equations(9.7.1) and the equations of the constraints, *viz.*

$$G_j(x,\ y_1,\ y_2, \quad \cdots, y_n) = 0, \quad j = 1,\ 2, \ \cdots, m$$

9.5.3 More general variational problem with constraints

In this case we have to find the extremal curves $y = y(x)$ which extremizes

$I[y] = \int_{x_1}^{x_2} F(x,\ y,\ y')dx$ with endpoint conditions $y(x_1) = y_1,\ y(x_2) = y_2$ subject to

$J[y] = \int_{x_1}^{x_2} G(x,\ y,\ y')dx = \text{constant}$

We assume (as in the case without constraint) that F and G have continuous second order derivatives w.r.t. their arguments; similarly y is supposed to have second order continuous derivative. We consider a 2-parameter family of curves represented by

$y(x,\ \epsilon_1,\ \epsilon_2) = y(x) + \epsilon_1\eta_1(x) + \epsilon_2\eta_2(x)$

The end point conditions on the curves $y(x,\ \epsilon_1,\ \epsilon_2)$ and $y(x)$ require that

$\eta_1(x_i) = 0, \quad \eta_2(x_i) = 0, \quad i = 1, 2$

Because of dependence of y on ϵ_1 and ϵ_2, we have

$$I(\epsilon_1,\ \epsilon_2) = \int_{x_1}^{x_2} F(x,\ y + \epsilon_1\eta_1 + \epsilon_2\eta_2,\ y' + \epsilon_1\eta_1' + \epsilon_2\eta_2')\ dx$$

$$\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} G(x, y+\epsilon_1\eta_1+\epsilon_2\eta_2, y'+\epsilon_1\eta_1'+\epsilon_2\eta_2')\,dx = \text{constant}$$

e vary ϵ_1, and ϵ_2, the function $I(\epsilon_1, \epsilon_2)$ takes different values but change and ϵ_2 conspire to keep $J(\epsilon_1, \epsilon_2)$ to the constant value of k. (This wi be possible in case of a single parameter).

suppose that the stationary value of $I(\epsilon_1, \epsilon_2)$ corresponds to $\epsilon_1 = \epsilon_2 = 0$. ce we must have

$$\left.\frac{\partial I}{\partial \epsilon_1}\right|_{\epsilon_1=\epsilon_2=0} = 0 = \left.\frac{\partial I}{\partial \epsilon_2}\right|_{\epsilon_1=\epsilon_2=0}$$

$$J(\epsilon_1 = 0, \epsilon_2 = 0) = k$$

s is equivalent to a problem in the calculus of constrained extrema in which have to extremize the function

$, \epsilon_2)$ subject to $J(\epsilon_1, \epsilon_2) = k$.

ce the solution corresponds to $\epsilon_1 = \epsilon_2 = 0$, we must have

$$\left.\left(\frac{\partial I}{\partial \epsilon_1} + \lambda\frac{\partial J}{\partial \epsilon_1}\right)\right|_{\epsilon_1=\epsilon_2=0} = 0, \quad \left.\left(\frac{\partial I}{\partial \epsilon_2} + \lambda\frac{\partial J}{\partial \epsilon_2}\right)\right|_{\epsilon_1=\epsilon_2=0} = 0$$

d

$$(J-k)|_{\epsilon_1=\epsilon_2=0} = 0$$

e first equation is equivalent to

$$\frac{\partial}{\partial \epsilon_1}\int_{x_1}^{x_2} [F(x, y+\epsilon_1\eta_1+\epsilon_2\,\eta_2, y'+\epsilon_2\eta_1'+\epsilon_2\,\eta_2')$$
$$+ \lambda G(x, y+\epsilon_1 eta_1+\epsilon_2\eta_2, y'+\epsilon_1\eta_1'+\epsilon_2\eta_2')]\,dx\,|_{\epsilon_1=0=\epsilon_2} = 0$$

$$\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y}\eta_1 + \frac{\partial F}{\partial y'}\eta_1' + \lambda\left(\frac{\partial G}{\partial y}\eta_1 + \frac{\partial G}{\partial y'}\eta_1'\right)\right] dx = 0$$

integrating the second and the fourth terms by parts, and using the end-int conditions $\eta_1(x_1) = 0 = \eta_1(x_2)$, we obtain

$$\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \lambda\left(\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial y'}\right)\right]\eta_1(x)\,dx = 0$$

milarly we have

$$\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \lambda\left(\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial y'}\right)\right]\eta_2(x)\,dx = 0$$

$$\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) \neq 0$$

general, (because the functional J is not an extremum for $\epsilon_1 = \epsilon_2 = 0$
e can choose $\eta_1(x)$ such that

$$\int_{x_1}^{x_2} \left(\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial y'}\right) \eta_1(x)\, dx \neq 0$$

hich is always possible when

$$\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) \neq 0$$

his relation can be used to define λ. Using this value of λ in equation (2), e obtain

$$\int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right) + \lambda\left(\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial y'}\right)\right] \eta_2(x)\, dx = 0$$

where $\eta_2(x)$ is arbitrary function which vanishes at the end-points. Invoking the fundamental theorem of the calculus of variations, we have the necessary condition

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \lambda\left[\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)\right] = 0$$

or

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\frac{\partial H}{\partial y'} = 0$$

which is the Euler-Lagrange equation for $H = F + \lambda G$, with the end-point conditions $y(x_1) = y_1, \quad y(x_2) = y_2$.

In actual calculations, λ is determined from the side condition
$\int_{x_1}^{x_2} G(x,\, y,\, y')\, dx = k$, a constant.

9.5.4 Illustrative examples

The following solved examples illustrate the variational problems with constraints.

Example 1 (Dido's problem)
Find the closed curve of given length which encloses maximum area.
Solution

Geodesic Problems

odesic is the curve of shortest length joining two points in space. Geodesic lems are similar to variational problems with constraints but sometimes be reduced to problems without constraint(s).

1 General geodesic problem

eodesic connects two points in space or on a surface. Here we are concerned n geodesics on surfaces. The geodesic problem for a surface can be stated ollows:

ven points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ on a surface $z = z(x, y)$ find the arc of shortest length connecting A and B.

re we have to minimize

$$\ell = \int_A^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

$$= \int_{x_1}^{x_2} \sqrt{1 + y'^2 + (z_x + z_y\, y')^2}\, dx$$

here we have used the result

$$z = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy = (z_x + z_y y')\, dx$$

ubject to the conditions

$(x_1) = y_1,\ y(x_2) = y_2,\ z(x_1) = z_1,\ z(x_2) = z_2$

.6.2 Illustrative examples

We illustrate some well-known geodesic problems with examples.

Example 1

Find the curve of shortest length between two given points in a plane, using polar coordinates (r, θ).

Solution

The arc length ds is given by

$$\ldots 2\,d\theta^2 = \sqrt{r'^2 + r^2}\, d\theta, \quad r' = \frac{dr}{d\theta}$$

…us we have to minimize $\int_A^B ds = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + r'^2}\, d\theta$ subject to $r(\theta_1) =$ …stant, and $r(\theta_2) =$ constant.

…e $F(\theta, r, r') = \sqrt{r^2 + r'^2}$

…e there is no explicit dependence on the independent variable θ, we use …first integral of the E-L equation, (Beltrami's identity): $F - r'\,(\partial F/\partial r') =$ …tant, which becomes

$$\sqrt{r^2 + r'^2} - r' \frac{r'}{\sqrt{r^2 + r'^2}} = c_1, \quad \text{a constant}$$

…h on simplification reduces to $r^2 = c_1 \sqrt{r^2 + r'^2}$.

…last equation gives

$$r'^2 = \frac{r^4 - c_1^2 r^2}{c_1^2} \quad \text{or} \quad \frac{dr}{d\theta} = \frac{r}{c_1} \sqrt{r^2 - c_1^2}$$

$$\int \frac{\pm c_1\, dr}{r\sqrt{r^2 - c_1^2}} = \theta + c_2 \quad \Rightarrow \quad \sec^{-1} \frac{r}{c_1} = \theta + c_2$$

…efrom

$$\begin{aligned} c_1 &= r\cos(\theta + c_2) = r(\cos\theta \cos c_2 - \sin\theta \sin c_2) \\ &= (\cos c_2)\, x - (\sin c_2)\, y \end{aligned}$$

…h is a linear equation in x, y of the form $\alpha x + \beta y + \gamma = 0$ and represents …ight line.

…mple 2

…d the curve of shortest length (geodesic) on the surface of a sphere.

…ion

… and B be two points on the sphere S. Here the problem is to minimize …egral

$\int^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

…t to the constraint $x^2 + y^2 + z^2 = a^2$ where a is the radius of the sphere. …roblem can also be solved by using spherical polar coordinates (r, θ, ϕ). … θ, ϕ) and $(a, \theta + d\theta, \phi + d\phi)$ be the coordinates of two neighbouring …on the curve through A and B and lying on the sphere. Then the above …n is equivalent to minimizing the integral

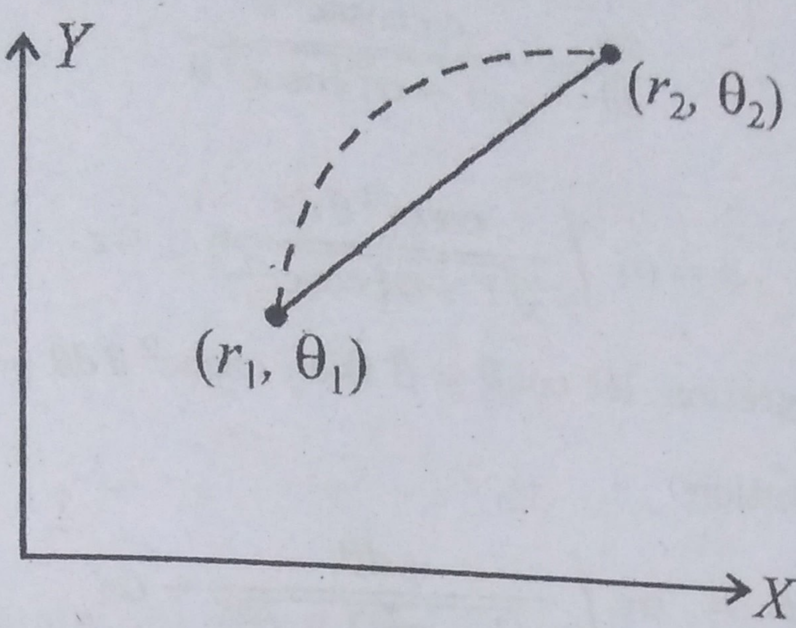

Figure 9.7: Points A and B on the surface of a sphere and the curve of shortest distance between them.

In case of a sphere $r = a,\ dr = 0$. Therefore the problem reduces to

$$\int_A^B \sqrt{a^2\,(d\theta)^2 + a^2\sin^2\theta\,(d\phi)^2} \ \rightarrow\ \min$$

or

$$a\int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\ d\theta \ \rightarrow\ \min$$

Here

$$F \equiv F(\theta; r; r') = \sqrt{1+\sin^2\theta\,\phi'^2},\quad \phi' = \frac{d\phi}{d\theta}$$

The required curve satisfies the E-L equation, (with $\phi' = d\phi/d\theta$)

$$\frac{\partial F}{\partial \phi} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \phi'}\right) = 0$$

or

$$0 - \frac{d}{d\theta}\left[\frac{1}{2}(1+\sin^2\theta\phi'^2)^{-1/2}\ 2\sin^2\theta\ \phi'\right] = 0$$

which gives

$$\frac{\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\phi'^2}} = \alpha_1 \ \text{ or }\ \sin^4\theta\phi'^2 = \alpha_1^2(1+\sin^2\theta\,\phi'^2)$$

or

$$\phi' = \frac{\alpha_1}{\sin\theta_1\sqrt{\sin^2\theta - \alpha_1^2}} \ \text{ or }\ \phi' = \frac{\alpha_1}{\sin^2\theta\sqrt{1-\alpha_1^2\,\mathrm{cosec}^2\,\theta}}$$

$$\frac{d\phi}{d\theta} = \frac{\alpha_1 \operatorname{cosec}^2 \theta}{\sqrt{1 - \alpha_1^2 \operatorname{cosec}^2 \theta}}$$

herefore

$$\phi = \alpha_1 \int \frac{\operatorname{cosec}^2 \theta \, d\theta}{\sqrt{1 - \alpha_1^2 \operatorname{cosec}^2 \theta}} + \alpha_2$$

perform the integration, let $\cot \theta = \beta$ then $\operatorname{cosec}^2 \theta \, d\theta = -d\beta$.

erefore on substitution

$$\begin{aligned}
\phi &= \alpha_1 \int \frac{-d\beta}{\sqrt{1 - \alpha_1^2 (1 + \beta^2)}} + \alpha_2 \\
&= -\int \frac{\alpha_1 d\beta}{\sqrt{(1 - \alpha_1^2) - \alpha_1^2 \beta^2}} + \alpha_2 \\
&= -\int \frac{d\beta}{\sqrt{(1 - \alpha_1^2)/\alpha_1^2 - \beta^2}} + \alpha_2
\end{aligned}$$

ontinuing further

$$\begin{aligned}
\phi &= -\int \frac{d\beta}{\sqrt{\alpha_3^2 - \beta^2}} + \alpha_2, \quad \alpha_3^2 = (1 - \alpha_1^2)/\alpha_1^2 \\
&= \cos^{-1} \frac{\beta}{\alpha_3} + \alpha_2 = \cos^{-1} \frac{\cot \theta}{\alpha_3} + \alpha_2
\end{aligned}$$

$$\frac{\cot \theta}{\alpha_3} = \cos(\phi - \alpha_2) = \cos \alpha_2 \cos \phi + \sin \alpha_2 \sin \phi$$

n can also be written as

$$\cot \theta = \frac{\cos \theta}{\sin \theta} = \gamma_1 \cos \phi + \gamma_2 \sin \phi$$

γ_1 and γ_2 are new constants.

ally

$$a \cos \theta = a \gamma_1 \sin \theta \cos \phi + a \gamma_2 \sin \theta \sin \phi$$

ssing to the Cartesian coordinates $z = \gamma_1 x + \gamma_2 y$ which is the equation plane through the centre of the sphere. Hence the curve of shortest joining A and B is the arc of the great circle through A and B.

nple 3

he geodesic curve for the cylinder $x^2 + y^2 = a^2$.

ion

In cylin

We have

subject t

Here $F =$

which in

whose sol

On passin

The inter extremal c

Example

Find the s $B(2, 1,$ –

Solution

Here we ha

ℓ

subject to

ylindrical coordinates $x = a\cos\theta,\ y = a\sin\theta,\ z = z$.

have to minimize

$$
\begin{aligned}
l &= \int_A^B ds = \int_A^B \sqrt{a^2\,(d\theta)^2 + dz^2} \\
&= \int_{\theta_1}^{\theta_2} \sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta \\
&= \int_{\theta_1}^{\theta_2} \sqrt{a^2 + z'^2}\, d\theta, \quad z' = \frac{dz}{d\theta}
\end{aligned}
$$

ect to no constraint, (where θ_1 and θ_2 correspond to A and B).

$F = \sqrt{a^2 + z'^2}$. The E-L equation in this case is

$$\frac{\partial F}{\partial z} - \frac{d}{d\theta}\frac{\partial F}{\partial z'} = 0$$

ch in this case reduces to

$$0 - \frac{d}{d\theta}(2z') = 0 \ \text{ or } \ \frac{d^2 z}{d\theta^2} = 0$$

se solution is given by $z = \alpha_1 + \alpha_2\theta$.

passing to Cartesian coordinates

$$z = \alpha_1 + \alpha_2 \tan^{-1}\frac{y}{x} \quad \text{or} \quad \tan\left(\frac{z - \alpha_1}{\alpha_2}\right) = \frac{y}{x}$$

intersection of this surface with the given cylinder gives the required emal curve.

mple 4

the shortest distance between the points $A(1,\ -1,\ 0)$ and 1, -1) in the plane $15x - 7y + z - 22 = 0$.

ution

we have to minimize

$$\ell = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \int_{x_1=1}^{x_2=2} \sqrt{1 + y'^2 + z'^2}\, dx$$

ect to the constraint that the points A, B be on the plane

$$G \equiv 15x - 7y + z - 22 = 0$$

ere the auxiliary function is given by

$$H = F + \lambda G = \sqrt{1 + y'^2 + z'^2} + \lambda\,(15x - 7y + z - 22)$$

e corresponding E-L equations are

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \text{ and } \frac{\partial H}{\partial z} - \frac{d}{dx}\left(\frac{\partial H}{\partial z'}\right) = 0$$

$$-7\lambda - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0 \qquad (1)$$

d

$$\lambda - \frac{d}{dx}\left(\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0 \qquad (2)$$

e have to solve (1) and (2) using the condition

$$15x - 7y + z - 22 = 0 \qquad (3)$$

e endpoint conditions satisfied by the functions $y = y(x)$ and $z = z(x)$ are

$$y(1) = -1, \quad y(2) = 1, \quad z(1) = 0, \quad z(2) = -1 \qquad (4)$$

om (1) and (2), by eliminating λ

$$-\frac{d}{dx}\left[\frac{y' + 7z'}{\sqrt{1 + y'^2 + z'^2}}\right] = 0$$

ch gives

$$\frac{y' + 7z'}{\sqrt{1 + y'^2 + z'^2}} = c_1 \qquad (5)$$

m (3)

$$z' = 7y' - 15 \qquad (6)$$

stituting for z' from (6) into (5), we obtain

$$\frac{y' + 7(7y' - 15)}{\sqrt{1 + y'^2 + (7y' - 15)^2}} = c_1$$

$$50y' - 105 = c_1\left[1 + y'^2 + 49y'^2 - 210y' + 225\right]^{1/2}$$

$$25(10y' - 21)^2 = c_1^2\left[50y'^2 - 210y' + 226\right]$$

$$25(100y'^2 - 420y' + 441) = c_1^2(50y'^2 - 210y' + 226)$$