Chapter 8

Green's Function and the Boundary Value Problems

Green's functions provide an important tool in the study of boundary val problems. They also have intrinsic value for a mathematician. We begin o study of Green's functions with a brief discussion of the Dirac delta functio

8.1 The Dirac Delta Function

8.1.1 Motivation and background

The Dirac-delta function can be regarded as as the generalization of the Kr necker delta δ_{ij}, when the discrete integer variables i, j are replaced by t continuous variables x, x'.

indexDirac delta function! properties of The Kronecker delta δ_{ij} has the fo lowing two well-known properties.

(i) $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

(ii) $\delta_{ij} A_i = A_j$

These properties are passed over to the Dirac delta function in the followi form.

$$\delta(x - t) = 0, \quad t \neq x$$
$$\delta(x - t) = \infty, \quad t = x$$

$\int \delta(x-t)f(t)dt = f(x)$

where the point x lies in the interval of integration and $f(x)$ is a continuous function.

To understand the significance of the Dirac delta function in physical situations we consider those situations in which a large effect lasting for a short duration or acting over a small stretch of length. The examples are an impulsive force or a load over a very small part of a beam.

From a rigorous mathematical point of view the Dirac delta function is a particular case of a class of functions known as generalized functions.

Here we will introduce it in an intuitive way.

Consider the sequence of functions, $\{f_n(x)\}$, defined over the interval $(-\infty, +\infty)$ as follows.

$$f_n(x) = \begin{cases} n/2, & x \in [-1/n, \ +1/n] \\ 0, & x \notin [-1/n, \ +1/n] \end{cases}$$

Then the integral $\int_{-\infty}^{+\infty} f_n(x)dx$ is the total area under the graph of the function $f_n(x)$ and can be written as

$$\int_{-\infty}^{+\infty} f_n(x)dx = \int_{-1/n}^{+1/n} \frac{n}{2}dx = 1$$

Before proceeding further we make use of the mean-value theorem for an integral.

It states that under certain suitable conditions on a function defined over the interval $[a, \ b]$, there exists a point $\xi_n \in [a, \ b]$ such that

$$\int_a^b g(x)dx = (b-a)g(\xi_n) \tag{8.1.1}$$

Using this mean-value theorem

$$\int_{-\infty}^{+\infty} f_n(x)g(x)dx = \frac{n}{2}\int_{-1/n}^{+1/n} g(x)dx = \frac{n}{2} \times \frac{2}{n} \times g(\xi_n) = g(\xi_n) \tag{8.1.2}$$

where $\xi_n \in\in [-1/n, \ +1/n]$.

Taking the limit as $n \to \infty$, we note that $\xi_n \to 0$ because the interval $[-1/n, \ +1/n]$ shrinks to zero. Hence from (11.1.2)

$$\lim_{n\to\infty} \int_{\infty}^{+\infty} f_n(x)g(x)dx = \lim g(\xi_n) = g(0) \tag{8.1.3}$$

ate the integral

$$\int_{-\infty}^{+\infty} \cos x\, \delta(x^2 - \pi^2)\, dx$$

tion

we note that the range of integration is from it 0 to 2π rather than $-\infty$
o. Therefore out of the two zeros of the Dirac delta function at $x = \pm\pi$
se consider only the zero at $x = \pi$.

$\delta[x - \pi^2) = (1/2\pi)\,[\delta(x - \pi) + \delta(x + \pi)]$

efore

$$\int_0^{2\pi} \cos x\, \delta(x^2 - \pi^2)\, dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \delta(x + 2/3)\, x^2\, dx$$

$x = -\pi$ is outside the interval of integration we can write

$$\text{given integral} = \frac{1}{2\pi} \int_0^{2\pi} \cos x\, \delta(x - \pi)\, dx = \frac{1}{2\pi} \cos\pi = -\frac{1}{2\pi}$$

6 Exercises

alculate the integral $\int_0^\pi \sin x\, \delta(x - \pi/2)\, dx$.

alculate the integral $\int_0^\pi (5x - 4x + 2)\, \delta[(x - 2)/3)\, \exp(2x)]\, dx$.

alculate the integral $\int_0^\pi x^3\, \delta(x + \pi/3)\, dx$.

alculate the integral $\int_0^\infty (5x - 4x + 2)\, \delta(x^2 - 5x + 6)\, \exp(-3x)]\, dx$

alculate the integral $\int_0^\pi (x^2 - 2x + 7)\, \delta(x^2 - 16)\, dx$

alculate the integral $\int_{-\infty}^{+\infty} (3x^2 - 7x + 2)\, \delta(x^2 - 5x + 6)\, \exp(-3x)]\, dx$

Green's Functions

.1 Motivation for Green's function

sically *Green's function* (also called *response function*) is the response cor-
ponding to unit source. For example in electromagnetic theory the potential
a field point **r** due to a unit charge (unit source) at a point **r**′ is the Green
ction $G(\mathbf{r},\ \mathbf{r}')$, defined by

$G(\mathbf{r},\ \mathbf{r}') = k\,q/|\mathbf{r} - \mathbf{r}'|$

where q is the charge at the field point. Similarly the gravitational potential at a point due to a unit mass (unit source) is the Green function in this case.

8.2.2 Formal definition and connection with linear inhomogeneous DEs

Let L be a linear differential operator (of order 2 or more), $L = L(x)$, acting on a point t, of the interval of definition, which may be taken as $[a,\ b]$. Then Green's function for the operator L, denoted by $G(x,\ t)$, is any solution of

$$L\,G(x,\ t) = \delta(x - t) \tag{8.2.1}$$

where $\delta(x - t)$ is the Dirac delta function. This defining property of Green's function can be used to solve the inhomogeneous linear equation

$$L\,G(x,\ t) = f(x) \tag{8.2.2}$$

Because of $\delta(x - t)$, we note that. like the Dirac delta function, Green's functions are generalized functions rather than ordinary functions.

Green's functions are useful tools in solving initial-boundary value problems associated with heat and wave equations.

Sometimes a Green's function is defined by $L\,G(x,\ t) = -\delta(x - t)$ with a negative sign on the right side, but this does not significantly change any of the properties of the Green's function.

If the operator L is invariant under translation, which will be the case when the coefficients are constant w.r.t. x, then Green's function can be written as as $G(x,\ t) = G(x - t)$, i.e. a convolution operator.

8.2.3 Solution of the inhomogeneous linear ODE in terms of Green's function

If Green's function corresponding to the operator L exists, then we can multiply (8.2.1) with $f(t)$ and integrate w.r.t. t (over the interval of definition) and obtain

$$\int L\,G(x,\ t)\,f(t)\,dt = \int L\,\delta(x - t)\,f(t)\,dt = f(x)$$

But from (8.2.2) we find that $f(x) = Lu$. Therefore

$$L\,u(x) = \int L\,G(x,\ t)\,f(t)\,dt \tag{8.2.3}$$

nce the operator $L = L(x)$ is linear and acts on the variable x alone (an ot the variable t), we can take the operator L out of the integral on the righ de of (3), thereby obtaining

$$L\,u(x) = L\,\left(\int G(x,\ t)\,f(t)\,dt\right)$$

hich suggests that

$$u(x) = \int G(x,\ t)\,f(t)\,dt \tag{8.2.4}$$

The above equation shows that we can obtain the solution $u(x)$ on the bas of our knowledge of Green's function in (8.2.1), and the source term on t right side in (8.2.2). This process depends on the linearity of the operator

Green's function may not exist for every operator L. A Green's function c also be thought of as a right inverse of L. Apart from the difficulties of findi a Green's function for a particular operator, it may not be easy to evalua the integral in equation (8.2.4). However the method gives an exact solutic

8.2.4 Green's function for solving inhomogeneous BV Pro lems

The foremost use of Green's functions in Applied Mathematics is to solve inh mogeneous boundary value problems. Let L be the Sturm-Liouville operat

$$L) = (d/dx)\ \{p(x)\,d/dx\} + q(x)$$

and let $\mathcal{B}$, the boundary condition operator be defined by

$$\mathcal{B}\,u = \begin{cases} \alpha_1\,u(a) + \alpha_2\,u'(a) \\ \beta_1\,u(b) + \beta_2\,u'(b) \end{cases}$$

We suppose that the function $f(x)$ is continuous on the interval $[a,\ b]$. also suppose that the B.V. problem defined by the equations

$$Lu(x) = f(x),\quad \mathcal{B}\,u = 0$$

is a regular system. This requires that only the trivial solution will exist the associated homogeneous system.

The following theorem gives the relationship between the solution and Gree function for a regular SL system.

There is a unique solution for the problem defined by the equations

$$Lu(x) = f(x), \quad \mathcal{B}\, u = 0$$

and is given by

$$u(x) = \int G(x,\ t)\, f(t)\, dt$$

where $G(x,\ t)$ is Green's function. It possesses the following properties.

(i) $G(x,\ t)$ is continuous in x and t.

(ii) For $x \neq t$, it satisfies the DE $LG(x,\ t)$. In other words in each of the subintervals $[a,\ t)$ and $(t,\ b]$.

(iii) It satisfies the BCs $\mathcal{B}\, G(x,\ t) = 0$ for $x \neq t$.

(iv) DE $G'(x,\ t)$, considered as a function of x has a jump discontinuity at $x = t$. This result can be stated as follows:

$$G'(t+0,\ t) - G'(t-0,\ t) = 1/p(t)$$

(v) It has the symmetry property: $G(x,\ t) = G(t,\ x)$, (when $G(x,\ t)$ is real).

8.3 Green's Functions for Initial/Boundary Value Problems

Sturm Liouville

We consider the SL equation

$$\frac{d}{dx}\left\{p(x)\,\frac{dy}{dx}\right\} + q(x)\, y = f(x) \tag{8.3.1}$$

If $y_1,\ y_2$ are linearly independent solutions of the homogeneous DE associated with (8.3.1), then $y = c_1 y_1 + c_2 y_2$ will be complementary function of the DE (8.3.1).

Using the method of variation of parameters[1] we can find a particular solution in the form

$$\begin{aligned} y_p &= y_2(x)\int_{x_1}^{x} \frac{f(t)y_1(t)}{p(t)W(t)}\, dt \\ &+\ y_1(x)\int_{x_2}^{x} \frac{f(\xi)y_2(t)}{p(t)W(t)}\, dt \end{aligned} \tag{8.3.2}$$

where $W(t)$ is the Wrnskian of the functions $y_1(t),\ y_2(t)$.

We now discuss initial and boundary value problems. For each type of problem, we will show that its solution can be written as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \int_a^b G(x,\ t)\, f(t)\, dt \tag{8.3.3}$$

where $(x,\ t)$ will be found to be Green's function. The knowledge of Green's function in each type of problem will enable us to determine the solution.

8.3.1 Green's function for the initial-value problem

Here we start with the same DE, *viz.*, (8.3.1) but with initial conditions $y(0) = y_0,\ y'(0) = v_0$ We have taken the DE in the form of an S L equation because every second order linear ODE can be written in this form.

We first recall that any linear inhomogeneous ODE of the form $Ly = f(x)$ where L is a linear differential operator of S L type, then its general solution will be $y = y_c + y_p$ where y_c (complementary function) is general solution of the associated homogeneous ODE $Ly = 0$ and y_p is a particular solution of the given inhomogeneous ODE: $Ly = f(x)$.

Because of this general result, we can reduce this I.V. problem to two separate I.V. problems. To do that we will assume that the homogeneous S L equation satisfies the original initial conditions. In other words we will first solve the following problem:

$$\frac{d}{dx}\left\{p(x)\,\frac{dy_c}{dx}\right\} + q(x)\, y_c = 0,\ \ y_c(0) = y_0,\ \ y_c'(0) = v_0$$

We further assume that particular solution y_p is a solution of the following problem

$$\frac{d}{dx}\left\{p(x)\,\frac{dy_p}{dx}\right\} + q(x)\, y_p = f(x),\ y_p(0) = 0,\ \ y_p'(0) = 0 \tag{8.3.4}$$

The above I.Cs. for y_p can be obtained from the original I.Cs. for $y(x)$ as follows.

$y(0) = y_c(0) + y_p(0) = y_0 + y_p(0)$ wherefrom $y_p(0) = 0$

$v_0 = y'(0) = y_c'(0) + y_p'(0) = v_0 + y_p'(0)$ wherefrom $y_p'(0) = 0$.

For the complementary function we have $y_c = c_1 y_1 + c_2 y_2$ where $y_1,\ y_2$ are two linearly independent solution of the homogeneous S L equation $Ly = 0$

Next we will determine particular integral y_p, and using the initial conditions or y_p determine the associated Green's function. To find y_p we use method of variation of parameters and start with $y_p = v_1(x)\,y_1(x) + v_2(x)\,y_2(x)$, where v_1, v_2 are unknown functions to be determined. In this method we can show that

$$v_1'(x) = -\frac{y_2(x)\,f(x)}{p(x)\,W(x)}, \quad v_2'(x) = \frac{y_1(x)\,f(x)}{p(x)\,W(x)}$$

wherefrom we can obtain expressions for $v_1(x)$, $v_2(x)$. (The function $W(x) = y_1y_2' - y_2y_1'$ is the Wronkian of the solutions y_1, y_2). Therefore on substitution in $y_p = v_1(x)\,y_1(x) + v_2(x)\,y_2(x)$, we obtain

$$y_p(x) = y_1(x)\int \frac{-y_2(x)f(x)}{p(x)W(x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{p(x)W(x)}\,dx$$

which can also be put in the form

$$y_p(x) = y_2(x)\int_{x_1}^{x} \frac{y_1(t)f(t)}{p(t)W(t)}\,dt - y_1(x)\int_{x_2}^{x} \frac{y_2(t)f(t)}{p(t)W(t)}\,dt \tag{8.3.5}$$

where the lower limits can be determined from the initial conditions $y_p(0) = 0$, $y_p'(0) = 0$. To do this we put $x = 0$ in (8.3.5) and obtain

$$y_p(0) = y_2(0)\int_{x_1}^{0} \frac{y_1(0)f(t)}{p(t)W(t)}\,dt - y_1(0)\int_{x_2}^{0} \frac{y_2(t)f(t)}{p(t)W(t)}\,dt \tag{8.3.6}$$

In (8.3.6) y_1, y_2 are linearly independent solutions of the homogeneous DE. Let us suppose that $y_1(0) = 0$ and $y_2(0) \neq 0$. Then from (8.3.6)

$$y_p(0) = y_2(0)\int_{x_1}^{0} \frac{y_1(0)f(t)}{p(t)W(t)}\,dt \tag{8.3.7}$$

In (8.3.7), if we put $x_1 = 0$, we get $y_p(0) = 0$.

Next we will consider the case when $y_p'(0) = 0$. Then from (8.3.5) by differentiation we obtain

$$\begin{aligned} y_p'(x) &= y_2'(x)\int_{0}^{x} \frac{f(t)y_1(t)}{p(t)W(t)}\,dt + y_2(x)\,\frac{p(x)y_1(x)}{p(x)W(x)} \\ &- y_1'(x)\int_{x_2}^{x} \frac{f(t)y_2(t)}{p(t)W(t)}\,dt - y_1(x)\,\frac{p(x)y_2(x)}{p(x)W(x)} \end{aligned}$$

which on simplification becomes

where we have used Leibnitz formula for differentiation under the integral

$$\frac{d}{dx}\int_0^{\alpha(x)} f(t)dt = \alpha'(x)\, f(\alpha(x))$$

If we put $x = 0$ on both sides of (8.3.8), we have

$$y_p'(0) = -y_1'(0)\int_{x_2}^{0} \frac{f(t)y_2(t)}{p(t)W(t)}\, dt \tag{8.3}$$

The condition $y_p'(0) = 0$ will give $x_2 = 0$, provided that $y_1'(0) \neq 0$.

Therefore we have found that

$$y_p(x) = y_2(x)\int_0^x \frac{y_1(t)f(t)}{p(t)W(t)}\, dt - y_1(x)\int_0^x \frac{f(t)y_2(t)}{p(t)W(t)}\, dt$$

which can also be put in the form

$$y_p(x) = \int_0^x \frac{y_1(t)y_2(x) - y_1(x)\, y_2(t)}{p(t)W(t)}\, f(t)\, dt \tag{8.3.}$$

The above expression for particular function $y_p(x)$ can also be written as

$$y_p(x) = \int_0^x G(x,\ t)\, f(t)\, dt \tag{8.3.}$$

where

$$G(x,\ t) = \frac{y_1(t)y_2(x) - y_1(x)\, y_2(t)}{p(t)W(t)} \tag{8.3.1}$$

is Green's function associated with the initial value problem.

Summary

The material discussed above can be summarized in the following theore which contains the procedure for constructing Green's function.

Theorem

The solution of the initial-value problem (8.3.4) is given by

$$y(x) = y_c(x) + \int_0^x G(x,\ t)\, f(t)\, dt \tag{8.3.13}$$

where

$$G(x,\ t) = \frac{y_1(t)y_2(x) - y_1(x)\, y_2(t)}{p(t)W(t)} \tag{8.3.14}$$

and solution of the homogeneous DE is $y_c = c_1y_1 + c_2y_2$ subject to initia conditions $y_c(0) = y_0,\ y_c'(0) = v_0$.

We illustrate the method by examples bel

8.3.2 Green's functions associated with B.V. problems

In this subsection we will discuss Green's function associated with the inhomogeneous S L equation

$$\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\}+q(x)\,y=f(x),\quad a<x<b \tag{8.3.15}$$

with B.Cs. in the form $y(a)=0,\ y(b)=0$.

It may be pointed out that general theory will be applicable to other forms of homogeneous B.Cs. We start with the solution in the form $y=y_c+y_p=c_1y_1+c_2y_2+y_p$, or written in full

$$y(x)=c_1y_1+c_2y_2+y_2(x)\int_{x_1}^{x}\frac{y_1(t)f(t)}{p(t)W(t)}\,dt-y_1(x)\int_{x_2}^{x}\frac{y_2(t)f(t)}{p(t)W(t)}\,dt$$

We will absorb the coefficients $c_1,\ c_2$ into the integrals with limits $x_1,\ x_2$ in such a way that the solution can be written as a single integral involving Green's function. In other words

$$y(x)=y_2(x)\int_{x_1}^{x}\frac{y_1(t)f(t)}{p(t)W(t)}\,dt-y_1(x)\int_{x_2}^{x}\frac{y_2(t)f(t)}{p(t)W(t)}\,dt \tag{8.3.16}$$

In this form the constants $x_1,\ x_2$ are new constants.

Next we will impose conditions on the solutions of the homogeneous S L equation such that $y_1(a)=0,\ y_2(b)=0$ and $y_2(a)\neq 0,\ y_1(b)\neq 0$.

From (8.3.16)

$$\begin{aligned} y(a) &= y_2(a)\int_{x_1}^{a}\frac{y_1(t)f(t)}{p(t)W(t)}\,dt-y_1(a)\int_{x_2}^{a}\frac{y_2(t)f(t)}{p(t)W(t)}\,dt\\ &= y_2(a)\int_{x_1}^{a}\frac{y_1(t)f(t)}{p(t)W(t)}\,dt \end{aligned} \tag{8.3.17}$$

When $x=b$ we find that

$$y(b)=-y_1(b)\int_{x_2}^{b}\frac{y_2(t)f(t)}{p(t)W(t)}\,dt$$

The two equations for $y(a),\ y(b)$ will vanish when $x_1=a$ and $x_2=b$. In view of these results we can write

$$y(x)=y_2(x)\int^{x}\frac{y_1(t)f(t)}{p(t)W(t)}\,dt-y_1(x)\int_{b}^{x}\frac{y_2(t)f(t)}{p(t)W(t)}\,dt \tag{8.3.18}$$

Now we construct Green's function in such a way that the solution can be expressed as a single integral. From (8.3.18) with interchange of limits in the second integral on the right side, we have

$$\begin{aligned} y(x) &= y_2(x)\int_a^x \frac{y_1(t)f(t)}{p(t)W(t)}\,dt + y_1(x)\int_x^b \frac{y_2(t)f(t)}{p(t)W(t)}\,dt \\ &= \int_a^b G(x,\, t)\, f(t)\, dt \end{aligned} \tag{8.3.19}$$

where

$$G(x,\, t) = \begin{cases} y_1(t)y_2(x)/p(t)W(t), & a < t < x \\ y_1(x)y_2(t)/p(t)W(t), & x < t < b \end{cases} \tag{8.3.20}$$

8.3.3 Illustrative examples

In this subsection we will illustrate the method of constructing Green's function for initial and boundary value problems, discussed in the subsections above.

by Simple meth
y = 4 cos x + x Sin

Example 1

Find the solution of the forced harmonic oscillator problem.

$y''(t) + y(t) = 2\cos t$ subject to the initial conditions $y(0) = 4, \quad y'(0) = 0,$ where dash denotes differentiation w.r.t. time t.

Solution

The problem can be solved by one of the standard methods of solving a linear inhomogeneous ODE. However we will the method based on Green's function.

This method consists of two parts. First we solve the associated homogeneous DE, *viz.* $y_c''(t) + y_c(t) = 0$ subject to the given initial conditions $y_c(0) = 4, \quad y_c'(0) = 0$.

General solution of the homogeneous DE $y_c''(t) + y_c(t) = 0$ is $y_c = c_1 \cos t + c_2 \sin t$. Applying the I.C. $y_c(0) = 4$ we get $c_1 = 4$. Therefore $y_c = 4\cos t + c_2 \sin t$. The I.C. $y_c'(0) = 0$ gives $c_2 = 0$. Therefore $y_c = 4\cos t$.

Next we construct Green's function associated with the homogenous problem: $y'' + y = 0, \quad y(0) = 0, \quad y'(0) = 0$. For this we need two linearly independent solutions $y_1(t) = \sin t, \quad y_2(t) = \cos t$. These should satisfy the conditions $y_1(0) = 0, \quad y_2'(0) = 0$.

In this problem $p(t) = 1$ and $W(t) = y_1 y_2' - y_2 y_1' = -1$. Therefore

$$G(t,\ \tau) = \frac{y_1(\tau) y_2(t) - y_1(t) y_2(\tau)}{p(t) W(t)} = \sin t\ \cos\tau - \cos t\ \sin\tau$$ = Sin (t

Green's function of the given inhomogeneous problem is

$$\begin{aligned}
y_p(t) &= \int_0^t G(t,\ \tau)\, f(t)\, dt = \int_0^t (\sin t\ \cos\tau - \cos t\ \sin\tau)\,(2\cos\tau)\, d\tau \\
&= 2\sin t \int_0^t \cos^2\tau\, d\tau - 2\cos t \int_0^t \sin\tau\ \cos\tau\, d\tau \\
&= 2\sin t \int_0^t \frac{1+\cos 2\tau}{2}\, d\tau - 2\cos t \int_0^t \sin 2\tau\, d\tau \\
&= \sin t \left(t + \frac{\sin 2t}{2}\right) - \cos t \left.\frac{-\cos 2\tau}{2}\right|_0^t \\
&= t\sin t + \frac{1}{2}(\sin t\ \sin 2t + \cos t\ \cos 2) - \frac{1}{2}\cos t \\
&= t\sin t + \frac{1}{2}\cos(2t - t) - \frac{1}{2}\cos t \\
&= t\sin t
\end{aligned}$$

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Example 2

Solve the problem

$y''(x) = x^2$ with B.Cs. $y(0) = 0,\ y(1) = 0$, using the Green function associated with S L boundary value problem..

Solution

We first solve the homogeneous B.V. problem. The homogeneous $y'' = 0$ has solutions $y = c_1 x + c_2$ in each of the subintervals $0 \le x < t$ and $t < x \le 1$. We need two linearly independent solutions $y_1,\ y_2$, one for each subinterval. The B.C. $y_1(0) = 0$ gives $c_2 = 0$. Therefore $y_1 = c_1 x$. We take $c_1 = 1$ and therefore $y_1 = x$.

Again for the interval $t < x \le 1,\ \ y_2 = c_1 x + c_2$. The B.C. $y(1) = 0$ gives $c_2 = -c_1$. Therefore $y_2 = c_1(x-1)$. We take $c_1 = -1$ and obtain $y_2 = 1 - x$.

For the given DE $p(x) = 1$ and $W(x) = y_1\, y_2' - y_2\, y_1' = 1$. Therefore

$$G(x,\ t) = \begin{cases} -t(1-x), & 0 \le t < x \\ -x(1-t), & x < t \le 1 \end{cases}$$

e solution is given by

$$\begin{aligned} y(x) &= \int_0^1 G(x,\ t)\, f(t)\, dt \\ &= -\int_0^x t(1-x)\, t^2\, dt - \int_x^1 x(1-t)\, t^2\, dt \\ &= -\left(t^4/4 - xt^4/4\right)\Big|_0^x - \left(xt^3/3 - xt^4/4\right)\Big|_x^1 \\ &= (x^4 - x)/12 \end{aligned}$$

Example 3

Construct Green's function the following B.V. problem

$y''(x) + k^2 y(x) = f(x)$ with B.Cs. $y(0) = 0,\ \ y(\ell) = 0$, using the method discussed in subsection 8.3.*.

Solution

We first find two linearly independent solutions $y_1(x),\ y_2(x)$ for the intervals $0 \le x < t$ and $t < x \le \ell$. Then $G(x,\ t) = y_1(t)\, y_2(x)/p(t)\, W(t)$

Here $L = d^2/dx^2 + k^2$, $p(x) = 1$ and the DE $L\, G(x,\ t) = \delta(x - t)$ shows that $y'' + k^2 y = 0$ for $x \neq 0$. General solution of this equation can be written as

$y_c(x) = c_1(t) \cos kx + c_2(t) \sin kx$. Therefore we can write

$y_1 = c_1(t) \cos kx + c_2(t) \sin kx,\ 0 \le x < t$

and

$y_2(x) = c_3(t) \cos kx + c_4(t) \sin kx,\ t < x \le \ell$

The B.C. $y_1(0) = 0$ gives $c_1 = 0$, therefore $y_1 = c_2(t) \sin kx$.

The B.C. $y_2(\ell) = 0$ gives $c_3 \cos k\ell + c_4 \sin k\ell = 0$ wherefrom $c_4 = -c_3(\cos k\ell)/(\sin k\ell)$.

On substitution for c_4 we obtain

$$y_2 = (c_3/\sin k\ell)\, \sin k(x - \ell) = d(t)\, \sin k(x - \ell)$$

Using the formula

$$G(x,\ t) = \begin{cases} y_1(t)\, y_2(x)/p(t)\, W(t), & 0 \le x < t \\ y_1(x)\, y_2(t)/p(x)\, W(x), & t < x \le \ell \end{cases}$$

we obtain

$$G(x,\ t) = \begin{cases} c_2(t) \sin kx, & 0 \le x < t \end{cases}$$

Now we apply the continuity condition at $x = t$ and obtain

$c_2(t)\sin kt = d(t)\sin k(t-\ell)$ which can be expressed as

$$\frac{c_2(t)}{\sin k(t-\ell)} = \frac{d(t)}{\sin kt} = \lambda$$

where the constant λ can be used to express each of the constants c_2, d in terms of it. We obtain $c_2 = \lambda\, sink(t-\ell)$, $d = \lambda \sin kt$. Hence Green's function can be written as

$$G(x,\ t) = \begin{cases} \lambda \sin kx \sin k((t-\ell), & 0 \leq x < t \\ \lambda \sin k((x-\ell) \sin kt, & t < x \leq \ell \end{cases}$$

To determine the constant λ we use the jump discontinuity and obtain

$\lambda k[\cos k(t-\ell)\sin kt - \sin k(t-\ell)\cos kt] = 1$ or $\lambda k \sin(kt - kt + k\ell) = 1$

wherefrom $\lambda = 1/(k \sin k\ell)$. Therefore finally

$$G(x,\ t) = \begin{cases} \sin kx \sin k((t-\ell)/1/(k \sin k\ell), & 0 \leq x < t \\ \sin k((x-\ell) \sin kt/1/(k \sin k\ell), & t < x \leq \ell \end{cases}$$

We note that $g(x,\ t) = G(t,\ x)$. This symmetry property has followed from the continuity property and is not an independent property.

8.3.4 Exercises

Q1 7 Q3

✓ 1. Use the formula of Green's functions associated with initial value problems, discussed in subsections 8.3.1, to solve the following problems.

(a) $y'' + 2y' - 8y = 2\exp(3x),\ y(0) = 1,\ y'(0) = -2/7$

(b) $y'' + 2y' + y = \exp(2x),\ y(0) = 0,\ y'(0) = 0$

(c) $xy'' - 3y' = 4x - 6,\ y(1) = 0,\ y'(1) = 1$

✓ 2. Use the formula of Green's functions associated with boundary value problems, discussed in subsections 8.3.2, to solve the following problems.

(a) $y'' + y = f(x),\ y(0) = 0,\ y(b) = 0$

What happens in the case when $b = n\pi$, where n is an integer?

Use the Green's function obtained above to solve the problem $y'' + y' = x,\ y(0) = 0,\ y(\pi/2) = 0$

✓ 3. Find the solution of the following initial value problems, by method of Green's function.

$3y' + 2y = 20\ \exp(-2x), \quad y(0) = 0,\ y'(0) = 6.$

$y = 2\sin 3x, \quad y(0) = 5,\ y'(0) = 0.$

$y = 1 + 2\cos x, \quad y(0) = 2,\ y'(0) = 0.$

$- 2xy + 2y = 3x^2 - x, \quad y(1) = \pi,\ y'(1) = 0.$

the problem $y'' = \sin x,\ y'(0) = 5,\ y(\pi) = 0$

the solution of the problem by direct integrating and then using the

ermine the associated Green's function and then solve the problem.

hen the above B.Cs. are replaced with $y'(0) = 5,\ y(\pi) = -3$ then he calculations of (1) and (ii).

n the problem

$x^2 = \delta(x - x_0), \quad (\partial G/\partial x)\,(0,\ x_0) = 0, \quad G(\pi,\ x_0) = 0.$

ve by direct integration.

nd Green's function and compare it to that determined in part (ii) of se 4 above.

en the B.V. problem $y'' - y = x$ with B.Cs. $y(0) = 0,\ y(1) = 0.$

nd a solution in closed form without using Green's function.

ind Green's function and compare it to that determined in part (ii) of ise 4 above.

Green's Function in the General Case

ave constructed Green's functions associated with the specific initial and ndary value problems, using the formulas of subsections (8.3.1) and (8.3.2).

now extend this method to the case of an SL system of the form $y + \lambda y =$) where $L = (d/dx)\ [p(x)d/dx] + q(x)$

note that if $\lambda = 0$, the problem becomes $y + \lambda y = f(x)$ with the same S L erator L. If the solution of the associated homogeneous problem is trivial each case and this can be seen by first writing the general solution and that $\lambda = 0$ is **not** an eigenvalue of the problem,

Then the general theorem which guarantees the existence of the Green functio in such problems is stated below.

Theorem

If the homogeneous problem associated with the SL problem

$$(py')' + q(x)\, y = f(x)$$

with usual B.Cs. has trivial solution, then Green's function exists.

In other words if $\lambda = 0$, is not an eigenvalue for $L(y) + \lambda\, r(x)\, y = 0$, with usua B.Cs., then Green's function exists.

We have to solve the problem associated with nonhomogeneous differential equation

$$L\{y(x)\} + \lambda\, r(x)\, y(x) = f(x) \qquad (8.4.1)$$

where $L = (d/dx)\ [p(x)d/dx] + q(x)$ and $y(x)$ satisfies suitable boundary conditions.

The solution of the nonhomogeneous differential equation (8.2.2) subject to B.Cs. is closely related to the existence of Green's function associated with the homogeneous equation.

$$L(y) + \lambda\, r(x)\, y = 0$$

If a function $G(x,\ t,\ \lambda)$ which does not depend on the source function $f(x)$ exists, then the solution of (8.4.1) can be written as

$$y(x) = \int_a^b G(x,\ t,\ \lambda)\ f(t)\ dt$$

$G(x,\ t,\ \lambda)$ is called *Green's function* and satisfies the equation $L(G) = \delta(x-t)$.

8.4.1 Green's function associated with regular S.L. system

Let

$$L(y) + \lambda\, r(x)\, y = 0 \qquad (8.4.2)$$

be the S-L equation with endpoint conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1\, y(b) + \beta_2\, y'(b) = 0 \qquad (8.4.3)$$

$$\mathcal{B}_{1(y)} = 0, \text{ and } \mathcal{B}_{2(y)} = 0$$

where $L = (d/dx)\,[p(x)(d/dx)] + q(x)$ and $\mathcal{B}$ is the boundary conditi

tor.

$\mathcal{B}_{1(y)} = \alpha_1 + \alpha_2\,(\partial/\partial x)$ and $\mathcal{B}_{2(y)} = \beta_1 + \beta_2\,(\partial/\partial x)$

The equations (8.4.1) and (8.4.2) define a regular S. L. system. U assumption that $\lambda = 0$ is not an eigenvalue of this system, i.e. it g ial solution, the Green function $G(x,\ t)$ associated with the systen following properties.

1. $G(x,\ t)$ considered as a function of x satisfies the differential $L\{G(x,\ t)\} = 0$ in each of the subintervals $[a,\ t)$ and $(t,\ b]$.

2. $G(x,\ t)$ is continuous for each value of x in the whole interval [if we take limit as x approaches t for each piece of the solution in t subintervals, then the limits should be equal.

3. $G(x,\ t)$ as a function of x satisfies the end-point conditions $\mathcal{B}_1$ and $\mathcal{B}_2\,(G) = 0$.

4. $G' \equiv dG(x,\ t)/dx$ is discontinuous as $x \to t$ and moreover

$$\lim_{x \to t+0} G'(x,\ t) - \lim_{x \to t-0} G'(x,\ t) = \frac{1}{p(t)}$$

8.4.2 Illustrative examples

Example 1 ✓

Construct Green's function associated with the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0$$

Solution

Here $p(x) = 1$, therefore $p(t) = 1$.

(i). First we verify if $\lambda = 0$ is not an eigenvalue. With $\lambda = 0$ we have $y'' = 0$ which gives $y = Ax + B$.

Now we apply the B.Cs. The B.C. $y(0) = 0$ gives $0 = 0 + B$ i.e. Similarly $y(1) = 0$ gives $A = 0$.

Therefore $y = 0$ is the solution of the problem corresponding to $\lambda = 0$. Th
fore $\lambda = 0$ is not an eigenvalue.

(ii). $G(x,\ t)$ satisfies the differential equation $[d^2G(x,\ t)/dx^2] = 0$, in e
of the subintervals $[0,\ t]$ and $(t,\ 1]$. Therefore we have

$$G(x,\ t) = \begin{cases} Ax + B, & 0 \le x \le t \\ A'x + B', & t < x \le 1 \end{cases}$$

(iii). $G(x,\ t)$ is continuous everywhere and in particular at $x = t$. Therefo

$At + B = A't + B'$ or $B' = (A - A')t + B$. The Green function therefore
be written as

$$G(x,\ t) = \begin{cases} Ax + B, & 0 \le x < t \\ A'x + (A - A')t + B, & t < x \le 1 \end{cases}$$

(iv). $G(x,\ t)$ satisfies the endpoint conditions $G(0,\ t) = 0$ and $G(1,\ t) =$

These give $A \times 0 + B = 0$ i.e. $B = 0$.

and $A' + (A - A')t = 0$ or $A = [(t-1)/t]\, A'$.

Therefore on substitution for A and A', we have

$$G(x,\ t) = \begin{cases} A'\,(t - 1)x/t, & 0 \le x < t \\ A'x + (-A'/t)t = A'x - A', & t < x \le 1 \end{cases}$$

(v). The discontinuity condition for $G'(x,\ t)$ gives

$$G'(t+0,\ t) - G'(t-0,\ t) = [1/p(t)]$$

or $A' - (A'/t)\ (t-1) = 1/p(t)$ or $A't - A't + A' = t$ i.e. $A' = t$.

On substitution we get

$$G(x,\ t) = \begin{cases} (1-t)x, & 0 \le x \le t \\ t(1-x), & t < x \le 1 \end{cases}$$

Example 2

Construct Green's function associated with the problem

$$y'' + k^2\, y = 0, \quad y(0) = 0, \quad y(\pi/2k) = 0$$

Solution

Here $p(x) = 1$, $L \equiv \partial^2/\partial x^2 + k^2$. Green's function $G(x,\ t)$ for the linea
operator L is defined as solution to the DE

$$G''(x,\ t) + k^2 G(x,\ t) = \delta(x - t)$$

with B.Cs. $G(0,\ t) = 0,\ G(\pi/2k,\ t) = 0$. Since $\delta(x - t) = 0$ for above equation is equivalent to the DE $G''(x,\ t) + k^2 G(x,\ t) = 0$ in the subintervals $[0,\ t)$ and $(t,\ \pi/2k]$.

General solution for each subinterval will be of the form $y = C$ $D \sin kx$.

For the subinterval $[0,\ t)$ we take general solution as $G(x,\ t) = c_1$ $c_2 \sin kx$ and apply the B.C. $G(0,\ t) = 0$ to it. This gives $c_1 = 0$. Hen

$G(x,\ t) = c_2 \sin kx,\quad 0 \le x < t$

For the subinterval $(t,\ \pi/2k]$ we take general solution as $G(x,\ t) = c_3$ $c_4 \sin kx$ and apply the B.C. $G(\pi/2k,\ t) = 0$ to it. This gives

$c_3 \cos(\pi/2) + c_4 \sin(\pi/2) = 0$, wherefrom $c_4 = 0$. Hence

$G(x,\ t) = c_3 \cos kx,\quad t < x \le \pi/2k$

The above two solutions can be combined in the form

$$G(x,\ t) = \begin{cases} c_2 \sin kx, & x < t \\ c_3 \cos kx, & x > t \end{cases}$$

To determine the unknown constants $c_2,\ c_3$ we will use the continuity con and the jump discontinuity condition.

The continuity condition gives $c_2 \sin kt = c_3 \cos kt$, wherefrom $c_3 = ($ Therefore

$$G(x,\ t) = \begin{cases} c_2 \sin kx, & x < t \\ c_2 \tan kt \cos kx, & x > t \end{cases}$$

Finally we apply the discontinuity condition $G'(t + 0,\ t) - G'(t -$ $1/p(t)$ and obtain $c_2 = -(\cos kt)/k$. Hence we obtain Green's function problem as

$$G(x,\ t) = \begin{cases} -(\cos kt \sin kx)/k, & x < t \\ -(\sin kt \cos kx)/k, & x > t \end{cases}$$

Example 3

Find Green's function associated with the problem

$$xy'' + y' + \lambda r(x)\, y = 0, \quad y(0) \text{ is finite and } y(1) = 0$$

Solution

This is a singular SL- system with $p(x) = x$.

(i). First we check if $\lambda = 0$ is an eigenvalue.

$\lambda = 0$ implies that $x\,y'' + y' = 0$ or $(d/dx)\ (x\,y') = 0$ which gives

$x\,y' = A$ or $y' = A/x$ or $y = A \ln x + B$, where $A,\ B$ are constants.

Now we apply the B.Cs. The B.C. $y(0)$ is finite gives $A = 0$. The B.C. $y(1) = 0$ gives $B = 0$.

Therefore $y = 0$ is the only possible solution. Hence $\lambda = 0$ is not an eigenvalue.

(ii). $G(x,\ t)$, regarded as a function of x satisfies the given D.E. *i.e.* $xG'' + G' = 0$ in each of the sub-intervals $(0,\ t]$ and $(t,\ 1]$. Therefore we can write

$$G(x,\ t) = \begin{cases} A + B \ln x, & 0 < x < t \\ A' + B' \ln x, & t < x \leq 1 \end{cases}$$

(iii). $G(x,\ t)$ as a function of x satisfies the B.Cs. $G(0,\ t)$ is finite, and $G(1,\ t) = 0$.

The first condition gives $B = 0$, and the second condition gives $A' + B' \cdot 0 = 0$ or $A' = 0$. Hence

$$G(x,\ t) = \begin{cases} A, & 0 < x \leq t \\ B' \ln x, & t < x \leq 1 \end{cases}$$

(iv). $G(x,\ t)$ as a function of x is continuous at all points and in particular at $x = t$. This gives $A = B' \ln t$.

or $A/\ln t = B'/1 = \rho$ which gives $A = \rho \ln t,\ B' = \rho$.

Therefore

$$G(x,t) = \begin{cases} \rho \ln t, & 0 < x \leq t \\ \rho \ln x, & t < x \leq 1 \end{cases}$$

(v). $G'(t - 0,\ t) - G'(t + 0,\ t) = 1/p(t)$ or $0 - \rho/t = 1/t$ i.e. $\rho = -1$.

e finally

$$G(x,\ t) = \begin{cases} -\ln t, & 0 < x < t \\ -\ln x, & t < x \le 1 \end{cases}$$

nple 4

truct Green's function for the B.V.P.

$y' - (n^2/x)\,y + \lambda r(x)\,y = 0$, $y(0)$ is finite and $y(1) = 0$.

tion

$p(x) = x$, $q(x) = -n^2/x$.

o construct Green's function we first check if $\lambda = 0$ is an eigenvalue of the geneous problem (obtained by putting $\lambda = 0$ in the given problem).

$y' - (n^2/x)\,y = 0$ or $x^2\,y'' + x\,y' - n^2 y = 0$, $(n > 0)$

is the Euler - Cauchy equation. To solve it we make the transformation ^{p}, and obtain on substitution

$$\{p(p-1) + p - n^2\}x^p = 0$$

gives $p = \pm n$

e the general solution can be written as $y = A\,x^n + B\,x^{-n}$.

we apply the B.Cs. We find that the condition $y(0)$ is finite gives $B = 0$, $y(1) = 0$ gives $A = 0$.

e the only possible solution is the trivial solution $y = 0$, and therefore 0 is not an eigenvalue. Therefore we can associate Green's function to the m.

Green's function $G(x,\ t)$ regarded as a function of x satisfies the DE

$$xG'' + G' - (n^2/x)\,G = 0$$

ch of the subintervals $[0,\ t]$ and $(t,\ 1]$. Therefore it can be written as

$$G(x,\ t) = \begin{cases} Ax^n + Bx^{-n}, & 0 \le x \le t \\ A'x^n + B'x^{-n}, & t < x \le 1 \end{cases}$$

$G(x,\ t)$ regarded as a function of x satisfies the given B.Cs.

$t)$ is finite and $G(1,\ t) = 0$. These B.Cs. give $B = 0$ and $B' = -A'$. refore

$$G(x,\ t) = \begin{cases} Ax^n, & 0 \le x \le t \\ A'(x^n - x^{-n}), & t < x \le 1 \end{cases}$$

(iv). $G(x,\ t)$ as a function of x is continuous at all points and in particular at $x = t$. So

$$A\,t^n = A'(t^n - t^{-n}) \quad \text{or} \quad At^n = A'\frac{t^{2n}-1}{t^n} \quad \text{or} \quad A = A'\frac{t^{2n}-1}{t^{2n}}$$

Hence we write

$$G(x,\ t) \;=\; \begin{cases} A'\,x^n\,(1 - t^{-2n}), & 0 \le x \le t \\ A'\,(x^n - x^{-n}), & t < x \le 1 \end{cases}$$

(v). $G'(t-0,\ t) - G'(t+0,\ t) = 1/p(t)$. Or

$$n\,A'\,t^{n-1}\,t^{-2n} - nA'\,(t^{n-1} + t^{-n-1}) = \frac{1}{t}$$

which on simplification gives $A' = -t^n/2n$.

Therefore finally

$$G(x,\ t) = \begin{cases} -(1/2n)\ (t^n - t^{-n})\ x^n, & 0 \le x \le t \\ -(1/2n)\ (x^n - x^{-n})\ t^n, & t < x \le 1 \end{cases}$$

Example 5

Construct Green's function for the B.V.P.

$(d/dx)\{(1-x^2)y'\} - [(h^2/(1-x^2)]\,y + \lambda\,r(x)\,y = 0,\ \ y(\pm 1)$ are finite.

Solution

This is a singular SL system with $p(x) = 1 - x^2$.

(i) First we check if $\lambda = 0$ is an eigenvalue, i.e. we solve the DE:

$$\frac{d}{dx}\{(1-x^2)y'\} - \frac{h^2}{1-x^2}\,y = 0$$

or

$$(1-x^2)\,y'' - 2xy' - \frac{h^2}{1-x^2}\,y = 0$$

Making the substitution

$t = \ln[(1+x)/(1-x)] = \ln(1+x) - \ln(1-x)$, we have

$$\frac{dt}{dx} = \frac{1}{1+x} - (-1)\frac{1}{1-x} = \frac{2}{1-x^2}$$

Therefore

$$\frac{d}{dx} = \frac{2}{1-x^2}\frac{d}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{1}{2}(1-x^2)$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\frac{dy}{dx} \\
&= \frac{2}{1-x^2}\frac{d}{dt}\frac{2}{1-x^2}\frac{dy}{dt} \\
&= \frac{2}{1-x^2}\frac{2}{1-x^2}\frac{d^2y}{dt^2} \\
&+ \frac{dx}{dt}\frac{d}{dx}\frac{2}{1-x^2}\frac{dy}{dt} \\
&= \frac{2}{1-x^2}\frac{2}{1-x^2}\frac{d^2y}{dt^2} \\
&+ \frac{4x}{(1-x^2)^2}\left(\frac{dx}{dt}\right)^2\frac{dy}{dt}
\end{aligned}$$

n substituting the value of dx/dt we have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{2}{1-x^2}\frac{2}{1-x^2}\frac{d^2y}{dt^2} \\
&+ \frac{4x}{(1-x^2)^2}\frac{(1-x^2)^2}{4}\frac{dy}{dt} \\
&= \frac{4}{(1-x^2)^2}\left[\frac{d^2y}{dt^2} + x\frac{dy}{dt}\right]
\end{aligned}$$

tituting for dy/dt and d^2y/dt^2 in the given DE, and simplifying, w
$^2y/dt^2 - (h^2/4)\,y = 0$.

solution of this equation can be written as

$$\begin{aligned}
y &= A\exp(ht/2) + B\exp(-ht/2) \\
&= A\,[(1+x)/(1-x)]^{h/2} + B\,[(1-x)/(1+x)]^{h/2}
\end{aligned}$$

apply the B.Cs.

C. $y(1) =$ finite gives $A = 0$, and $y(-1) =$ finite gives $B = 0$.

$u = 0$ is the only solution for the associated homogeneous BV problem.