$$\mathcal{B}_{1(y)} = 0, \quad \text{and} \quad \mathcal{B}_{2(y)} = 0$$

where $L = (d/dx)\,[p(x)(d/dx)] + q(x)$ and $\mathcal{B}$ is the boundary conditi tor.

$\mathcal{B}_{1(y)} = \alpha_1 + \alpha_2\,(\partial/\partial x)$ and $\mathcal{B}_{2(y)} = \beta_1 + \beta_2\,(\partial/\partial x)$

The equations (8.4.1) and (8.4.2) define a regular S. L. system. U assumption that $\lambda = 0$ is not an eigenvalue of this system, i.e. it g ial solution, the Green function $G(x,\ t)$ associated with the systen following properties.

1. $G(x,\ t)$ considered as a function of x satisfies the differential $L\{G(x,\ t)\} = 0$ in each of the subintervals $[a,\ t)$ and $(t,\ b]$.

2. $G(x,\ t)$ is continuous for each value of x in the whole interval [if we take limit as x approaches t for each piece of the solution in t subintervals, then the limits should be equal.

3. $G(x,\ t)$ as a function of x satisfies the end-point conditions $\mathcal{B}_1$ and $\mathcal{B}_2\,(G) = 0$.

4. $G' \equiv dG(x,\ t)/dx$ is discontinuous as $x \to t$ and moreover

$$\lim_{x\to t+0} G'(x,\ t) - \lim_{x\to t-0} G'(x,\ t) = \frac{1}{p(t)}$$

8.4.2 Illustrative examples

Example 1

Construct Green's function associated with the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0$$

Solution

Here $p(x) = 1$, therefore $p(t) = 1$.

(i). First we verify if $\lambda = 0$ is not an eigenvalue. With $\lambda = 0$ we have $y'' = 0$ which gives $y = Ax + B$.

Now we apply the B.Cs. The B.C. $y(0) = 0$ gives $0 = 0 + B$ i.e. B Similarly $y(1) = 0$ gives $A = 0$.

Therefore $y = 0$ is the solution of the problem corresponding to $\lambda = 0$. Th fore $\lambda = 0$ is not an eigenvalue.

(ii). $G(x,\ t)$ satisfies the differential equation $[d^2G(x,\ t)/dx^2] = 0$, in e of the subintervals $[0,\ t]$ and $(t,\ 1]$. Therefore we have

$$G(x,\ t) = \begin{cases} Ax + B, & 0 \le x \le t \\ A'x + B', & t < x \le 1 \end{cases}$$

(iii). $G(x,\ t)$ is continuous everywhere and in particular at $x = t$. Theref

$At + B = A't + B'$ or $B' = (A - A')t + B$. The Green function therefore be written as

$$G(x,\ t) = \begin{cases} Ax + B, & 0 \le x < t \\ A'x + (A - A')t + B, & t < x \le 1 \end{cases}$$

(iv). $G(x,\ t)$ satisfies the endpoint conditions $G(0,\ t) = 0$ and $G(1,\ t) =$

These give $A \times 0 + B = 0$ i.e. $B = 0$.

and $A' + (A - A')t = 0$ or $A = [(t-1)/t]\,A'$.

Therefore on substitution for A and A', we have

$$G(x,\ t) = \begin{cases} A'(t-1)x/t, & 0 \le x < t \\ A'x + (-A'/t)t = A'x - A', & t < x \le 1 \end{cases}$$

(v). The discontinuity condition for $G'(x,\ t)$ gives

$$G'(t+0,\ t) - G'(t-0,\ t) = [1/p(t)]$$

or $A' - (A'/t)\ (t-1) = 1/p(t)$ or $A't - A't + A' = t$ i.e. $A' = t$.

On substitution we get

$$G(x,\ t) = \begin{cases} (1-t)x, & 0 \le x \le t \\ t(1-x), & t < x \le 1 \end{cases}$$

Example 2

Construct Green's function associated with the problem

$$y'' + k^2 y = 0, \quad y(0) = 0, \quad y(\pi/2k) = 0$$

Solution

Here $p(x) = 1$, $L \equiv \partial^2/\partial x^2 + k^2$. Green's function $G(x,\ t)$ for the linea operator L is defined as solution to the DE

$$G''(x, t) + k^2 G(x, t) = \delta(x - t)$$

with B.Cs. $G(0, t) = 0$, $G(\pi/2k, t) = 0$. Since $\delta(x - t) = 0$ for x above equation is equivalent to the DE $G''(x, t) + k^2 G(x, t) = 0$ in the subintervals $[0, t)$ and $(t, \pi/2k]$.

General solution for each subinterval will be of the form $y = C$ $D \sin kx$.

For the subinterval $[0, t)$ we take general solution as $G(x, t) = c_1$ $c_2 \sin kx$ and apply the B.C. $G(0, t) = 0$ to it. This gives $c_1 = 0$. Hen

$G(x, t) = c_2 \sin kx, \quad 0 \le x < t$

For the subinterval $(t, \pi/2k]$ we take general solution as $G(x, t) = c_3$ $c_4 \sin kx$ and apply the B.C. $G(\pi/2k, t) = 0$ to it. This gives

$c_3 \cos(\pi/2) + c_4 \sin(\pi/2) = 0$, wherefrom $c_4 = 0$. Hence

$G(x, t) = c_3 \cos kx, \quad t < x \le \pi/2k$

The above two solutions can be combined in the form

$$G(x, t) = \begin{cases} c_2 \sin kx, & x < t \\ c_3 \cos kx, & x > t \end{cases}$$

To determine the unknown constants c_2, c_3 we will use the continuity con and the jump discontinuity condition.

The continuity condition gives $c_2 \sin kt = c_3 \cos kt$, wherefrom $c_3 = (c_2$ Therefore

$$G(x, t) = \begin{cases} c_2 \sin kx, & x < t \\ c_2 \tan kt \cos kx, & x > t \end{cases}$$

Finally we apply the discontinuity condition $G'(t + 0, t) - G'(t -$ $1/p(t)$ and obtain $c_2 = -(\cos kt)/k$. Hence we obtain Green's function problem as

$$G(x, t) = \begin{cases} -(\cos kt \sin kx)/k, & x < t \\ -(\sin kt \cos kx)/k, & x > t \end{cases}$$

Example 3

Find Green's function associated with the problem

$$xy'' + y' + \lambda r(x)\, y = 0, \quad y(0) \text{ is finite and } y(1) = 0$$

Solution

This is a singular SL- system with $p(x) = x$.

(i). First we check if $\lambda = 0$ is an eigenvalue.

$\lambda = 0$ implies that $x\,y'' + y' = 0$ or $(d/dx)\ (x\,y') = 0$ which gives

$x\,y' = A$ or $y' = A/x$ or $y = A \ln x + B$, where $A,\ B$ are constants.

Now we apply the B.Cs. The B.C. $y(0)$ is finite gives $A = 0$. The B.C. $y(1) = 0$ gives $B = 0$.

Therefore $y = 0$ is the only possible solution. Hence $\lambda = 0$ is not an eigenvalue.

(ii). $G(x,\ t)$, regarded as a function of x satisfies the given D.E. *i.e.* $xG'' + G' = 0$ in each of the sub-intervals $(0,\ t]$ and $(t,\ 1]$. Therefore we can write

$$G(x,\ t) = \begin{cases} A + B \ln x, & 0 < x < t \\ A' + B' \ln x, & t < x \leq 1 \end{cases}$$

(iii). $G(x,\ t)$ as a function of x satisfies the B.Cs. $G(0,\ t)$ is finite, and $G(1,\ t) = 0$.

The first condition gives $B = 0$, and the second condition gives $A' + B' \cdot 0 = 0$ or $A' = 0$. Hence

$$G(x,\ t) = \begin{cases} A, & 0 < x \leq t \\ B' \ln x, & t < x \leq 1 \end{cases}$$

(iv). $G(x,\ t)$ as a function of x is continuous at all points and in particular at $x = t$. This gives $A = B' \ln t$.

or $A/\ln t = B'/1 = \rho$ which gives $A = \rho \ln t,\ B' = \rho$.

Therefore

$$G(x,t) = \begin{cases} \rho \ln t, & 0 < x \leq t \\ \rho \ln x, & t < x \leq 1 \end{cases}$$

(v). $G'(t-0,\ t) - G'(t+0,\ t) = 1/p(t)$ or $0 - \rho/t = 1/t$ i.e. $\rho = -1$.

e finally

$$G(x,\ t) = \begin{cases} -\ln t, & 0 < x < t \\ -\ln x, & t < x \le 1 \end{cases}$$

nple 4

ruct Green's function for the B.V.P.

$y' - (n^2/x)\, y + \lambda r(x)\, y = 0$, $y(0)$ is finite and $y(1) = 0$.

tion

$p(x) = x,\ \ q(x) = -n^2/x.$

construct Green's function we first check if $\lambda = 0$ is an eigenvalue of the geneous problem (obtained by putting $\lambda = 0$ in the given problem).

$+ y' - (n^2/x)\, y = 0$ or $x^2\, y'' + x\, y' - n^2 y = 0,\ (n > 0)$

is the Euler - Cauchy equation. To solve it we make the transformation ^{p}, and obtain on substitution

$$\{p(p-1) + p - n^2\}x^p = 0$$

gives $p = \pm n$

e the general solution can be written as $y = A\, x^n + B\, x^{-n}$.

we apply the B.Cs. We find that the condition $y(0)$ is finite gives $B = 0$, $y(1) = 0$ gives $A = 0$.

e the only possible solution is the trivial solution $y = 0$, and therefore 0 is not an eigenvalue. Therefore we can associate Green's function to the m.

Green's function $G(x,\ t)$ regarded as a function of x satisfies the DE

$$xG'' + G' - (n^2/x)\, G = 0$$

ch of the subintervals $[0,\ t]$ and $(t,\ 1]$. Therefore it can be written as

$$G(x,\ t) = \begin{cases} Ax^n + Bx^{-n}, & 0 \le x \le t \\ A'x^n + B'x^{-n}, & t < x \le 1 \end{cases}$$

$G(x,\ t)$ regarded as a function of x satisfies the given B.Cs.

$t)$ is finite and $G(1,\ t) = 0$. These B.Cs. give $B = 0$ and $B' = -A'$. refore

$$G(x,\ t) = \begin{cases} Ax^n, & 0 \le x \le t \\ A'(x^n - x^{-n}), & t < x \le 1 \end{cases}$$

(iv). $G(x,\ t)$ as a function of x is continuous at all points and in particular at $x = t$. So

$$A\,t^n = A'(t^n - t^{-n}) \quad \text{or} \quad At^n = A'\frac{t^{2n}-1}{t^n} \quad \text{or} \quad A = A'\frac{t^{2n}-1}{t^{2n}}$$

Hence we write

$$G(x,\ t) \;=\; \begin{cases} A'\,x^n\,(1-t^{-2n}), & 0 \le x \le t \\ A'\,(x^n - x^{-n}), & t < x \le 1 \end{cases}$$

(v). $G'(t-0,\ t) - G'(t+0,\ t) = 1/p(t)$. Or

$$n\,A'\,t^{n-1}\,t^{-2n} - nA'\,(t^{n-1} + t^{-n-1}) = \frac{1}{t}$$

which on simplification gives $A' = -t^n/2n$.

Therefore finally

$$G(x,\ t) = \begin{cases} -(1/2n)\,(t^n - t^{-n})\,x^n, & 0 \le x \le t \\ -(1/2n)\,(x^n - x^{-n})\,t^n, & t < x \le 1 \end{cases}$$

Example 5

Construct Green's function for the B.V.P.

$(d/dx)\{(1-x^2)y'\} - [(h^2/(1-x^2)]\,y + \lambda\,r(x)\,y = 0,\ \ y(\pm 1)$ are finite.

Solution

This is a singular SL system with $p(x) = 1 - x^2$.

(i) First we check if $\lambda = 0$ is an eigenvalue, i.e. we solve the DE:

$$\frac{d}{dx}\{(1-x^2)y'\} - \frac{h^2}{1-x^2}\,y = 0$$

or

$$(1-x^2)\,y'' - 2xy' - \frac{h^2}{1-x^2}\,y = 0$$

Making the substitution

$t = \ln[(1+x)/(1-x)] = \ln(1+x) - \ln(1-x)$, we have

$$\frac{dt}{dx} = \frac{1}{1+x} - (-1)\frac{1}{1-x} = \frac{2}{1-x^2}$$

Therefore

$$\frac{d}{dx} = \frac{2}{1-x^2}\frac{d}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{1}{2}(1-x^2)$$

e

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\frac{dy}{dx} \\
&= \frac{2}{1-x^2}\frac{d}{dt}\frac{2}{1-x^2}\frac{dy}{dt} \\
&= \frac{2}{1-x^2}\frac{2}{1-x^2}\frac{d^2y}{dt^2} \\
&+ \frac{dx}{dt}\frac{d}{dx}\frac{2}{1-x^2}\frac{dy}{dt} \\
&= \frac{2}{1-x^2}\frac{2}{1-x^2}\frac{d^2y}{dt^2} \\
&+ \frac{4x}{(1-x^2)^2}\left(\frac{dx}{dt}\right)^2\frac{dy}{dt}
\end{aligned}$$

n substituting the value of dx/dt we have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{2}{1-x^2}\frac{2}{1-x^2}\frac{d^2y}{dt^2} \\
&+ \frac{4x}{(1-x^2)^2}\frac{(1-x^2)^2}{4}\frac{dy}{dt} \\
&= \frac{4}{(1-x^2)^2}\left[\frac{d^2y}{dt^2} + x\frac{dy}{dt}\right]
\end{aligned}$$

tituting for dy/dt and d^2y/dt^2 in the given DE, and simplifying, w
$^2y/dt^2 - (h^2/4)\,y = 0$.

solution of this equation can be written as

$$\begin{aligned}
y &= A\exp(ht/2) + B\exp(-ht/2) \\
&= A\,[(1+x)/(1-x)]^{h/2} + B\,[(1-x)/(1+x)]^{h/2}
\end{aligned}$$

apply the B.Cs.

C. $y(1)$ = finite gives $A = 0$, and $y(-1)$ = finite gives $B = 0$.

$u = 0$ is the only solution for the associated homogeneous BV problem.

(ii) The associated Green function $G(x,\ t)$ satisfies the DE

$$\frac{d}{dx}\{(1-x^2)G'\} - \frac{h^2}{1-x^2}G = 0$$

in each of the subintervals $-1 \le x < t$ and $t < x \le 1$. Therefore we can write

$$G(x,\ t) = \begin{cases} A\,[(1+x)/(1-x)]^{h/2} + B\,[(1-x)/(1+x)]^{h/2}, & x < t \\ A'\,[(1+x)/(1-x)]^{h/2} + B'\,[(1-x)/(1+x)]^{h/2}, & x > t \end{cases}$$

(iii) The Green function $G(x,\ t)$ satisfies the B.Cs.: $G(\pm 1,\ t)$ are finite in each of the two subintervals $[-1,\ t)$ and $(t,\ 1]$. In view of these conditions, we must have

$$G(x,\ t) = \begin{cases} A\,[(1+x)/(1-x)]^{h/2}, & -1 \le x < t \\ B'\,[(1-x)/(1+x)]^{h/2}, & t < x \le 1 \end{cases}$$

(iv) $G(x,\ t)$ is continuous at each value of x in the interval $[-1,\ 1]$. In particular it is continuous at $x = t$. This condition gives

$$A\left(\frac{1+t}{1-t}\right)^{h/2} = B'\left(\frac{1-t}{1+t}\right)^{h/2} = \rho, \quad \text{(say)}$$

Therefore

$$A = \rho\left(\frac{1-t}{1+t}\right)^{h/2}, \quad B' = \rho\left(\frac{1+t}{1-t}\right)^{h/2}$$

Hence

$$G(x,\ t) = \begin{cases} \rho\,[(1-t)/(1+t)]^{h/2}\,(1+x)/(1-x)]^{h/2}, & x < t \\ \rho\,[(1+t)/(1-t)]^{h/2}\,(1-x)/(1+x)]^{h/2}, & x > t \end{cases}$$

(v) To apply the discontinuity condition, we first calculate

$$\begin{aligned} G'(t-0,\ t) &= \lim_{x\to t-0} G'(x,\ t) \\ &= \rho\left(\frac{1-t}{1+t}\right)^{h/2}\left[\frac{h}{2}\cdot\left(\frac{1+x}{1-x}\right)^{h/2-1}\cdot\frac{2}{(1-x)^2}\right]\Bigg|_{x=t} \\ &= \rho\,h\,\frac{1-t}{1+t}\cdot\frac{1}{(1-t)^2} = h\,\rho/(1-t^2) \end{aligned}$$

and

$$G'(t+0,\ t) = \lim_{x\to t+0} G'(x,\ t)$$

$$\Big|_{x=t}$$

$$= -\rho h \, \frac{1+t}{1-t} \cdot \frac{1}{(1+t)^2} = -\frac{h\rho}{1-t^2}$$

fore $G'(t-0,\ t) - G'(t+0,\ t) = 1/p(t)$

$/(1-t^2) = 1/(1-t^2)$; which gives $\rho = 1/(2h)$.

$$x,\ t) = \begin{cases} (1/2h)\,[(1-t)/(1+t)]^{h/2}\,[(1\,\text{[illegible]}\,x)/(1-x)]^{h/2}, & x \le t \\ (1/2h)\,[(1+t)/(1-t)]^{h/2}\,[(1-x)/(1+x)]^{h/2}, & x > t \end{cases}$$

ple 6

een's function for the B.V.P. defined by the equations

$= 0,\ \ y(0) + y'(1) = 0,\ \ y(1) + 2y'(0) = 0$

on

$x) = 1,\ a = 0,\ b = 1,\ q(x) = 0,\ r(x) = 1.$

onstruct Green's function we first check if $\lambda = 0$ is an eigenvalue. The 0 obtained by substituting $\lambda = 0$ in the given DE has the solution $+ Bx$.

the given B.Cs., we obtain $A + B = 0$ and $A + B + 2B = 0$.

se two equations we obtain $A = B = 0$. Therefore $y = 0$ is the only olution possible. Hence $\lambda = 0$ is not an eigenvalue.

en's function $G(x,\ t)$ satisfies the equation $G'' = 0$ in each of the l $[0,\ t)$ and $(t,\ 1]$. Therefore we can write

$$G(x,\ t) = \begin{cases} A + Bx, & 0 \le x \le t \\ A' + B'x, & t < x \le 1 \end{cases}$$

$,\ t)$ as a function of x satisfies the given B.Cs.

$G'(1,\ t) = 0$, and $G(1,\ t) + 2G'(0,\ t) = 0$. Now

A and $G(1,\ t) = A' + B',\ \ G'(0,\ t) = B,\ \ G'(1,\ t) = B'$.

the boundary condition $G(0,\ t) + G'(1,\ t) = 0$ gives $B' = -A$, and $(1,\ t) + 2G'(0,\ t) = 0$ gives $A' = A - 2B$.

$$A - 2B - Ax, \quad t < x \leq 1$$

iv). $G(x,\ t)$ as a function of x is continuous at all points and in particular at $x = t$. So

$A + Bt = A - 2B - At$ or $B\,(t + 2) = -\,At$.

Putting $B(t+2) = -At = \rho$, we obtain $A = -(\rho/t)$, $B = \rho/(t+2)$.

Therefore on substitution for constants, we have

$$G(x,\ t) = \begin{cases} -\rho/t + \rho x/(2+t), & 0 \leq x \leq t \\ -\rho/t - 2\rho/(t+2) + \rho x/t, & t < x \leq 1 \end{cases}$$

v). Next the discontinuity condition $G'(t+0,\ t) - G'(t-0,\ t) = -1/p(t)$ yields $\rho/(2+t) - \rho/t = 1$ wherefrom $\rho = -t(t+2)/2$. Hence finally on substitution and simplification

$$G(x,\ t) = \begin{cases} -(tx + t + 2)/2, & 0 \leq x \leq 1 \\ -(tx + 2x - 3t - 2)/2, & t < x < 1 \end{cases}$$

Example 7

Find Green's function for the problem defined by the equations

$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0$

Solution

Here $p(x) = 1$

i). To construct Green's function $G(x,\ t)$ we first check if $\lambda = 0$ is an eigenvalue. In this case we have to solve the the DE $D^2 u = 0$ subject to the same boundary conditions.

This gives $y = A + Bx$ and $y' = B$.

Now we apply the B.Cs. The B.C. $y'(0) = 0$ gives $B = 0$, and the B.C. $y(1) = 0$ gives $A + B \cdot 1 = 0$ i.e. $A = 0$.

Hence we obtain the trivial solution $u = 0$, which implies that $\lambda = 0$ is not an eigenvalue.

ii). Green's function $G(x,\ t)$ as a function of x satisfies the equation $G'' = 0$ in each of the subintervals $[0,\ t]$ and $(t,\ 1]$. Therefore we can write

$$G(x,\ t) = \begin{cases} A + Bx, & 0 \leq x < t \\ A' + B'x, & t < x \leq 1 \end{cases}$$

(iii). $G(x,\ t)$ as a function of x satisfies the given B.Cs $G'(0,\ t) = 0$ and $G(1,\ t) = 0$.

Now the condition $G'(0,\ t) = 0$ gives $B = 0$, and the condition $G(1,\ t) = 0$ gives $A' + B' = 0$ i.e. $B' = -A'$.

Therefore we can write

$$G(x,\ t) = \begin{cases} A, & 0 \leq x \leq t \\ A'(1-x), & t < x \leq 1 \end{cases}$$

(iv). $G(x,\ t)$ as a function of x is continuous at all points and in particular at $x = t$. This condition gives $A = A' - A't$. Hence

$$G(x,\ t) = \begin{cases} A'(1-t), & 0 \leq x \leq t \\ A'(1-x), & t < x \leq 1 \end{cases}$$

(v). The discontinuity condition

$$G'(t-0,\ t) - G'(t+0,\ t) = 1/p(t)$$

gives $0 + A' = 1$. Hence finally we have

$$G(x,\ t) = \begin{cases} 1-t, & 0 \leq x < t \\ 1-x, & t < x \leq 1 \end{cases}$$

3.4.3 Exercises

Verify the property

$$\lim_{\epsilon \to 0}[G'(x,\ x+\epsilon) - G'(x,\ x-\epsilon)] = \frac{1}{p(x)}$$

f Green's functions where $p(x) = 1$ and prime denotes differentiation w.r.t. , and the Green function is given by

$$G(x,\ x') = \begin{cases} x^3x'/2 + xx'^3/2 - 9xx'/5 + x, & 0 \leq x < x' \\ x^3x'/2 + xx'^3/2 - 9xx'/5 + x', & x' \leq x \leq 1 \end{cases}$$

Verify the property

$$\lim_{\epsilon \to 0}[G'(x,\ x+\epsilon) - G'(x,\ x-\epsilon)] = \frac{1}{p(x)}$$

Green's functions where $p(x) = 1 - x^2$ and prime denotes differentiation r.t. x, and the Green function is given by

$$G(x,\ x') = \begin{cases} (-1/2)\ln|1-x|\,|1+x'| + \log 2 - 1/2, & -1 \leq x < x' \\ (-1/2)\ln|1+x|\,|1-x'| + \log 2 - 1/2, & x' < x < 1 \end{cases}$$

ind the solution of the I.V. problem using Green's function $y''(t) - y(t) =$ with B.Cs. $y(0) = 0,\ y'(0) = 0$.

ind the solution of the B.V. problem using Green's function $y'' - y = x$ h B.Cs. $y(0) = 0,\ y(1) = 0$.

ind the solution of the B.V. problem using Green's function $y''+y/4 = f(x)$ h B.Cs. $y(0) = 0,\ y(\pi) = 0$.

o solve the problem when $f(x) = \sin 2x$ and $f(x) = x/2$. $f(x) = \sin 2x$

ind the solution of the B.V. problem using Green's function $y'' = f(x)$ h B.Cs. $y(0)0,\ y'(1) = 0$.

o solve the problem when $f(x) = \sin 2x$ and $f(x) = x/2$. $f(x) = \sin 2x$

Find the solution of the B.V. problem using Green's function $y'' + y =$ in $2\pi x$ with B.Cs. $y(0) = 0,\ y'(1) = 0$.

5 Solution of the B.V. Problem with Inhomogeneous B.Cs.

5.1 Solution of S L system with homogeneous B.Cs.

start with the equations

$$Ly = f(x) \tag{8.5.1}$$

d

$$Ly = \delta(x-t) \tag{8.5.2}$$

ere $L \equiv (d/dx)\,\{p(x)\,d/dx\} + q(x)$.

ltiplying (8.5.1) with $G(x,\ t)$ and (8.5.2) with $y(x)$, and subtracting we ain

$$GLy - yLG(x,\ t) = G(x,\ t)f(x) - \delta(x-t)y(x)$$

w we integrate both sides of the last equation w.r.t. x from a to b, and ain

$$\int_a^b [GLy - yLG(x,\ t)]\,dx = \int_a^b G(x,\ t)f(x)\,dx - \int_a^b \delta(x-t)y(x) \tag{8.5.3}$$

xt we simplify both sides of (8.5.3).

$$\text{L.H.S. of (3)} \quad = \quad \int_a^b [GLy - yLG(x,\ t)]\,dx$$

$$y(x) = \int_a^b f(t)G(x,\ t)\,dt - \beta p(b)G(x,\ b) + \alpha p(a)G'(x,\ a)$$

e illustrate the problems with inhomogeneous B.Cs. with examples in the xt subsection.

5.4 Illustrative examples

cample 1

lve the problem

$+ y = f(x),\ \ y(0) = 0,\ \ y(1) = 0$ using Green's function.

lution

st we will find the associated Green's function. Regarded as a function of t satisfies the homogeneous DE $G''(x,\ t) + G(x,\ t) = 0$, where dash denotes ivative w.r.t. x. On solving the DE, we can write

$$G(x,\ t) = \begin{cases} c_1(t)\cos x + c_2(t)\sin x & 0 \le x \le t \\ c_3(t)\cos x + c_4(t)\sin x & t \le x \le 1 \end{cases}$$

e B.C. $G(0,\ t) = 0$ gives $c_1 =$ and the B.C. $G(1,\ t) = 0$ gives $c_3 \cos 1 +$ in $1 = 0$ wherefrom $c_4 = -(c_3/\sin 1)\sin(x-1)$. Therefore we can write

$$G(x,\ t) = \begin{cases} A(t)\sin x & 0 \le x < t \\ B(t)\sin(x-1) & t \le x \le 1 \end{cases}$$

plying the continuity condition we obtain $A(t)\sin t = B(t)\sin(t-1)$ or $)/\sin(t-1) = B(t)/\sin t = \lambda$, say. Because of this we obtain the symmetric m of $G(0,\ t)$ as

$$G(x,\ t) = \begin{cases} \lambda \sin x \sin(t-1) & 0 \le x < t \\ \lambda \sin t \sin(x-1) & t \le x \le 1 \end{cases}$$

e discontinuity condition gives $\lambda = -1/\sin 1$. Therefore

$$G(x,\ t) = \begin{cases} -\sin x \sin(t-1)/\sin 1, & 0 \le x < t \\ -\sin t \sin(x-1)/\sin 1, & t \le x \le 1 \end{cases}$$

ng the formula

$= \int_0^1 G(x,\ t)\,f(t)\,dt$, we obtain the solution as

$$y(x) = \frac{\sin(x-1)}{\sin 1}\int_0^x \sin t\, f(t)\,dt + \frac{\sin x}{\sin 1}\int^1 \sin(t-1)\,f(t)\,dt$$

Example 2

Solve the problem

$y'' = x^2, \;\; y(0) = 1, \;\; y(1) = 2$ using associated Green's function.

Solution

The problem can be solved by first general solution $y = x^4/12 + c_1 x + c$ through direct integration. The constants c_1, c_2 can be determined with th help of B.Cs.

Here we will solve the problem to illustrate the use of Green's function. Th associated Green function is found to be

$$y(x) = \begin{cases} t(x-1), & 0 \le t \le x \\ x(t-1), & x \le t \le 1 \end{cases} \qquad (1)$$

The solution is given by [equation (8.5.6)]

$$\begin{aligned} y(x) &= \int_a^b f(t) G(x,\, t)\, dt \\ &- \left[p(t) \left\{ y(t) G'(x,\, t) - G(x,\, t) y'(t) \right\} \right] \Big|_a^b \end{aligned}$$

Here $a - 0, \; b = 1, \; f(t) = t^2, \; p(t) = 1, \; \alpha = 1, \; \beta = 2.$

On substituting the value of $f(t)$ and of different constants, we have

$$\begin{aligned} y(x) &= \int_0^1 t^2\, G(x,\, t)\, dt - [y(1) G'(x,\, 1) - G(x,\, 1) y'(1)] \\ &+ \; [y(0) G'(x,\, 0) - G(x,\, 0) y'(0)] \end{aligned}$$

8.5.5 Exercises

8.6 Modified Green's Function

8.6.1 Why modified Green's function ?

When the solution of homogeneous B.V. problem associated with an S L system $L[y] = f(x), \;\; B_1(y) = 0, \;\; B_2(y) = 0$

is nontrivial; or in other words when $\lambda = 0$ is an eigenvalue of the SL systen defined by

$] + \lambda r(x) y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0$

en the associated Green's function is called *modified Green's function*. It not be constructed by the method used in the earlier examples.

e modified Green's function, to be denoted by $G_M(x, t)$, will exist if a tain solvability condition is satisfied by the source term $f(x)$. First we te this condition and then outline the method of its construction.

$y_0(x)$ be the nontrivial solution of the homogeneous problem

$$_0 = 0, \quad B_1(y_0) = 0, \quad B_2(y_0) = 0$$

en

$$=< y, Ly_0 >=< Ly, y_0 >=< f, y_0 > \text{ i.e. } \int_a^b f(x)\, y_0(x)\, dx = 0.$$

ich is the solvability condition.

e above observations can be summarized in the form of a theorem.

eorem (Fredholm's alternative for the S L system)

Either the problem $Ly = f(x), \quad B_1(y) = 0, \quad B_2(y) = 0$ has exactly one ation,

There is a nontrivial solution of the corresponding homogeneous problem $= 0, \quad B_1(y_0) = 0, \quad B_2(y_0) = 0.$

ase of alternative (ii) a solution exists if and only if the solvability condition $, y_0 >= 0$ is satisfied. The solution in this case will be undetermined to multiplicative constant. follows.

thod of constructing modified Green's function

Let $y_0(x)$ be the normalized eigenfunction corresponding to $\lambda =$ means that

$$, y_0 > = \int_a^b y_0(x)\, y_0(x)\, dx = 1$$

The modified Green function $G_M(x, t)$ satisfies the equation

$_M(x, t)] = y_0(x) y_0(t)$ in each of the subintervals $[a, t), \quad (t, b]$.

$G_M(x, t)$, considered as a function of x, satisfies the B.Cs.

$$_M(a, t)] = 0, \quad B_2[G_M(b, t)] = 0$$

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$G''_M(x,$

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(iv). $G_M(x, t)$ is continuous everywhere in the interval $[a, b]$, and in particular at $x = t$.

(v). The modified Green function satisfies the discontinuity condition

$G'_M(t+0, t) - G'_M(t-0, t) = 1/p(t)$

(vi). The modified Green function satisfies the orthogonality condition

$\int_a^b G_M(x, t) y_0(x) dx = 0$

The above conditions uniquely determine the modified Green function.

8.6.2 Illustrative examples

Example 1

Construct Green's function associated with the system

$y'' + \lambda r(x) y = 0, \quad y'(0) = 0 = y'(1)$

Solution

1. First we check if $\lambda = 0$ is an eigenvalue.

$\lambda = 0$ which implies $y'' = 0, \quad y = Ax + B$

Now we apply the B.Cs. The B.C. $u'(0) = 0$ gives $A = 0$. The other boundary condition does not yield additional information. Therefore $y = B$ is the (non-trivial) solution corresponding to the eigenvalue $\lambda = 0$.

Normalized eigenfunction is therefore $y_0(x) = 1$.

(ii). $G_M(x, t)$ satisfies the DE

$G''_M(x, t) = y_0(x) y_0(t)$ in each of the intervals $[0, t]$ and $(t, 1]$. On integration $G''_M(x, t) = x + A$ and $G_M(x, t) = (x^2/2) + Ax + B$. Therefore we can write

$$G_M(x, t) = \begin{cases} x^2/2 + Ax + B, & 0 \le x < t \\ x^2/2 + A'x + B', & t < x \le 1 \end{cases}$$

iii). $G_M(x, t)$ satisfies the B.Cs. $G'(0, t) = 0 = G'(1, t)$. On applying hese conditions we obtain $A = 0, A' = -1$.

efore we can write

$$G_M(x,\ t) = \begin{cases} x^2/2 + B, & 0 \le x \le t \\ x^2/2 + A'x + B', & t < x \le 1 \end{cases}$$

$G_M(x,\ t)$ is continuous at each point of the interval, and in particular $= t$. This condition gives

$t^2 + B = (1/2)t^2 - t + B'$ or $B' = B + t$.

$$G_M(x,\ t) = \begin{cases} x^2/2 + B, & 0 \le x \le t \\ x^2/2 - x + B + t, & t < x \le 1 \end{cases}$$

The discontinuity condition does not help in determining the unknown ants.

Using the orthogonality condition $\int_0^1 G_M(x,\ t)\ y_0(x)\ dx = 0$,

ve

$/2)x^2 - x + B]\ dx + \int_t^1 [(1/2)x^2 + B + t]\ dx = 0$

$$((x^3/6) + Bx)\Big|_0^t + ((x^3/6) - (x^2/2) + Bx + tx)\Big|_t^1 = 0$$

$- Bt + 1/6 - 1/2 + B + t - t^3/6 + t^2/2 - Bt - t^2 = 0$

$+ t - (t^2/2) + 1/6 - 1/2 = 0$

$+ (6t - 3t^2 + 1 - 3)/6 = 0$ or $B = -t + t^2/2 + 1/3$.

finally after simplification

$$G_M(x,\ t) = \begin{cases} x^2/2 + t^2/2 - t + 1/3, & 0 \le x \le t \\ t^2/2 + x^2/2 - x + 1/3, & t < x \le 1 \end{cases}$$

ple 2

reen's function associated with the problem

$$y'' + \lambda y = 0, \quad y(0) = y(1), \quad y'(0) = y'(1)$$

e that this is a periodic SL system.

on

In this problem $a = 0,\ b = 1,\ p(x) = 1.$

(i) $\lambda = 0$ implies that $y'' = 0$ whose general solution is $y = Ax + B$.

Applying the B.C. $y(0) = y(1)$, we obtain $B = A + B$, which gives $A = 0$, and he solution reduces to $y = 0$.

Now the second boundary condition $y'(0) = y'(1)$ leads to $A = A$, which does ot give any new information. So in general $B \neq 0$.

Therefore we obtain a nontrivial solution corresponding to $\lambda = 0$. We denote he normalized eigenfunction corresponding to $\lambda = 0$ by $y_0(x) = 1$.

ii) The modified Green function $G_M(x,\ t)$ satisfies the equation

$G''_M(x,\ t) = u_0(x)\,u_0(t)$ in each of the subintervals $[0,\ t)$ and $(t,\ 1]$. Hence we an write

$$G_M(x,\ t) = \begin{cases} x^2/2 + Ax + B, & 0 \leq x \leq t \\ x^2/2 + A'x + B', & t < x \leq 1 \end{cases}$$

iii) $G_M(x,\ t)$ satisfies the given end-point conditions

$$G_M(0,\ t) = G_M(1,\ t),\quad G'_M(0,\ t) = G'_M(1,\ t)$$

These give $B = 1/2 + A' + B',\ A = 1 + A'$. From these equations we obtain $B' = B - 1/2 - A',\ \ A' = A - 1$. Substituting for A' we have $B' = B - A + 1/2$.

Therefore we can write

$$G_M(x,\ t) = \begin{cases} x^2/2 + Ax + B, & 0 < x < t \\ x^2/2 + (A-1)x + B - A + 1/2, & t < x \leq 1 \end{cases}$$

v) $G_M(x,\ t)$ is continuous for all x and in particular at $x = t$. After mplification we obtain $A = 1/2 - t$ and

$$G_M(x,\ t) = \begin{cases} x^2/2 + (1/2 - t)x + B, & 0 \leq x \leq t \\ x^2/2 - (1/2 + t)x + t + B, & t < x \leq 1 \end{cases}$$

e discontinuity condition does not help because when we take derivative of $G_M(x,\ t)$ the constant b disappears.

To determine B we use the orthogonality condition $\int_0^1 G_M(x,\ t)u_0(x)\,dx =$

$$\int_0^t \left(\frac{1}{2}x^2 + \frac{1}{2}x - tx + B\right) dx$$

$$+\int_t^1\left(\frac{1}{2}x^2-\frac{1}{2}x-tx+B+t\right)dx=0$$

$$/4-t^3/2+Bt+1/6-1/4-t/2+B+t-t^3/6+t^2/4+t^3/2-Bt-t^2=0$$

simplification $B=t^2/2-t/2+1/12$

$$(x,\ t)=\begin{cases} x^2/2+(1/2-t)x+t^2/2-t/2+1/12, & 0\le x\le t\\ x^2/2-(1/2+t)x+t/2+t^2/2+1/2, & t<x\le 1\end{cases}$$

le 3

een's functions for the following problem

$=0,\ \ y(-1)=y(1),\ \ y'(-1)=y'(1)$

periodic SL system.)

on

construct Green's function we first check if $\lambda=0$ is an eigenvalue.

nplies $y''=0$, whose solution is given by $y=Ax+B$.

we apply the B.Cs. The B.C. $yu(-1)=y(1)$ gives $A=0$. The $(-1)=y'(1)$ also gives $A=0$, which means that $B\neq 0$. Hence the eneous problem $y=B$, i.e. nontrivial solution.

$\lambda=0$ is an eigenvalue, and the normalized eigenfunction over the $[-1,\ 1]$ is given by $y_0(x)=1/\sqrt{2}$.

$_M(x,\ t)$ satisfies the DE

$$G''_M(x,\ t)=y_0(t)\,y_0(x)=1/2$$

of the intervals $[-1,\ t)$ and $(t,\ 1]$.

$t)=(1/2)\,x+A$ and therefore $G_M(x,\ t)=(1/4)\,x^2+Ax+B$.

$$G_M(x,\ t)=\begin{cases} x^2/4+Ax+B, & -1\le x\le t\end{cases}$$

$$\left\{ x^2/4+(A-1)x+B-2A+1, \quad t<x\le 1 \right.$$

The continuity condition at $x=t$ gives $A=(1-t)/2$. Hence on tution

$$G_M(x,\,t)=\begin{cases} x^2/4+x(1-t)/2+B, & -1\le x\le t \\ x^2/4-x(1+t)/2+B+t, & t<x\le 1 \end{cases}$$

The discontinuity condition does not help in finding the value of B'.

In order to determine B, we use the condition $\int_{-1}^{+1} G_M(x,\,t)y_0(x)dx=0$ gives

$$\begin{aligned} &\int_{-1}^{t}(x^2/4+x/2-xt/2+B)\;dx \\ &+\int_{1}^{t}(x^2/4-x/2-xt/2+B+t)\;dx=0 \\ &[t^3/6+x^2/4-x^2\,t/4+Bx]\big|_0^t \\ +\;&[x^3/12-x^2/4+x^2\,t/4+B\,x+t\,x]=0 \end{aligned}$$

simplification

$/2+t+2B-1/3=0$ or $B=(3t^2-6t+2)/12$. Hence finally

$$G_M(x,\,t)=\begin{cases} x^2/4+t^2/4-x\,t/2+x/2-t/2+1/6, & -1\le x\le t \\ x^2/4+t^2/4-x\,t/2-x/2-t/2+1/6, & t<x\le 1 \end{cases}$$

ample 4

d Green's function associated with the problem

$$y''+\lambda y=0,\quad y'(0)=0,\quad y(2)=0$$

ution

a regular SL system with $p(x)=1$.

With $\lambda=0$, the DE becomes $y''=0$ whose solution is $y=Ax+B$.

w we apply the B.Cs. The B.C. $y'(0)=0$ gives $A=0$, and the B.C. $)=0$ gives $2A+B=0$ or $B=0$.

nce $u=0$ is the only solution of the homogeneous problem. Therefore $\lambda=0$ not an eigenvalue.

$$G(x,\ t) = \begin{cases} Ax + B, & 0 \le x \le t \\ A'x + B', & t < \le 2 \end{cases}$$

$G(x,\ t)$ satisfies the B.Cs. $G'(0,\ t) = 0$ and $G(2,\ t) = 0$.

rst condition gives $A = 0$. The second gives $2A' + B' = 0$ or $B' = -2A'$.

we can write

$$G(x,\ t) = \begin{cases} B, & 0 \le x \le t \\ A'(x-2), & t < x \le 2 \end{cases}$$

$G(x,\ t)$ is continuous for all values of x, and in particular at $x = t$.
ore we have $B = A't - 2A'$. and

$$G(x,\ t) = \begin{cases} A'(t-2), & 0 \le x < t \\ A'(x-2), & t < x \le 2 \end{cases}$$

he discontinuity condition

$0,\ t) - G'(t+0,\ t) = [1/p(t)]$ gives $0 - A' = 1$ or $A' = -1$.

finally

$$G(x,\ t) = \begin{cases} 2-t, & 0 \le x \le t \\ 2-x, & t < x \le 2 \end{cases}$$

Exercises

d modified Green's function for the B.V. problem.

$,\ y(0) = 0,\ y(1) - y'(1) = 0$

s.

$$G(x,\ t) = \begin{cases} x^3t/2 + xt^3/2 - 9xt/5 + x, & 0 \le < t \\ x^3t/2 + xt^3/2 - 9xt/5 + t, & t \le < x \le 1 \end{cases}$$

d Green's function for the B.V. problem.

$y'' = 0,\ y(0) = 0,\ y(1) = 0$

$y'' = 0,\ y(-1) = 0,\ y(1) = 0$

ıs. (a) $G(x,\ t) = \begin{cases} x & x \le < t \\ t, & x > t \end{cases}$

$$G(x,\ t) = \begin{cases} -(t - x + xt - 1)/2, & x \le t \\ -(x - t + xt - 1)/2, & x > t \end{cases} \quad)$$

Green's function for the BV problem.

$y' = 0, \quad y(1) = 0, \quad |y(0)| < \infty.$

Green's function for the BV problem.

$y' - \mu^2 y/x = 0, \quad y(1) = 0, \quad |y(0)| < \infty.$

d Green's function for the BV problem.

$y(-1) - y(1) = 0, \quad y(-1) - y'(1) = 0.$

nstruct Green's function for the problem:

$/4 = 0, \quad y(0) = 0, \quad y'(1) = 0$

ns. $G(x,\ t) = \begin{cases} \sin 2x, & x < t \\ \cos 2(1 - x), & x > t \end{cases} \quad)$

nstruct Green's function for the problem:

$$(xy')' = 0, \quad y(0) = 0, \quad y'(1) = 0$$

nstruct Green's function for the problem:

$$y'' + y/4 = 0, \quad y(0) = 0, \quad y'(1) = 0$$

ns. $G(x,\ t) = \begin{cases} \sin 2x, & x < t \\ \cos 2(1 - x), & x > t \end{cases} \quad)$

lve each problem.

$y'' = f(x), \ y'(0) = a, \ y(1) = b$

$y'' - k^2 y = f(x), \ y(0) - y'(0) = a, \ y(1) = b$

$(xy')' = f(x), \ y(0) = 0, \ y'(1) = b$

Find Green's for the problem defined by the DE $y'' + k^2 y = 0$ subject to B.Cs: $y(0) = y(1), \ y'(0) = y'(1)$.

Explain that it is not possible to find Green's function for the problem
ned by the DE $y'' + k^2 y = 0$ subject to the B.Cs. $y(0) = y(1), \ y'(0) =$
(1).