$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

$= y(x)$ is a continuous function having continuous first and second vatives satisfying the following endpoint conditions.

$$y(x_1) = y_1, \quad y(x_2) = y_2$$

pposed to have continuous first and second order derivatives w.r.t. ents, then the function $y(x)$ will extremise the given integral if it he differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

proof of this theorem involves a result known as fundamental theorem lculus of variations. We discuss this theorem in the next subsection.

Fundamental theorem of the calculus of variations

nent: (one independent variable)

s continuous in the interval (x_1, x_2) and the integral $\int_{x_1}^{x_2} f(x)\, g(x)\, dx$ cally zero i.e. $\int_{x_1}^{x_2} f(x)\, g(x)\, dx \equiv 0$, where $g(x)$ satisfies the conditions

it is an arbitrary function with continuous derivatives in the interval .

$1) = g(x_2) = 0$

$x) \equiv 0$ for all $x \in [x_1, x_2]$.

ve by contradiction. If possible let $f(x) \neq 0$ in (x_1, x_2). Then there ast one point x_0 in (x_1, x_2) such that $f(x_0) \neq 0$. Then because of ity of $f(x)$ in (x_1, x_2) there must exist an interval $(x_0 - \delta, x_0 + \delta)$, $(\delta > 0)$ surrounding x_0 such that $f(x) > 0$ for all $x \in [x_0 - \epsilon, x_0 + \epsilon]$. (x) is arbitrary it can be taken as

$$g(x) = \begin{cases} (x - x_0 + \delta)^2 (x - x_0 - \delta)^2, & x_0 - \delta \le x \le x_0 + \delta \\ 0, & \text{otherwise} \end{cases}$$

lear that $g(x)$ vanishes at the endpoints of the interval $(x_0 - \delta, x_0 + \delta)$ as continuous derivative inside the interval

The integral $\int_{x_1}^{x_2} f(x)\, g(x)\, dx$ then becomes

$$\int_{x_0-\delta}^{x_0+\delta} f(x)(x - x_0 + \delta)^2\, (x - x_0 - \delta)^2\, dx > 0$$

This contradicts the assumption that

$$\int_{x_1}^{x_2} f(x)\, g(x)\, dx = 0$$

Hence

$$f(x) \equiv 0 \;\; \forall x \in (x_1,\, x_2)$$

Fundamental theorem of calculus of variations for two independen variables

Theorem

Let a function $f(x,\, y)$ be continuous in a region D of the XY-plane, an $g(x,\, y)$ be an arbitrary function with continuous partial derivatives in D, an let $g(x,\, y)$ vanish on the boundary curve C of the domain D.

If

$$\iint_D f(x,\, y)\, g(x,\, y)\, dx\, dy = 0$$

then $f(x,\, y) \equiv 0$ for all $(x,\, y)$ in the domain D.

Proof

If possible, let $f(x,\, y) \neq 0$ in D. Then there is at least one point $(x_0,$ of the region D such that $f(x_0,\, y_0) \neq 0$. Without loss of generality we t $f(x_0,\, y_0) > 0$. Since $f(x,\, y)$ is continuous, there exists a circular dom centred at $(x_0,\, y_0)$ and with radius $\epsilon > 0$ i.e. $C_0 :\; (x - x_0)^2 + (y - y_0)$ ϵ^2 such that $f(x,\, y) > 0$ in this domain. Now since $g(x,\, y)$ is arbitrary continuous, we can choose it such that

$$g(x,\, y) = \begin{cases} k\{(x - x_0)^2 + (y - y_0)^2\}, & (x,\, y) \in D,\;\; k > 0 \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\iint_D f(x,\, y)\, g(x,\, y)\, dx\, dy = \iint_D f(x,\, y) \times k\, \left[(x - x_0)^2 + (y - y_0)^2\right]\, dx\, dy > 0$$

which is a contradiction. Hence the theorem.

$$= \frac{\partial y'}{\partial \alpha}(x, 0) + \frac{\partial}{\partial \alpha}\alpha\eta'(x) = \eta'(x)$$

erefore

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2}\left[\eta(x)\frac{\partial F}{\partial y} + \eta'(x)\frac{\partial F}{\partial y'}\right] dx \qquad (9.2.3)$$

xt we consider

$$\int_{x_1}^{x_2} \eta'(x)\frac{\partial F}{\partial y'} dx = \eta(x)\frac{\partial F}{\partial y'}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$
$$= -\int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

erefore

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \eta(x)\frac{\partial F}{\partial y} dx - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$
$$= \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'})\right]\eta(x)\, dx$$

r the extreme value, $(\partial I/\partial \alpha) = 0$. This condition gives

$$\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)\, dx = 0$$

ow $\eta(x)$ is an arbitrary function of x which vanishes at the endpoints of the nterval $[x_1, x_2]$ and the expression within square brackets is a continuous unction of x. Therefore by invoking the fundamental theorem of the calculus f variations, we obtain

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad (9.2.4)$$

E.L.q.

The differential equation (9.2.4) is known as the *Euler-Lagrange equation.*

It is easy to show that Euler-Lagrange equation is a second order ODE in . Since $F = F(x, y, y')$, it follows that $\partial F/\partial y$ and $\partial F/\partial y'$ are also functions f x, y, y'. Therefore by chain rule

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial^2 F}{\partial x\partial y'}\frac{dx}{dx} + \frac{\partial^2 F}{\partial y\partial y'}\frac{dy}{dx} + \frac{\partial^2 F}{\partial y'^2}\frac{dy'}{dx}$$
$$= F_{xy'} + F_{yy'}\, y' + F_{y'y'}\, y''$$

Hence on substitution in the Euler-Lagrange equation

$$F_{xy'} + F_{yy'}\, y' + F_{y'y'}\, y'' = F_y$$

which is a second order ODE.

...pendent of y then from (9.2.4)

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \text{ which imlpies } \frac{\partial F}{\partial y'} = \text{constant}$$

...result is also called Beltrami's identity, after the Italian mathema... ...enio Beltrami (1835–1900) who first derived it.

If F is not explicitly dependent on x i.e. F is independent of x, ...$/\partial x) = 0$ Also the Euler-Lagrange equation in this case becomes

$$\begin{aligned}\frac{\partial F}{\partial y} &= \frac{d}{dx}\frac{\partial F}{\partial y'} \\ &= \frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} = y'\frac{d}{dy}\left(\frac{\partial F}{\partial y'}\right)\end{aligned}$$

...erefore

$$\left(\frac{\partial F}{\partial y}\right) dy = y'\, d\left(\frac{\partial F}{\partial y'}\right)$$

...n taking the total differential of $F = F(y, y')$ and using the above re... ...e have

$$\begin{aligned}dF &= \frac{\partial F}{\partial y}\, dy + \frac{\partial F}{\partial y'}\, dy' \\ &= y'\, d\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}\, dy' \\ &= d\left(y'\frac{\partial F}{\partial y'}\right)\end{aligned}$$

...ence in this case the Euler-Lagrange equation takes the simpler form

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}$$

which in fact is its first integral.

9.2.3 Short hand procedure for obtaining variation of the fun... tional

We know that for a function $y = f(x)$, the increment (or variation) δy given by $\delta y \approx f'(x)\,\delta x$. Therefore from (9.2.3) we can obtain $\delta I \approx (dI/d\alpha)\,\delta$... where $\delta\,\alpha$ denotes an increment in the parameter α and will correspond to... neighbouring curve.

Using this result we obtain from (9.2.3)

$$\frac{\partial I}{\partial \alpha}\delta\alpha \equiv \delta I = \int_{x_1}^{x_2}\left[\eta(x)\frac{\partial F}{\partial y} + \eta'(x)\frac{\partial F}{\partial y'}\right]\delta\alpha\, dx \tag{9.2.5}$$

Now we use relations (9.2.1) and (9.2.2) and obtain

$$y(x,\ \alpha) - y(x,\ 0) \equiv \delta y \approx \delta\alpha\,\eta(x)$$

$$y'(x,\ \alpha) - y'(x,\ 0) \equiv \delta y' \approx \delta\alpha\,\eta'(x)$$

In view of these relations, (9.2.5) can be written as

$$\delta I = \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right]dx \tag{9.2.6}$$

The procedure sketched above for obtaining the variation δI can be regarded as a short-hand form of the method described earlier.

9.2.4 Illustrative examples

Application of the simplest form of the Euler-Lagrange equation i.e. in the case in which the functional has the form $I[y] = \int_{x_1}^{x_2} F(x, y, y')dx$ is illustrated by examples.

Example 1

Use calculus of variations to prove that a straight line is the curve with shortest distance between two points in a plane.

Solution

The element of length along a curve $y = y(x)$ is given by $ds = \sqrt{x^2 + y^2}$ or $ds = \sqrt{1 + y'^2}\,dx$. Therefore we have to minimize $I = \int_a^b \sqrt{1 + y'^2}\,dx$ subject to the endpoint conditions $y(a) = y_0$ and $y(b) = y_1$.

Here $F = \sqrt{1 + y'^2}$. I will be minimum if $y(x)$ satisfies the Euler-Lagrange equation (9.2.4). On substituting for $\partial F/\partial y$ in (9.2.4), we obtain

$$\frac{\partial F}{\partial y'} = c \implies \frac{y'}{\sqrt{1 + y'^2}} = c$$

which on simplification gives

$$y'^2 = \frac{c^2}{1 - c^2} = a^2, \text{ say}$$

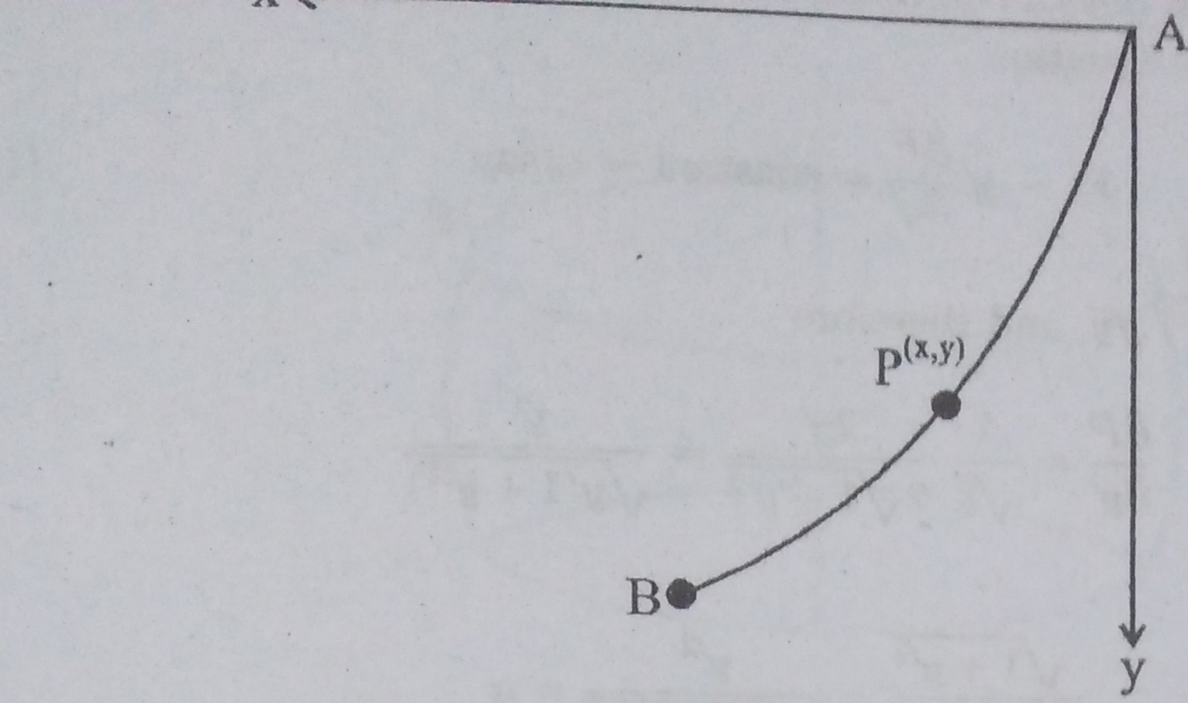

Figure 9.2: Particle falling under gravity from one point to another, not lying on the vertical.

Example 3

Find the equation of the path in space down which a particle will fall from one point to another in the shortest possible time.

Solution

This problem is called *brachistochrone problem* and is one of the earliest problems of the calculus of variations. It was first proposed by John Bernoulli in 1696 and solved by himself, his elder brother Jacob (James), Newton, Leibniz and de l'Hôpital.

Let a particle fall from a point A to another point B. There are infinite number of paths between A and B, but we are to consider that path only along which the time taken is minimum.

We choose coordinate axes as shown in figure 9.2. Let (x, y) be the position of the particle at time it. If ds denotes the arc element of the curve $y = y(x)$, then the total time taken by the particle in falling from A to B is given by

$$\tau = \int_A^B dt \quad = \int \frac{dt}{ds}\, ds = \int \frac{ds}{v}$$

where the time increment dt is related to the arc element ds by $v = ds/dt$.

Therefore

$$\tau = \int_A^B \frac{ds}{v} = \int_A^B \frac{ds}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}} \int_A^B \frac{ds}{\sqrt{y}} = \frac{1}{\sqrt{2g}} \int_A^B \frac{\sqrt{1+y'^2}}{\sqrt{y}}\, dx$$

$v^2 - u^2 = 2gy \Rightarrow v^2 = 2gy \Rightarrow v = \sqrt{2gy}$

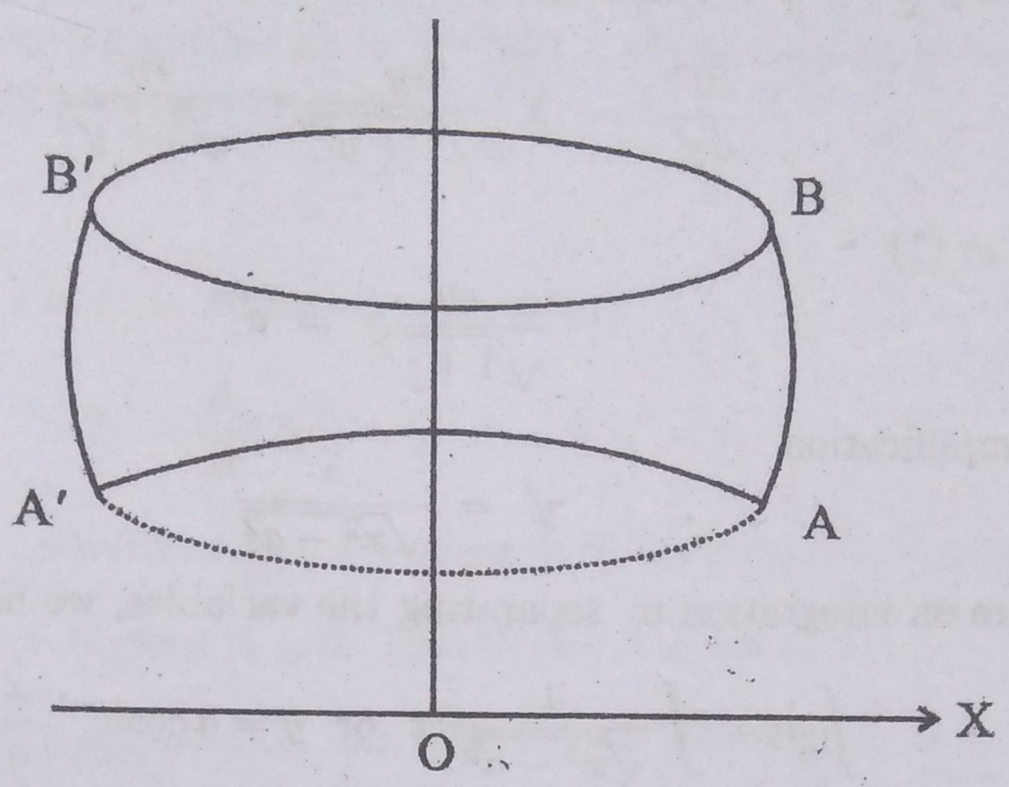

Figure 9.8: Curve connecting two fixed points in space A and B, whose surface of revolution is also shown.

Example 4

Find the curve joining the points $(x_1,\ y_1)$ and $(x_2,\ y_2)$ which gives minimum area of the surface of revolution generated around (i) y-axis, (ii) x-axis.

Solution

Let $A(x_1,\ y_1)$ and $B(x_2,\ y_2)$ be two points in xy- plane. We want to find a curve which gives the minimum area of the surface of revolution.

(i) Let us consider the case when the curve revolves about the y - axis. In this case, area of the surface of revolution will be given by

$$\text{Area} = \int_A^B 2\pi\, x\, ds$$

$$= 2\pi \int_A^B x\, ds = 2\pi \int_A^B x\, \sqrt{1+y'^2}\, dx$$

For the minimum value it must satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

which in this case is equivalent to (because F is independent of y)

$$\frac{\partial F}{\partial y'} = \text{constant}, \quad a, \text{ say} \qquad (1)$$

Here $F = x\sqrt{1+y'^2}$. Therefore

$$\frac{\partial F}{\partial y'} = x\frac{2y}{2\sqrt{1+y'^2}} = \frac{xy'}{\sqrt{1+y'^2}}$$

Putting in (1)

$$\frac{xy'}{\sqrt{1+y'^2}} = a$$

or on simplification

$$y' = \frac{a}{\sqrt{x^2-a^2}}$$

Therefore on integration by separating the variables, we have

$$\int dy = \int \frac{a}{\sqrt{x^2-a^2}}dx \quad \text{or} \quad y = a\cosh^{-1}\frac{x}{a} + c$$

(ii)

$$\text{Area} = \int_A^B 2\pi\, y\, ds = 2\pi\int_A^B y\sqrt{1+y'^2}\, dx$$

Since we want a curve which gives minimum area of the surface of revolution generated about the x - axis, so it must satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

which (because of no explicit dependence on x) is equivalent to

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}$$

In this case $F = y\sqrt{1+y'^2}$. Therefore

$$\frac{\partial F}{\partial y'} = \frac{yy'}{\sqrt{1+y'^2}}$$

The Euler-Lagrange equation becomes

$$y\sqrt{1+y'^2} - \frac{yy'}{\sqrt{1+y'^2}} = a, \text{ say}$$

or on simplification

$$\frac{dy}{dx} = \frac{\sqrt{y^2-a^2}}{a}$$

On integration

$$\int dx = \int \frac{a}{\sqrt{y^2-a^2}}dy \quad \text{or} \quad x = a\cosh^{-1}\frac{y}{a} + c$$

Example 5

On what curves can the functional $I = \int_0^{\pi/2}(y'^2 - y^2)\,dx$ with endpoint conditions $y(0) = 0,\ y(\pi/2) = 1$ be extremized.

Solution

Here $F = y'^2 - y^2$. The E.L. equation is given by

$$-2y - \frac{d}{dx}(2y') = 0 \text{ or } y + y'' = 0$$

whose solution can be written as $y = A\cos x + B\sin x$.

Now the B.C. $y(0) = 0$ gives $A = 0$. Therefore $y = B\sin x$.

Next we apply the second B.C. *viz.* $y(\pi/2) = 1$, which gives $B = 1$.

Hence the required extremal is $y = \sin x$.

9.2.5 Exercises

1. From among the curves connecting the points $A(1,\ 3)$ and $B(2,\ 5)$ find the extremal curve of the functional

$$I[y] = \int_1^2 y'(x)\left(1 + x^2 y'(x)\right)\,dx$$

Ans. $y = 7 - 4/x$).

Find the extremals of the problem

$$I[y] = \int_0^1 (y'^2 + 3y + 2x)\,dx,\ y(0) = 0,\ y(1) = 1$$

Ans. $y = (3/4)\,x^2 + (1/4)x$).

Find the extremals $y = y(x)$ subject to the given conditions and satisfying the given functionals and endpoint conditions.

(a) $I[y] = \int_0^1 xyy'\,dx,\quad y(0) = 0,\ y(1) = 1$

(b) $I[y] = \int_1^2 \frac{\sqrt{1+y'^2}}{x}\,dx,\ y(1) = 0,\ y(2) = 1$

(c) $I[y] = \int_{x_1}^{x_2} (y'^2 + y^2)\,dx,\ y(x_1) = y_1,\ y(x_2) = y_2$

4. Find the extremals for the functionals defined below where $y(x)$ is supposed to be constant at the end points.

$$\text{(a)} \int_a^b \left(\frac{y'^2}{x^3}\right) dx, \quad \text{(b)} \int_a^b (y^2 + y'^2 + 2ye^x)\, dx$$

5. Show that the E-L equation for the functional

$$I[y] = \int_{x_1}^{x_2} F(x,\, y)\, \sqrt{1+y'^2}\, dx$$

has the form

$$F_y - F_{xy'} - \frac{y''}{1+y'^2} F = 0$$

. Find extremals of the following functional subject to the given conditions: $I[y] = \int_0^1 (e^y + xy')\, dx$, $y(0) = 0$, $y(1) = a$ (**Ans.:** $y = 0$ if $a = 0$. For $a \neq 0$ there is no extremal).

. Find extremals of the following functional subject to the given conditions:

) $I[y] = \int_0^\pi (y'^2 - y^2)\, dx$, $y(0) = 1$, $y(\pi/4) = (\sqrt{2}/2)$

Ans.: $y = \cos x$).

) $I[y] = \int_0^\pi (y'^2 - y^2)\, dx$, $y(0) = 1$, $y(\pi) = -1$

Ans.: $y = \cos x + C \sin x$, where C is an arbitrary constant).

) $I[y] = \int_0^1 (x + y'^2)\, dx$, $y(0) = 1$, $y(1) = 2$

ns.: $y = x + 1$).

) $I[y] = \int_0^1 (y^2 + y'^2)\, dx$, $y(0) = 0$, $y(1) = 1$

ns.: $y = \sinh x / \sinh 1$).

$I[y] = \int_0^1 (y'^2 + 4y^2)\, dx$,

$) = e^2$, $y(1) = 1$

ns.: $y = e^2 e^{-2x}$).

$I[y] = \int_0^{\pi/2} (y^2 - y'^2 - 8y \cosh x)\, dx$, $y(0) = 2$, $y(\pi/2) = 2\cosh(\pi/2)$

ns.: $y = 2\cosh x$).

(h) $\int_{x_1}^{x_2}(y^2 + y'^2 + 2y\exp(x)\,dx \to$ stationary, $y(x_1) = y_1$, $y(x_2) = y_2$

(Ans. (a) $y = c_1 + c_2x^4$, (b) $y = (1/2)xe^x + c_1e^x + c_2\exp(-x)$).

8. Find the general solution of the E-L equation corresponding to the problem

$$\int_{x_1}^{x_2} f(x)\sqrt{1+y'^2}\,dx,\quad y(x_1) = y_1,\quad y(x_2) = y_2$$

and investigate the special cases $f(x) = \sqrt{x}$ and $f(x) = x$.

9. Among all curves of length ℓ in the upper half-plane passing through the points $(-a,\ 0)$ and $(a,\ 0)$, find the one which together with the interval $[-a,\ a]$ encloses the largest area.

10. Find the curve joining the points (0, 0) and (1, 0) for which the integral $\int_0^1 y''^2\,dx$ is minimum if

(a) $y'(0) = 0$, $y'(1) = b$, (b) No other conditions are prescribed.

11. Find the equilibrium position of a heavy flexible inextensible cord of given length, fastened at its end points.

12. Among all curves joining a given point $(0,\ b)$ on X-axis to a point on the X-axis and enclosing a given area A together with X-axis, find the curve which generates the surface of revolution of having the least area when rotated about the X-axis.

13. Find the Euler-Lagrange equation when the function F is given by

(a)) $F = x^2y^2 - y'^2$ (b) $F = \sqrt{xy} + y'^2$

(c) $F = \sin(x\,y')$ (d) $F = x^2y'/\sqrt{1+y'^2}$

(Ans (a) $x^2y + y'' = 0$ (b) $x - 4(xy)^{1/2}\,y'' = 0$.

(c) $\cos(xy') - x(y' + xy'')\sin(xy') = 0$.

(d) $2(1+y'^2) - 3x\,y'\,y'' = 0$, which is a second order nonlinear ODE).

9.3 Extensions of the Euler-Lagrange Equation with one Independent Variable

Here we discuss two extensions of the Euler-Lagrange equation in one inde-

Therefore all the unknown constants have been determined as $A = B$ = $C = 0$ and $E = 1$. On substitution in (3) and (4) we finally obtain

$$y = \sin x, \quad z = -\sin x$$

Example 2

Find the extremal of the functional

$$I[y(x)] = \int_0^{\pi/2} (y''^2 - y^2 + x^2)\, dx$$

subject to the B.Cs.

$$y(0) = 1, \quad y(\pi/2) = 0, \quad y'(0) = 0, \quad y'(\pi/2) = 1$$

Solution

The extremal curve $y = y(x)$ is obtained by solving the Euler-Lagrange equation, *viz.*

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \quad ($$

Since $F = (y'')^2 - y^2 + x^2$, we have

$$\frac{\partial F}{\partial y} = -2y, \quad \frac{\partial F}{\partial y'} = 0 \text{ and therefore } \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Also

$$\frac{\partial F}{\partial y''} = 2y'' \text{ and therefore } \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 2y^{(iv)}$$

On substitution (1) becomes $-2y - 0 + 2y^{(iv)} = 0$, which is equivalent to

$$(D^4 - 1)y = 0 \text{ or } \left[(D-1)(D+1)(D^2+1)\right] y = 0$$

whose solution is given by

$$y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

To determine the constants we use the given endpoint conditions.

$$y(0) = 1 \implies c_1 + c_2 + c_3 = 1$$

$$y(\pi/2) = 0 \implies c_1 e^{\pi/2} + c_2 e^{-\pi/2} + c_4 = 0$$

Now since $y' = c_1 e^x - c_2 e^{-x} - c_3 \sin x + c_4 \cos x$, therefore

$$y'(0) = 0 \implies c_1 - c_2 + c_4 = 0$$