Chapter 9

Variational Methods

The subject of calculus of variations or variational methods is similar to but more general than the subject of maxima and minima in Calculus. In the former we have to find the extreme or stationary values of a function of one or more variables whereas in the latter we have to determine the extreme value of a functional, which is usually in the form of an integral.

9.1 Preliminaries

Functional

Let M be the set of functions defined over the interval $[a, b]$. If there is a rule which assigns each function of M to a real number J, then J is called the functional from M to $\mathbb{R}$. Extremal is the curve along which the functional takes the stationary value.

Stationary Value: The maximum or minimum value of a function or functional is called *stationary value*.

Extremal: The curve $y = f(x)$ along which a functional J takes the stationary value is called the *extremal*.

9.1.1 Some examples of variational problems

Here we discuss some important problems whose attempted solutions have led to the development of the subject of Calculus of Variations. At this stage we will only discuss and formulate the problems. The solutions will be given after

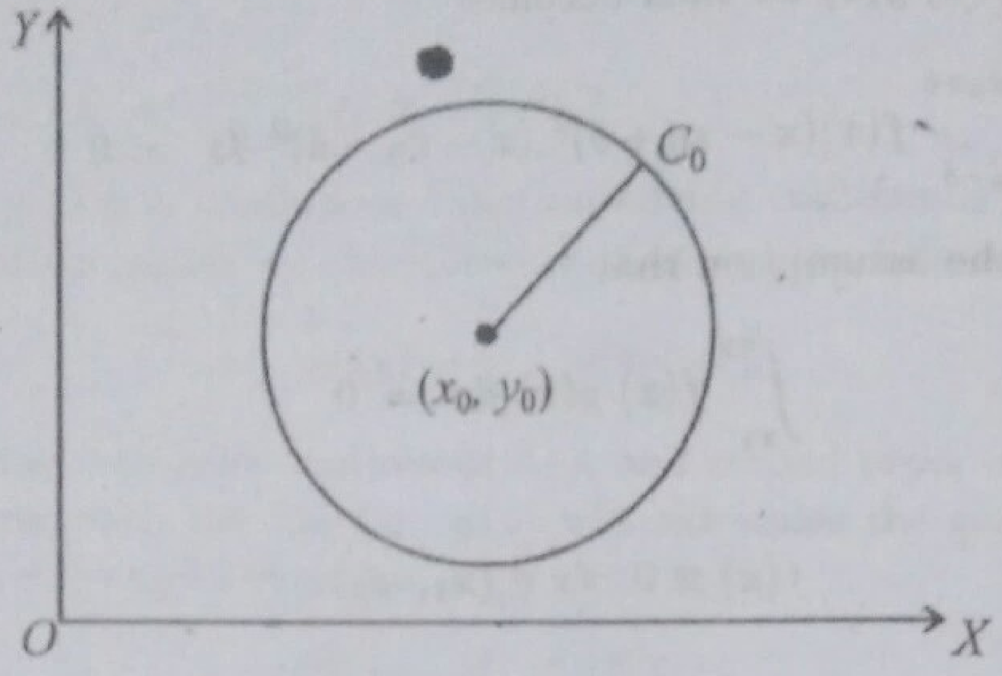

Figure 9.1: Circular region of radius ϵ centred at $(x_0,\ y_0)$ lying in the region D

Proof of the theorem for the extremal curve

It is clear that between the fixed points $A(x_1,\ y_1)$ and $B(x_2,\ y_2)$, there are infinite number of curves giving infinite number of values to the functional I. We have to find the curve which gives stationary value of the functional. Let the family of curves passing through fixed points A and B be defined as

$$y(x,\ \alpha) = y(x,\ 0) + \alpha\,\eta(x) \tag{9.2.1}$$

where $\eta(x)$ denotes the deviation from the curve $y = y(x) = y(x,\ 0)$ and α is a parameter labelling different paths, and it is independent of x. We suppose that the extremal curve corresponds to the value $\alpha = 0$. Since all curves pass through the endpoints $(x_1,\ y_1)$ and $(x_2,\ y_2)$, it follows that $\eta(x_1) = \eta(x_2) = 0$.

Now

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial F}{\partial \alpha} dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha} \right) dx$$

From (9.2.1)

$$\frac{\partial y(x,\ \alpha)}{\partial \alpha} = \eta(x)$$

Also

$$y'(x,\ \alpha) = y'(x,\ 0) + \alpha\,\eta'(x) \tag{9.2.2}$$

and

$$\frac{\partial y'}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(y'(x,\ 0) + \alpha\eta'(x)\right)$$

On integration we get $y = \pm ax + b$, which is the equation of a straight line.

Example 2

Give the geometrical interpretation of the variational problem $\int_0^1 \sqrt{1+y'^2}\,dx$, minimum with the boundary conditions $y(0) = 0$ and $y(1) = 1$.

Solve the problem for the extremal. Find the stationary value of the integral and compare it with the values of the integral which are obtained for curves that join the same end points but are different from the extremal.

Solution

(i) Since arc element of a curve is $ds = \sqrt{1+y'^2}\,dx$, (in view of example 1) it follows that the given problem can be stated as

$$\int_{(0,0)}^{(1,1)} ds \to \text{minimum}$$

This is equivalent to finding the curve of shortest length through the points $(0,0)$ and $(1,1)$; which is a straight line.

(ii) If V_0 denote the stationary value, then

$$V_0 = \int_0^1 \sqrt{1+y'^2}\,dx = \int_0^1 \sqrt{1+1}\,dx = \sqrt{2}\int_0^1 dx = \sqrt{2}$$

(iii) For the values of the integral which are obtained for curves that join the same end points and are different from the extremal, we have for $y = x^2$, $y' = 2x$.

$$\begin{aligned} I &= \int_0^1 \sqrt{1+y'^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx = \frac{1}{2}\int_0^1 \sqrt{1+(2x)^2}\,2dx \\ &= \frac{1}{2}\left[\frac{1}{2}2x\sqrt{1+4x^2} + \frac{1}{2}\ln\left(2x + \sqrt{1+4x^2}\right)\right]_0^1 \\ &= \frac{1}{2}\left[\sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5})\right] = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(2+\sqrt{5}) \end{aligned}$$

where we have used the result

$$\int_a^b \sqrt{x^2+a^2}\,dx = \left(\frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2\ln(x+\sqrt{x^2+a^2})\right)\Big]_a^b$$

Comparison

This value is always greater than $\sqrt{2}$ which shows that $\sqrt{2}$ is the stationary value. If we take any other curve, its value will always be greater than $\sqrt{2}$

We want to find out the extremal curve through A and B. It must satisfy the Euler-Lagrange equation

$$F - y'\frac{\partial F}{\partial y'} = \text{constant} = a, \text{say} \tag{1}$$

Here $F = \sqrt{1+y'^2}/\sqrt{y}$, and therefore

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{y}}\frac{2y'}{2\sqrt{1+y'^2}} = \frac{y'}{\sqrt{y(1+y'^2)}}$$

Putting in (1)

$$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y(1+y'^2)}} = a$$

or on simplification

$$\frac{1}{y(1+y'^2)} = a^2, y' \equiv \frac{dy}{dx} = \sqrt{\frac{1-a^2y}{a^2y}}$$

or

$$dx = \sqrt{\frac{a^2y}{1-a^2y}}\,dy \tag{2}$$

To integrate (2), let $a^2 y = \sin^2 \theta/2$, or

$$y = \frac{1}{a^2}\sin^2\frac{\theta}{2} \tag{3}$$

Therefore

$$dy = \frac{1}{a^2}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\left(\frac{\theta}{2}\right) = \frac{1}{2a^2}\sin\theta\, d\theta$$

On substitution in (2)

$$\frac{1}{a^2}\sin^2\theta/2\,d\theta = dx \quad \text{or} \quad \frac{1}{2a^2}(1-\cos\theta)\,d\theta = dx$$

On integration

$$x = \frac{1}{2a^2}(\theta - \sin\theta) + b \tag{4}$$

and from (3)

$$y = \frac{1}{2a^2}(1-\cos\theta) \tag{5}$$

Equations (4) and (5) are the parametric equations of a cycloid; the constants a and b can be determined from the endpoint or initial conditions. Therefore the curve down which the particle takes the minimum time is a cycloid.

(i) The case $F = F(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n')$ i.e. when there is one independent variable and n dependent variables.

(ii) The case $F = F(x, y, y', y'', \cdots, y^{(n)})$ i.e. the integrand in the functional involves higher order derivatives.

9.3.1 Euler-Lagrange equation, one independent, many dependent variables

This is a direct generalization of the case studied in section 9.2 above. Here the functional is of the form

$$I = \int_{x_1}^{x_2} F(x, y_1, y_2, y_3, \cdots, y_n, y_1', y_2', y_3', \cdots y_n')\, dx$$

with the boundary conditions

$$y_k(x_1) = \text{constant}, \; y_k(x_2) = \text{constant}, \quad k = 1, 2, \cdots n$$

Using the shorthand procedure for finding the variation, δI of I when y_k varies by δy_k, $k = 1, 2, \cdots, n$, we have

$$\delta I = \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y_1}\delta y_1 + \frac{\partial F}{\partial y_2}\delta y_2 + \cdots + \frac{\partial F}{\partial y_n}\delta y_n \right) + \left(\frac{\partial F}{\partial y_1'}\delta y_1' + \frac{\partial F}{\partial y_2'}\delta y_2' + \cdots + \frac{\partial F}{\partial y_n'}\delta y_n' \right) \right] dx$$

or

$$\delta I = \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y_1}\delta y_1 + \frac{\partial F}{\partial y_1'}\delta y_1' \right) + \left(\frac{\partial F}{\partial y_2}\delta y_2 + \frac{\partial F}{\partial y_2'}\delta y_2' \right) + \cdots + \left(\frac{\partial F}{\partial y_n}\delta y_n + \frac{\partial F}{\partial y_n'}\delta y_n' \right) \right] dx. \tag{9.3.1}$$

Now consider

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y_k'}\delta y_k' \right) dx = \frac{\partial F}{\partial y_k'}\delta y_k \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \delta y_k \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'} \right) dx$$

$$= -\int_{x_1}^{x_2} \delta y_k \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'} \right) dx$$

Since $\delta y_k = 0$ at $x = x_1, x_2$, therefore

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y_k}\delta y_k + \frac{\partial F}{\partial y_k'}\delta y_k' \right) dx = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_k'} \right) \right] \delta y_k \, dx$$

$$k = 1,\ 2,\ \cdots, n \tag{9.3.2}$$

Therefore on substituting expressions (9.3.2) in (9.3.1), we obtain

$$\begin{aligned}\delta I &= \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y_1} - \frac{d}{dx_1} \frac{\partial F}{\partial y_1'} \right) \delta y_1 + \left(\frac{\partial F}{\partial y_2} - \frac{d}{dx} \frac{\partial F}{\partial y_2'} \right) \delta y_2 \right. \\ &+ \cdots + \left. \left(\frac{\partial F}{\partial y_n} - \frac{d}{dx} \frac{\partial F}{\partial y_n'} \right) \delta y_n \right] dx \end{aligned} \tag{9.3.3}$$

The extremal corresponds to $\delta I = 0$. Therefore putting $\delta I = 0$ in (9.3.3) we obtain

$$\int_{x_1}^{x_2} \sum_k \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_k{}'} \right) \right] \delta y_k \, dx = 0 \tag{9.3.4}$$

Since the curves (functions) y_k, $k = 1,\ 2,$ are independent of each other, each of the integrals on R.H.S. of (9.3.4) must be zero, (fundamental theorem of the calculus of variations).

Therefore we must have

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx} (\frac{\partial F}{\partial y_k{}'}) \right] \delta y_k \, dx = 0, \quad k = 1,\ 2,\ \cdots$$

Now invoking the fundamental theorem of the calculus of variations, we have

$$\left[\frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_k'} \right) \right] = 0, \quad k = 1,\ 2,\ \cdots$$

Since $\delta y_k \neq 0, k = 1, 2, \cdots$, we must have

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_k'} \right) = 0, \quad k = 1,\ 2,\ \cdots$$

which are the required equations for the extremal curves. Since there are n independent curves (functions), there are n E-L equations.

9.3.2 Euler-Lagrange equation, with higher order derivatives

Here the functional is of the form

$$I[y] = \int F\left(x,\ y,\ y',\ y'',\ \ldots,\ y^{(n)}\right) dx$$

with the endpoint conditions

$$y(x_1) = y'(x_1) = y''(x_1) = \cdots = y^{(n-1)}(x_1) = \text{constant}$$

The given function is $F = y'^2 + z'^2 + 2yz$. Therefore

$$\frac{\partial F}{\partial y} = 2z, \quad \frac{\partial F}{\partial z} = 2y, \text{ and } \frac{\partial F}{\partial y'} = 2y', \quad \frac{\partial F}{\partial z'} = 2z'$$

On substitution in (1) and (2)

$$2z - \frac{d}{dx}(2y') = 0 \implies z - y'' = 0$$

and

$$2y - \frac{d}{dx}(2z') = 0 \implies y - z'' = 0$$

On combining the DEs $z = y''$ and $y = z''$ we obtain $z'' = y^{(iv)}, \; y = z''$.

From these DEs, we have $y^{(iv)} = y$ or $(D^4 - 1)y = 0$, where $D = d/dx$. This DE can be written as

$$[(D-1)(D+1)(D^2+1)]\, y = 0$$

whose solution is given by

$$y = Ae^x + Be^{-x} + C\cos x + E\sin x \quad (3)$$

Also

$$\begin{aligned} z &= y'' = \frac{d^2}{dx^2}\left(Ae^x + Be^{-x} + C\cos x + E\sin x\right) \\ &= Ae^x + Be^{-x} - C\cos x - E\sin x \end{aligned} \quad (4)$$

Applying the given conditions to (3) and (4), we obtain

$$y(0) = 0 \implies 0 = A + B + C \quad (5)$$

$$y(\pi/2) = 1 \implies 1 = Ae^{\pi/2} + Be^{-\pi/2} + E \quad (6)$$

Similarly

$$z(0) = 0 \implies 0 = A + B - C \quad (7)$$

$$z(\pi/2) = -1 \implies -1 = Ae^{\pi/2} + Be^{-\pi/2} - E \quad (8)$$

Adding (5) and (7) $A + B = 0$ or $B = -A$. Also by subtraction from (5) and (7) $C = 0$.

Adding (6) and (8) $Ae^{\pi/2} + Be^{-\pi/2} = 0$, and by subtraction from (8) and (6) $E = 1$. Using the relation $B = -A$ we obtain

$$+ (-1)^n \delta y \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right)\Bigg] dx$$

For the extremal curve, $\delta I = 0$, therefore

$$0 = \int_{x_1}^{x_2} \Bigg[\frac{\partial F}{\partial y} + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + (-1)^3 \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) + \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right)\Bigg] \delta y \, dx$$

which implies that (invoking the fundamental theorem of the calculus of variations)

$$\Bigg[\frac{\partial F}{\partial y} + (-1)\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + (-1)^2 \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + (-1)^3 \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) + \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right)\Bigg] = 0$$

which is the Euler-Lagrange equation in generalised form.

9.3.3 Illustrative examples

Example 1

Find the extremal for

$$I = \int_0^{\pi/2} \left(y'^2 + z'^2 + 2yz\right) dx$$

where the boundary conditions are

$$y(0) = 0, \quad y(\pi/2) = 1, \quad z(0) = 0, \quad z(\pi/2) = -1$$

Solution

Since there are two unknown functions (extremal curves) y and z, there will be a pair of Euler-Lagrange equations. Note that here $y = y_1$ and $z = y_2$. The corresponding E-L equations are

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad (1)$$

and

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0 \qquad (2)$$

and

$$y(x_2) = y'(x_2) = y''(x_2) = \cdots = y^{(n-1)}(x_2) = \text{constant}$$

Now a variation δy of y produces a variation δI in the functional I given by

$$\delta I = \int_{x_1}^{x^2} \left(\frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y'' + \cdots + \frac{\partial F}{\partial y^n}\,\delta y^n \right) dx \tag{9.3.5}$$

Now consider

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\,\delta y'\,dx = \frac{\partial F}{\partial y'}\,\delta y\Big]_{x_1}^{x_2} - \int_{x_1}^{x_2} \delta y\,\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

Now since $y(x_1)$ and $y(x_2)$ are constant, δy must be zero at x_1 and x_2. Therefore

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y'}\right) \delta y'\,dx = -\int_{x_1}^{x_2} \delta y\,\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

Similarly integrating by parts twice and making use of the endpoint conditions for y and y' at x_1, x_2 we have

$$\begin{aligned}
\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y''}\,\delta y''\right) dx &= \frac{\partial F}{\partial y''}\,\delta y'\Big]_{x_1}^{x_2} - \int_{x_1}^{x_2} \delta y'\,\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) dx \\
&= -\int_{x_1}^{x_2} \delta y'\,\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) dx \\
&= -\delta y\,\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\Big]_{x_1}^{x_2} + (-1)^2 \int_{x_1}^{x_2} \delta y\,\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) dx \\
&= (-1)^2 \int_{x_1}^{x_2} \delta y\,\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) dx
\end{aligned}$$

Similarly

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y'''}\,\delta y'''\right) dx = (-1)^3 \int_{x_1}^{x_2} \delta y\,\frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) dx$$

$$\vdots$$

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y^{(n)}}\,\delta y^{(n)}\right) dx = (-1)^n \int_{x_1}^{x_2} \delta y\,\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) dx$$

Substituting for these expressions in (9.3.5), we have

$$\begin{aligned}
\delta I &= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\,\delta y + (-1)\delta y\,\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right. \\
&+ (-1)^2 \delta y\,\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) + (-1)^3 \delta y\,\frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) + \cdots
\end{aligned}$$

subject to $y(0) = 0,\ y(\pi) = 1,\ z(0) = 0,\ z(\pi) = -1$.

(**Ans.** $y = C\sin x - (x/\pi)\cos x,\ z = C\sin x + (2\sin x - x\cos x)/\pi$, where C is arbitrary).

8. Find extremals for the variational problems

(a) $\int_{-1}^{0}\left[240y - (y''')^2\right] dx \to$ minimum, $y(-1) = 1,\ y(0) = 0,\ y'(-1) = -(9/2),\ y'(0) = 0,\ y''(-1) = 16,\ y''(0) = 0$.

(b) $\int_a^b (y + y'')\, dx \to$ minimum, $y(a) = y_0,\ y(b) = y_1,\ y'(a) = y_0',\ y(b)' = y_1'$.

(**Ans.** There is no minimum).

9. Find the extremals of the following problems:

(a) $\int_0^1 (y'')^2\, dx \to$ minimum, $y(0) = 0,\ y'(0) = 0,\ y'(1) = 1$

(b) $\int_{-1}^{+1} (2xy - y'^2 + z'^2/3)\, dx \to$ stationary, $y(1) = 0,\ y(-1) = 2,\ z(1) = 1,\ z(-1) = -1$.

(c) $\int_0^{\pi/2} (y'^2 + z'^2 - 2yz)\, dx \to$ stationary, $y(0) = 0,\ y(\pi/2) = 1,\ z(0) = 0,\ z(\pi/2) = 1$,

(d) $\int_0^1 (y'^2 + z'^2 + 2y)\, dx \to$ minimum, $y(0) = 1,\ y(1) = (3/2),\ z(0) = 0,\ z(1) = 1$.

[**Ans.**: (a) $y = x^2/2$, (b) $y = -(x^3 + 5x - 6)/6,\ z = x$), (c) $y = \sin x,\ z = \sin x$, (d) $y = (x^2/2) + 1$]

10. Find the extremals of the functional $I[y] = \int_{x_1}^{x_2} \sqrt{x^2 + y^2}\,\sqrt{1 + y'^2}\, dx$

(**Hint:** Use polar coordinates.

Ans. $x^2\cos x + 2xy\sin\alpha - y^2\cos\alpha = \beta$, where α, β are constants).

11. Find the extremals of the following fixed endpoint problems.

(a) $\int_{x_1}^{x_2} (y'^2 + z'^2 + y'z')\, dx \to$ stationary.

(b) $\int_{x_1}^{x_2} (2yz - 2y^2 + y'^2 - z'^2)\, dx \to$ stationary.

12. Find the extremals of the problem $I[y] = \int_0^{\pi/2} (y''^2 - y^2 + x^2)\, dx$ subject to $y(0) = 1,\ y'(0) = 0,\ y(\pi/2) = 0,\ y'(\pi/2) = -1$.

13. Show that the E-L equation for the functional $I[y] = \int_{x_1}^{x_2} F(x,\ y,\ y',\ y'')\, dx$ with the usual endpoint conditions, has the first integral

Since u is prescribed on the boundary, δu must be zero, and therefore $I_2 = 0$. Next we simplify the first integral

$$\delta I = \int_R\int \left[\frac{\partial F}{\partial u}\delta u - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u\right] dx\, dy$$

For extremal $\delta I = 0$, and therefore

$$\int_R\int \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right] \delta u\, dx\, dy = 0$$

Since δu is arbitrary, by invoking generalized form of the fundamental theorem of the calculus of variations, we have

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$$

9.4.2 Euler-Lagrange equation for 3 independent variables

Theorem

Given the functional

$$I[u(x,\ y,\ z)] = \int\int_V\int F(x,\ y,\ z,\ u,\ u_x,\ u_y,\ u_z)\, dx\, dy\, dz$$

where u has different values in a three dimensional region V but is prescribed on the boundary surface S of the region; it is assumed that u has continuous partial derivatives up to the second order in the region V. Then a necessary condition for this functional to have an extremum is that $u = u(x,\ y,\ z)$ must satisfy the PDE

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) = 0$$

Proof

It is given that

$$I(u(x,\ y,\ z)) = \int\int_R\int F(x,\ y,\ z,\ u,\ u_x,\ u_y,\ u_z)\, dx\, dy\, dz$$

Considering the variation in the functional corresponding to $u \to u + \delta u$, we have

$$\delta I = \int\int_V\int \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial u_z}\delta u_z\right) dx\, dy\, dz \qquad (9.4.3)$$

Now since
$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\,\delta u\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u + \frac{\partial F}{\partial u_x}\,\delta u_x$$

Therefore
$$\frac{\partial F}{\partial u_x}\,\delta u_x = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$$

Similarly
$$\frac{\partial F}{\partial u_y}\,\delta u_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u$$

and
$$\frac{\partial F}{\partial u_z}\,\delta u_z = \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\delta u$$

Putting these values in (9.4.3), we obtain

$$\begin{aligned}\delta I &= \iint_V\!\!\int \left[\frac{\partial F}{\partial u}\,\delta u + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\,\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u\right.\\ &+ \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\,\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u\\ &+ \left.\frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\,\delta u\right)\right] dx\,dy\,dz\end{aligned}$$

or

$$\begin{aligned}\delta I &= \iint_V\!\!\int \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\right] dx\,dy\,dz\\ &+ \iint_V\!\!\int \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right)\right] dx\,dy\,dz\end{aligned}$$

Consider

$$\begin{aligned}I_2 &= \iint_V\!\!\int \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\,\delta u\right)\right] dx\,dy\,dz\\ &= \iint_V\!\!\int \operatorname{div}\left(\frac{\partial F}{\partial u_x}\,\delta u\,\mathbf{i} + \frac{\partial F}{\partial u_y}\,\delta u\,\mathbf{j} + \frac{\partial F}{\partial u_z}\,\delta u\,\mathbf{k}\right) dx\,dy\,dz\\ &= \iint_V\!\!\int \operatorname{div}\mathbf{G}\,dV\end{aligned}$$

where $dV = dx\,dy\,dz$, and

$$\begin{aligned}\mathbf{G} &= \frac{\partial F}{\partial u_x}\,\delta u\,\mathbf{i} + \frac{\partial F}{\partial u_y}\,\delta u\,\mathbf{j} + \frac{\partial F}{\partial u_z}\,\delta u\,\mathbf{k}\\ &= \left(\frac{\partial F}{\partial u_x}\,\mathbf{i} + \frac{\partial F}{\partial u_y}\,\mathbf{j} + \frac{\partial F}{\partial u_z}\,\mathbf{k}\right)\delta u\end{aligned}$$

$$F = \frac{\partial \mathbf{r}}{\partial x} \cdot \frac{\partial \mathbf{r}}{\partial y} = (\mathbf{i} + z_x\,\mathbf{k}) \cdot (\mathbf{j} + z_y\,\mathbf{k}) = z_x\,z_y$$

and

$$G = (\frac{\partial \mathbf{r}}{\partial y})^2 = (\mathbf{j} + z_y\,\mathbf{k})^2 = 1 + z_y^2 .$$

In terms of parametric coordinates $(u,\ v)$ the surface element dS is given by

$$dS = ds_1\ ds_2 = \sqrt{E}\,du\ \sqrt{G}\,dv\ \sin\theta = \sqrt{E\,G}\,du\,dv\ \sin\theta$$

where θ is the angle between the parametric coordinate curves $u =$ constant and $v =$ constant. and $\cos\theta = F/\sqrt{EG}$.

Therefore

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{F^2}{EG}} = \frac{\sqrt{EG - F^2}}{\sqrt{EG}}$$

and

$$dS = \sqrt{EG - F^2}\,du\,dv$$

When $(x,\ y)$ are used as parameters

$$dS = \sqrt{(1 + z_x^2)\,(1 + z_y^2) - z_x^2\,z_y^2}\,dx\,dy = \sqrt{1 + z_x^2 + z_y^2}\,dx\,dy$$

The problem is to minimize the integral

$$\int\!\!\int_S\!\int \sqrt{1 + z_x^2 + z_y^2}\,dx\,dy$$

subject to the condition $z = z_0$ on C. On simplifying the E-L equation we obtain

$$z_{xx}(1 + z_y^2) - 2z_x z_y z_{xy} + z_{yy}(1 + z_x^2) = 0$$

9.4.4 Exercises

1. Find the extremals corresponding to the following functionals, (assuming suitable endpoint conditions).

   (a) $I[u] = \int\!\!\int_R (x^2 u_x^2 + y^2 u_y^2)\,dx\,dy, \quad u = u(x,\ y)$

   (b) $I[u] = \int\!\!\int_R (u_t^2 - c^2 u_x^2)\,dx\,dt, \quad u = u(x, t), \quad c =$ constant

2. Derive the E-L equations for the functional

$$I[u] = \int\!\!\int_R F(x,\ y,\ u,\ u_x,\ u_y,\ u_{xx},\ u_{xy},\ u_{yy})\,dx\,dy, \quad u = u(x,\ y)$$

where the arguments of F are subject to some additional conditions or constraints such as

(i) $G(x, y_1, \cdots y_n) = \text{constant.}$

or (ii) $G(x, y_1, \cdots y_n, y_1', \cdots y_n') = \text{constant.}$

or (iii) $\int_{x_1}^{x_2} G(x, y_1, \cdots, y_n, y_1', \cdots y_n')\, dx = \text{constant.}$

Isoperimetrical problems are special cases of these problems.

9.5.1 Constrained maxima and minima problems in calculus

Let $u = f(x_1, x_2, \cdots, x_n)$ with side conditions

$$\phi_i(x_1,\ x_2,\ \cdots x_n) = 0,\quad i = 1,\ 2,\ \cdots m,\quad (m < n)$$

In Lagrange's method of multipliers, we consider an auxiliary function

$$w(x_1,\ x_2,\ \cdots x_n) = f(x_1,\ x_2,\ \cdots x_n) + \sum_{i=1}^{m} \lambda_i\, \phi_i(x_1,\ x_2,\ \cdots x_n)$$

(where λ_i are constant multipliers) and then try to find the extrema of the function w. To find stationary values of w we have to solve the system of equations

$$\frac{\partial w}{\partial x_j} = 0,\quad j = 1,\ 2,\ \cdots n$$

along with the equations of the constraints, *viz.*

$$\phi_i(x_1,\ x_2,\ \cdots x_n) = 0,\ i = 1,\ 2,\ \cdots m,\quad (m < n)$$

Both these equations involve $m + n$ unknowns $x_1, x_2, \cdots x_n, \lambda_1, \lambda_2, \cdots, \lambda_m$. Constrained variational problems are similar to the above problems in ordinary calculus.

9.5.2 The Euler-Lagrange equation for constrained extrema

The following theorem states the relevant result.

Theorem

The extremal curves $y_i = y_i(x),\ i = 1, 2, \cdots n$ of the functional

$$I[y_1(x),\ y_2(x),\ \cdots\ y_n(x)] = \int_{x_1}^{x_2} F(x,\ y_1,\ y_2,\ \cdots\ y_n,\ y_1',\ y_2' \cdots y_n')\, dx$$

storically three such fundamental problems are

The problem of geodesics i.e. to find the curve of minimum length joining o points on a given surface.

The brachistochrone problem i.e. to find the path of quickest descent, ning two points in space, for a particle moving under gravity.

Dido's problem i.e. the problem of finding curve of given length which loses maximum area by itself or with a given straight line.

ample 1

find the curve whose distance between two points on a surface is minimum.

scussion

s problem is called *geodesic* problem. Let A and B be any two points on urve C, lying on a surface S. The equation of the surface may be taken as $z(x, y)$. The distance between the points A and B on any curve $y = y(x)$ ich is the same as length l of the curve between A and B) is given by

$$\begin{aligned} \ell &= \int_A^B ds = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \\ &= \int_A^B \sqrt{1 + (dy/dx)^2 + (dz/dx)^2}\, dx \\ &= \int_A^B \sqrt{1 + y'^2 + z'^2}\, dx \end{aligned}$$

e ds is element of arc along the curve. In Calculus of Variations we have nd minimum value of l. When the points A and B lie in the X Y plane, the expression for l takes the simpler form $\ell = \int_A^B \sqrt{1 + y'^2}\, dx$.

mple 2

rticle of falls under gravity from A to B. Determine the curve along which taken by the particle will be minimum.

cussion

rmulate this problem as a variational problem we note that the time and acement elements dt, ds are related to the instantaneous velocity v by ds/dt. Therefore

$$\text{total time taken} = \int_A^B dt = \int_A^B \frac{dt}{ds}\, ds = \int_A^B \frac{1}{v}\, ds$$

$$= \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{1 + y'^2}{y}}\, dx$$

ch is in the form of a functional. By minimizing this functional we can the required time.

ample 3

find the curve $y = y(x)$ which has a given length l and encloses maximum a with, say, X-axis.

scussion

ice the area enclosed by the curve $y = y(x)$ between the lines $x = a$ and $= b$ and the X- axis is given by

$$\text{area} = \int_a^b y(x)\, dx \tag{9.1.1}$$

nd the length ℓ of the same curve between $x = a$ and $x = b$ is given by

$$\ell = \int_a^b ds = \int_a^b \sqrt{1 + y'^2}\, dx \tag{9.1.2}$$

hus the problem reduces to that of maximizing the area in equation (9.1.1) ubject to the condition given in (9.1.2).

.2 Euler-Lagrange Equation

n this and the following sections we derive the relevant DEs from which the equired extremal curves or surfaces can be obtained. In each case we have to xtremize a functional of the form

$$J[y(x)] = \int_C F(x,\, y,\, y',\cdots)\, dx$$

or

$$J[u(x,\, y,\, \cdots)] = \int_C F(x,\, y,\, \cdots, u,\, u_x,\, u_y,\cdots)\, dx\, dy\, \cdots$$

The integrand F will have different forms in different situations. The simplest corresponds to $F = F(x,\, y,\, y')$.

Theorem