$$= f(x)H(x-x_0)|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} H(x-x_0)\, f'(x)\, dx$$

$$= -\int_{-\infty}^{+\infty} H(x-x_0)\, f'(x)\, dx = -\int_{x_0}^{+\infty} f'(x)\, dx$$

$$= -f(x)|_{x_0}^{\infty} = f(x_0) = \int_{x_0}^{+\infty} \delta(x-x_0)\, f(x)\, dx$$

where we have used the result $\lim_{x\to\pm\infty} f(x) = 0$.

On comparison $H'(x-x_0) = \delta(x-x_0)$.

Fourier transforms of Dirac delta function

$$\mathcal{F}\{\delta(x-x_0)\} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \exp(\imath\xi x)\, \delta(x-x_0)\, dx = \frac{1}{\sqrt{2\pi}} \exp(\imath\xi x_0)$$

Fourier transforms of Heaviside unit step function

Using the results $\mathcal{F}\{f'(x)\} = \imath\xi F(\xi)$ and $H'(x-x_0) = \delta(x-x_0)$, we have

$$\mathcal{F}\{H'(x-x_0)\} = \imath\xi\, \mathcal{F}\{\delta(x-x_0)\} \tag{7.6.1}$$

In (7.6.1) we use the result $H'(x-x_0) = \delta(x-x_0)$ and obtain

$$\mathcal{F}\{\delta(x-x_0)\} = \imath\xi\, \mathcal{F}\{H(x-x_0)\} \tag{7.6.2}$$

or $\mathcal{F}\{H(x-x_0)\} = (-\imath/\sqrt{2\pi})\,\xi\, \exp(\imath\xi\, x_0)$

7.7 Examples and Exercises

7.7.1 Illustrative examples

In this subsection we will discuss the examples illustrating the applications of the convolution theorem and Parseval's identities (also called Plancherel's identities). In this way we will be able to evaluate certain integrals.

Example 1

Use Plancherel's identity for the function $f(x) = \exp(-|x|)$ to evaluate the integral $\int_{-\infty}^{+\infty} dx/(1+x^2)^2$.

Solution

By Plancheral's identity $\int_{-\infty}^{+\infty}|f(x)|^2dx = \int_{-\infty}^{+\infty}|F(\xi)|^2d\xi$, where $\{f(x), F(\xi)\}$ are a F.T. pair. When $f(x) = \exp(-|x|)$. Then

$$\begin{aligned}\int_{-\infty}^{+\infty}|f(x)|^2dx &= \int_{-\infty}^{+\infty}\exp(-2|x|)\,dx\\ &= 2\int_0^{\infty}\exp(-2x)\,dx = 2\cdot\left.\frac{\exp(-2x)}{-2}\right|_0^{\infty} = 1\end{aligned}$$

Next we calculate $\mathcal{F}\{\exp(-|x|)\}$. Using the definition

$$\begin{aligned}\mathcal{F}\{\exp(-|x|)\} &= F(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(\imath\xi x)\exp(-|x|)\,dx\\ &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}e^{\imath kx}e^{x} + \int_0^{\infty}\exp(\imath\xi x)\exp(-x)\right]dx\\ &= \frac{1}{\sqrt{2\pi}}\left(\int_0^{\infty}e^{(1+\imath k)x} + \int_0^{\infty}\exp(1-\imath\xi x)\right)dx\\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{\exp((1+\imath\xi x)}{-(1+\imath\xi)} + \frac{\exp(-(1-\imath\xi)x)}{-(1-\imath\xi)}\right]\Bigg|_0^{\infty}\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+\imath\xi} + \frac{1}{1-\imath\xi}\right) = \sqrt{\frac{2}{\pi}}\,\frac{1}{1+\xi^2}\end{aligned}$$

Now by Plancherel's identity $\int_{-\infty}^{+\infty}|f(x)|^2dx = \int_{-\infty}^{+\infty}|F(\xi)|^2d\xi$. Therefore on substituting for $f(x)$ and $F(\xi)$, we obtain

$$\int_{-\infty}^{+\infty}|f(x)|^2dx = 1 = \int_{-\infty}^{+\infty}\frac{2}{\pi}\frac{dk}{(1+\xi^2)^2}$$

which gives

$$\int_{-\infty}^{+\infty}\frac{dk}{(1+\xi^2)^2} = \frac{\pi}{2}$$

Example 2

Let a function $f(x)$ be defined as follows.

$$f(x) = \begin{cases} 1, & |x| \le 1\\ 0, & |x| > 1\end{cases}$$

compute the convolutions $f \star f$ and $f \star f \star f \star f$ and using the convolution theorem evaluate the integrals

$$\int_{-\infty}^{+\infty}\frac{\sin^2\xi}{\xi^2}d\xi \quad\text{and}\quad \int_{-\infty}^{+\infty}\frac{\sin^4\xi}{\xi^4}d\xi$$

Solution

By definition

$$\begin{aligned} f(x) \star f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x')\, f(x - x')dx' \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{-1} 0 + \int_{-1}^{+1} 1 + \int_{+1}^{+\infty} 0 \right) dx' \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} 1 dx' = \frac{1}{\sqrt{2\pi}}\, x' |_{-1}^{+1} = \sqrt{\frac{2}{\pi}} \end{aligned}$$

By convolution theorem $\mathcal{F}\{f \star f\} = F(\xi)\, F(\xi) = F^2(\xi)$, where

$$\begin{aligned} F(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, \exp(\imath\xi x)dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} \exp(\imath\xi x)dx \\ &= \frac{1}{\sqrt{2\pi}} \left. \frac{\exp(\imath\xi x)}{\imath\xi} \right|_{-1}^{+1} = \frac{1}{\sqrt{2\pi}} \frac{\exp(\imath\xi) - \exp(-\imath\xi)}{\imath\xi} = \sqrt{\frac{2}{\pi}} \frac{\sin\xi}{\xi} \end{aligned}$$

Hence using the result $\mathcal{F}\{f \star f\} = F^2(\xi)$, we obtain

$$\mathcal{F}^{-1}\{F^2(\xi)\} = f(x) \star f(x) \quad \text{or} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{2}{\pi} \exp(-\imath\xi x) \frac{\sin^2\xi}{\xi^2} d\xi = 2$$

or

$$\int_{-\infty}^{+\infty} \exp(-\imath\xi x) \frac{\sin^2\xi}{\xi^2} d\xi = \left(\frac{2}{\pi}\right)^{3/2}$$

To solve the second part of the problem, let

$$\begin{aligned} g(x) &= f(x) \star f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x') f(x - x')dx' \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} 1 dx' = \sqrt{\frac{2}{\pi}} \end{aligned}$$

Therefore

$$g(x') = \sqrt{2/\pi}, \quad g(x - x') = \sqrt{2/\pi}.$$

Hence

$$g \star g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x')\, g(x - x')dx' = \frac{4}{(2\pi)^{3/2}}$$

Now

$$G(\xi) = F(\xi)\, F(\xi) = F(\xi)^2 = (2\sin^2\xi)/(\pi\xi^2)$$

Hence using the convolution theorem

$$\mathcal{F}\{g(x) \star g(x)\} = \mathcal{F}\{f \star f \star f \star f\} = G(\xi)\, G(\xi) = (4\sin^4\xi/\pi^2\xi^4)$$

Therefore

$$\mathcal{F}^{-1}\left\{\frac{4\sin^4\xi}{\pi^2\xi^4}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{4\exp(-\imath\xi x)\sin^4\xi}{\pi^2\xi^4}d\xi = 2\pi^3$$

wherefrom on simplification we obtain $[(\sin^4\xi)/\xi^4]\,d\xi = 2\pi^3$.

Example 3

Let the functions $f_a(x)$, $f_b(x)$ be defined as follows.

$$f_a(x) = \begin{cases} 1, & |x| \le a, \quad a > 0 \\ 0, & |x| > a \end{cases}$$

$$f_b(x) = \begin{cases} 1, & |x| \le b, \quad b > 0 \\ 0, & |x| > b \end{cases}$$

Show that

$$F_a(\xi) = \sqrt{\frac{2}{\pi}}\frac{\sin a\xi}{\xi}, \quad F_b(\xi) = \sqrt{\frac{2}{\pi}}\frac{\sin b\xi}{\xi}$$

Using these results and Plancherel's second identity, show that

$$\int_{-\infty}^{+\infty}\frac{\sin a\xi\,\sin b\xi}{\xi^2}d\xi = \pi\min(a,b)$$

Solution

By definition

$$\begin{aligned} F_a(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(\imath\xi x)f_a(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{+a}\exp(\imath\xi x)f_a(x)dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-a}^{+a}\exp(\imath\xi x)dx = \frac{1}{\sqrt{2\pi}}\left.\frac{\exp(\imath\xi x)}{\imath\xi}\right|_{-a}^{+a} \\ &= \frac{1}{\sqrt{2\pi}}\frac{\exp(\imath\xi a)-\exp(-\imath\xi a)}{\imath\xi} = \sqrt{\frac{2}{\pi}}\frac{\sin\xi a}{\xi} \end{aligned}$$

Similarly

$$F_b(\xi) = \sqrt{\frac{2}{\pi}}\frac{\sin\xi b}{\xi}$$

By direct calculation, we find that

$$\int_{-\infty}^{+\infty}|f_a(x)|^2dx = \int_{-a}^{+a}1dx = 2a, \quad \int_{-\infty}^{+\infty}|f_b(x)|^2dx = \int_{-b}^{+b}1dx = 2b$$

From Plancherel's second identity, viz:

$$\int_{-\infty}^{+\infty}f(x)g(x)dx = \int_{-\infty}^{+\infty}F(\xi)G(\xi)d\xi$$

with $f(x)$ and $g(x)$ replaced by $f_a(x)$ and $f_b(x)$, we obtain

$$\int_{-\infty}^{+\infty} F_a(\xi)F_b(\xi)d\xi = \int_{-\infty}^{+\infty} f_a(x)f_b(x)dx$$

or on substituting the values for $f_a(x), F_a(\xi)$ *etc.*, we have

$$\begin{aligned}\frac{2}{\pi}\int_{-\infty}^{+\infty} \frac{\sin a\xi \;\sin b\xi}{\xi^2}d\xi &= \int_{-\infty}^{+\infty} f_a(x)f_b(x)dx \\ &= \int_{-c}^{+c} 1dx = 2c\end{aligned}$$

where c is the smaller of the two numbers a, b, i.e. $c = \min(a, b)$. Hence

$$\int_{-\infty}^{+\infty} \frac{\sin a\xi \;\sin b\xi}{\xi^2}dk = \pi \;\min(a, b)$$

Example 4

Find the F.T. of the function $f(x) = x/[(x^2+a^2)(x^2+b^2)]$, and hence calculate the integral

$$\int_{-\infty}^{+\infty} \frac{x\sin mxdx}{(x^2+a^2)(x^2+b^2)}, \quad (a, b \text{ real}),\; a > 0,\; b > 0,\; b > a$$

Solution

Let $f(z) = z/x/[(z^2+a^2)(z^2+b^2)]$. By definition

$$\mathcal{F}\{f(x)\} \equiv F(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(\imath\xi x)\frac{xdx}{(x^2+a^2)(x^2+b^2)} \tag{1}$$

To compute the integral in (1), we introduce the contour integral $\oint_{\Gamma_r} f(z)\exp(\imath\xi z)dz$, where Γ_r is the contour of figure (10.2). Then by the residue theorem

$$\oint_{\Gamma_r} f(z)\exp(\imath\xi z)dz = 2\pi\imath\sum_j R_j \tag{2}$$

The poles of $f(z)$ are given by $z = \pm\imath a$ and $z = \pm\imath b$. Only the poles at $z_1 = \imath a$ and $z_2 = \imath b$ lie inside the contour Γ_r. The residues R_1, R_2 of these poles are given by

$$\begin{aligned}R_1 &= \frac{1}{\sqrt{2\pi}}\cdot\lim_{z\to a\imath}\frac{(z-a\imath)z\exp(\imath\xi z)}{(z-a\imath)(z+a\imath)(z-b\imath)(z+b\imath)} \\ &= \frac{1}{2\sqrt{2\pi}}\cdot\frac{\exp(-\xi a)}{b^2-a^2}\end{aligned}$$

Similarly

$$R_2 = \frac{1}{\sqrt{2\pi}} \cdot \lim_{z \to bi} \frac{(z - bi) z e^{ikz}}{(z - ai)(z + ai)(z - bi)(z + bi)}$$
$$= \frac{1}{2\sqrt{2\pi}} \cdot \frac{e^{-kb}}{a^2 - b^2}$$

Therefore

$$R_1 + R_2 = \frac{1}{2\sqrt{2\pi}} \cdot \frac{\exp(-\xi a) - \exp(-\xi b)}{b^2 - a^2}$$

On substitution we obtain

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(i\xi x) f(x) dx = \sqrt{\frac{1}{2}} \frac{\exp(-\xi a) - \exp(-\xi b)}{b^2 - a^2} i \quad (3)$$

Computation of the integral

With $k = m$ in (3), we obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(imx) f(x) dx = \sqrt{\frac{1}{2}} \frac{\exp(-ma) - \exp(-mb)}{b^2 - a^2} i$$

On equating the imaginary parts on both sides, we have

$$\int_{-\infty}^{+\infty} \frac{x \sin mx}{(x^2 + a^2)(x^2 + b^2)} = \sqrt{\frac{\pi}{2}} \frac{\exp(-ma) - \exp(-mb)}{b^2 - a^2}$$

7.7.2 Exercises

1. Find the F.T. of each of the following functions.

   (a) $f(x) = 1/(a^2 + x^2)$ (b) $f(x) = \cos bx/(a^2 + x^2)$

   (c) $f(x) = \sin bx/(a^2 + x^2)$

2. Using results of the previous problem and Plancherel's identity, evaluate the following integrals

   (a) $\int_0^\infty du/[(a^2 + u^2)^2]$ (b) $\int_0^\infty [u^2 du/(a^2 + u^2)^2]$

   (c) $\int_0^\infty [(x \sin \pi x)/(1 - x^2)]\, dx$ (d) $\int_0^\infty [x \sin \pi x \cos \pi x)/(1 - x^2)]\, dx$

   (e) $\int_0^\infty [(x \sin \pi x)/(1 - x^2)^2]\, dx$

   (f) $\int_0^\infty [(\sin x - x \cos x)/x^2]^2\, dx$

3. Determine the F.T. of

$$f(x) = \begin{cases} +1, & 0 < x \leq a \\ -1, & -a < x \leq 0 \\ 0, & x > |a| \end{cases}$$

where $a > 0$ and use it to evaluate the integral

$$\int_0^\infty \frac{(\cos ax - 1)}{x} \sin bx \, dx, \quad b > 0$$

4. Using the generalized Plancherel identity evaluate the integral

$$\int_0^\infty \frac{dx}{(a^2 + x^2)(b^2 + x^2)}, \quad a0, \quad b > 0$$

5. Let $f(x)$ be a complex-valued function of the real variable x, and let $F(k)$ be its F.T. If

(a) $F(\xi) = \begin{cases} 1 - \xi^2, & |\xi| \leq 1 \\ 0, & |\xi| > 1 \end{cases}$

(b) $F(\xi) = \begin{cases} 1 - \xi, & |\xi| \leq 1 \\ 0, & |\xi| > 1 \end{cases}$

find $f(x)$.

6. Calculate the F.T. of the following function (also called the *two-sided exponential pulse*, when x is interpreted as time t)

$$f(x) = \begin{cases} e^{ax}, & x \leq 0 \\ e^{-ax}, & x > 0 \end{cases}, \quad (a > 0).$$

7. Calculate the F.T. of the 'on-off' pulse shown in the figure below. 8. Sketch the graph of the function below, calculate its F.T.

$$f(x) = \begin{cases} A(+x/X + 1), & -X \leq x \leq 0 \\ A(-x/X + 1), & 0 < x \leq X \end{cases}.$$

What is the relationship between this pulse and that of the previous problem?

9. Calculate the F.T. of the following function.

$$f(x) = \begin{cases} 2c, & |x| \leq 2 \\ 0, & |x| > 2 \end{cases}.$$

and

$$g(x) = \begin{cases} c, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}.$$

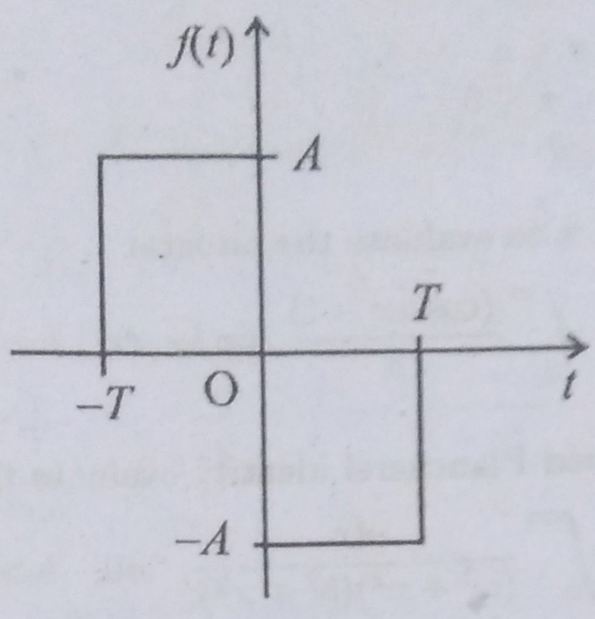

Figure 7.4: Graph of the on-off pulse.

Sketch he graph of the function $w(x) = f(x) - g(x)$ and calculate its F.T.

10. Calculate the F.T. of the off-on-off pulse represented by the function

$$f(x) = \begin{cases} 0, & x < -2 \\ -1, & -2 \le x < -1 \\ +1, & -1 \le x \le 1 \\ -1, & 1 < x \le 2 \\ 0, & x > 2 \end{cases}.$$

11. Show that the F.T. of the function

$$f(x) = \begin{cases} \sin ax, & |x| \le \pi/a \\ 0, & |x| > \pi/a \end{cases}.$$

is $12a\sin(\pi k/a)/(k^2 - a^2)$.

12. Find the the F.T. of the function

$f(x) = \exp(-ax)\ \sin\xi_0 x\ H(x)$.

13. Show that the Fourier sine and cosine transforms of the function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \le x \le a \\ 0, & x > a \end{cases}$$ are $(1 - \cos ax)/x$ and $(\sin ax)/x$ respectively.

14. Calculate the Fourier sine and cosine transforms of

$f(x) = \exp(-ax)\ H(x),\ \ a > 0$.

23. Let $f(x)$ and $F(k)$ denote the F.T. pair, with the condition that $f(x)$ is continuous and absolutely integrable. Given that

$$F(\xi) + \int_{-\infty}^{+\infty} F(\xi - u)\exp(-|u|)du = \begin{cases} \xi^2, & 0 \le u \le 1 \\ 0, & \text{for } u < 0,\ u > 1 \end{cases}$$

find $f(x)$.

24. Let $f(x)$ and $F(\xi)$ denote the F.T. pair, as in the previous problem. If $F(\xi) = 0$, for all $k \ge |\xi_0|$, then show that for all $a > |\xi_0|$

$$f(x) \star \left(\frac{\sin ax}{\pi x}\right) = f(x)$$

25. Let $F(\xi)$ be the F.T. of the function $f(x)$, defined as follows

$$f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{for all other values of } x \end{cases}$$

find the function $g(x)$ such that the F.T. $G(\xi)$ of $g(x)$ which satisfies $G(\xi) = |F(\xi)|^2$.

26. Use the formula

$$\mathcal{F}\{x^n f(x)\} = (-\imath)^n F^n(\xi)$$

to calculate $\mathcal{F}\{x\exp(-\alpha x^2)\}$, $\alpha > 0$.

(**Ans.**: $\mathcal{F}\{x\exp(-\alpha x^2)\} = (1/\sqrt{2\alpha})\exp[-\xi^2/(4\alpha)]$)

27. Find the the F.T. of the function

$$f(x) = \exp(-ax)\ \sin k_0 x\ H(x).$$

7.8 Use of Complex Fourier Transform in Solving B.V./I.V. Problems

When we have an I.V./B.V. problem in which the space coordinate x extends over the whole real line, we may use complex Fourier transform. For the validity of the formula for the derivatives it is required that both the unknown function u and its partial derivatives approach zero as x goes to $\pm\infty$.

7.8.1 Illustrative examples

Example 1

(a) Solve the problem by means of the Fourier transform method.

$$u_t = \kappa^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0 \tag{1}$$

$$u(x,\ 0) = f(x), \quad u(x,\ t),\ u_x(x,\ t) \to 0 \text{ as } x \to \pm\infty \tag{2}$$

(b) Obtain the solution of the problem in (a) when $u(x,\ 0) = \exp(-\alpha x^2)$.

Solution

The given PDE and the B.Cs. describe the conduction of heat through a rod or wire of infinite length. The initial temperature distribution is given by $f(x)$ and the temperature towards the end points gets smaller and smaller. The source of heat in the body is the initial temperature.

We assume that both u and $f(x)$ satisfy the conditions for the existence of their Fourier transforms. Taking the Fourier transforms of both sides of (1), and denoting the Fourier transform of $u(x,\ t)$ w.r.t. x by $U(\xi,\ t)$, we obtain the first order ODE

$$\frac{dU(\xi,\ t)}{dt} = -\kappa^2 \xi^2 U(\xi,\ t) \tag{3}$$

General solution of (3) is given by

$$U(\xi,\ t) = c \exp(-\kappa^2 \xi^2 t) \tag{4}$$

The I.C. in (2) can be transformed as $U(\xi,\ 0) = F(\xi)$. From this condition and (4), we obtain $c = F(\xi)$. Hence

$$U(\xi,\ t) = F(\xi) \exp(-\kappa^2 \xi^2 t)$$

Taking the inverse Fourier transform, we have

$$u(x,\ t) = \mathcal{F}^{-1}\{F(\xi)\ \exp(-\beta\xi^2)\}$$

where $\beta = \kappa^2 t$. Next using the convolution theorem, we can simplify the right side of the above relation as

$$u(x,\ t) = f(x) \star \mathcal{F}^{-1}\{\exp(-\beta\xi^2)\}$$

Now using the formula in equation (2) of example 3, *viz.*

$$\mathcal{F}^{-1}\{\exp(-\beta\xi^2)\} = \frac{1}{\sqrt{2\beta}} \exp\left(-\frac{x^2}{4\beta}\right)$$

we finally obtain

$$u(x,\ t) = \frac{1}{\sqrt{\pi t \kappa^2}} \int_{-\infty}^{+\infty} f(x') \exp[-(x - x')^2/(4\kappa^2 t)]$$

(b)

The initial temperature distribution in this case is the sharply peaked Gaussian function $f(x) = \exp(-\alpha x^2)$.

Physically it implies that if the rod is heated from the middle, the temperature gets smaller and smaller as one moves away from the middle of the rod.

On substituting for this value of $f(x)$ in (1), we have

$$\begin{aligned} u(x,\, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\exp(-x^2/(4\beta))}{\sqrt{2\beta}}\, f(x - x')\, dx' \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\beta}} \int_{-\infty}^{+\infty} \exp(-x^2/(4\beta)) \exp[-\alpha(x^2 + x'^2 - 2xx')]\, dx' \\ &= \frac{\exp(-\alpha x^2)}{2\sqrt{\pi\beta}} \int_{-\infty}^{+\infty} \exp(-x'^2/4\beta) \exp[(\alpha + 1/\beta)x'^2 + 2\alpha x x']\, dx' \end{aligned}$$

Now

$$\begin{aligned} \alpha'\xi^2 + \imath\xi x &= \alpha' \left(\xi^2 + \frac{\imath x}{\alpha'}\xi\right) \\ &= \alpha' \left[\left(\xi + \frac{\imath x}{2\alpha'}\xi\right)^2 + \frac{x^2}{4\alpha'^2}\right] \end{aligned}$$

On substitution we obtain

$$\begin{aligned} u(x,\, t) &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{+\infty} \exp\left[-\alpha' \left(\xi + \frac{\imath x}{2\alpha'}\right)^2\right] \exp(-x^2/4\alpha')\, d\xi \\ &= \frac{1}{2\sqrt{\pi\alpha}} \exp(-x^2/4\alpha') \int_{-\infty}^{+\infty} \exp(-\alpha' K^2)\, dK \\ &= \frac{1}{2\sqrt{\pi\alpha}} \exp(-x^2/4\alpha') \sqrt{\frac{\pi}{\alpha'}} = \frac{\exp(-x^2/4\alpha')}{4\alpha t + 1} \end{aligned}$$

Example 2

Solve the problem by means of the Fourier transform method.

$$u_t = \alpha^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0 \tag{1}$$

$$u_x(x,\, 0) = f(x), \quad |u(x,\, 0)| < \infty, \quad -\infty < x < +\infty \tag{2}$$

Solution

We assume that both u and $u_x \to 0$ as $x \to \pm\infty$, and further that both u and $f(x)$ are absolutely integrable over $(-\infty,\, +\infty)$. Taking the Fourier transform of both sides of (1), and denoting the Fourier transform of $u(x,\, t)$ w.r.t. x by $U(\xi,\, t)$, we obtain the first order ODE

$$dU(\xi,\ t)/dt = -\alpha^2\,\xi^2\,U(\xi,\ t) \tag{3}$$

General solution of (3) is given by

$$U(\xi,\ t) = c\,\exp(-\alpha^2\,\xi^2\,t) \tag{4}$$

The I.C. in (2) can be transformed as $U(\xi,\ 0) = F(\xi)$. From this condition and (4), we obtain $c = F(\xi)$. Hence

$$U(\xi,\ t) = F(\xi)\,\exp(-\alpha^2\,\xi^2\,t)$$

Taking the inverse Fourier transform, we have

$$u(x,\ t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(-\imath\xi x)\,F(\xi)\,\exp(-\alpha^2\,\xi^2\,t)\,d\xi$$

or

$$u(x,\ t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(-\imath\xi x)\,\exp(-\alpha^2\,\xi^2\,t)\cdot \times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(\imath\xi x')\,f(x')\,dx'\,d\xi$$

or on changing order of integration

$$u(x,\ t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x')\left[\int_{-\infty}^{+\infty} \exp[-\imath\xi(x-x') - \alpha^2\,\xi^2\,t]\,d\xi\right]dx'$$

or

$$u(x,\ t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x')\,I(x,\ x')\,dx' \tag{5}$$

where

$$\begin{aligned} I(x,\ x') &= \int_{-\infty}^{\infty} \exp[-\imath\xi(x-x') - \alpha^2\,\xi^2\,t]\,d\xi \\ &= \int_{-\infty}^{+\infty} \exp(-\imath\xi u - \beta\,\xi^2)\,d\xi \end{aligned}$$

where $\beta = \alpha^2 t$ and $u = x - x'$. By completing the squares, we can express the Gaussian integral as

$$\begin{aligned} I(x,\ x') &= \exp(-u^2/4\beta)\int_{-\infty}^{+\infty} \exp[-\beta(\xi + \imath u/2\beta)^2\,d\xi \\ &= \exp(-u^2/4\beta)\cdot\sqrt{\frac{\pi}{\beta}} \end{aligned}$$

Hence on substituting in (5), we have

$$u(x,\ t) = \frac{1}{2\pi}\,\frac{1}{\sqrt{\pi\,t\,\alpha^2}}\int_{-\infty}^{+\infty} f(x')\,\exp\left(-\frac{(x-x')^2}{4\alpha^4 t^2}\right)\,dx'$$

Example 3

Solve by the Fourier transform method

$$u_{xxxx} = (1/a^2)\, u_{tt} \tag{1}$$

where

$$u(x,\, 0) = f(x), \quad u_t(x,\, 0) = a g'(x) \tag{2}$$

(Suitable asymptotic behaviour for u and its derivatives, and for g is assumed); *i.e.* $g,\, u,\, u_x,\, u_{xx},\, u_{xxx} \to 0$ as $x \to \pm\infty$.

Solution

From (1) taking Fourier transform of both sides

$$\mathcal{F}\{u_{xxxx}\} = \mathcal{F}\{(1/a^2)\, u_{tt}\}$$

where $\mathcal{F}$ denotes F.T. operator w.r.t. x. On simplification the last equation becomes

$$(-\imath\xi)^4\, U(\xi,\, t) = \frac{1}{a^2}\frac{\partial^2}{\partial t^2} U(\xi,\, t) \tag{3}$$

where

$$U(\xi,\, t) = \mathcal{F}\{u(x,\, t)\} = (1/\sqrt{2\pi}) \int_{-\infty}^{+\infty} \exp(\imath\xi x)\, u(x,\, t)\, dx$$

From (3)

$$(d^2/dt^2)\, U(\xi,\, t) - a^2\xi^4\, U(\xi,\, t) = 0$$

or

$$(D_t^{\,2} - a^2\,\xi^4) U(\xi,\, t) = 0$$

Therefore

$$U(\xi,\, t) = A\exp(a\xi^2 t) + B\exp(-a\xi^2 t) \tag{4}$$

We now transform the initial conditions in terms of ξ by taking Fourier transform of both sides of conditions in (2) w.r.t. x.

From these equations

$$U(\xi,\, 0) = F(\xi) \tag{5a}$$

and

$$\left.\frac{d}{dt} U(\xi,\, t)\right|_{t=0} = a(-\imath\xi)\, G(\xi) = -\imath a\xi\, G(\xi) \tag{5b}$$

To find A, B in (4), we use I.Cs. (5). From these I.Cs. we obtain

$$U(\xi,\ 0) = F(\xi) = A + B \text{ or } A + B = F(\xi) \tag{6}$$

Again from (4)

$$\left.\frac{d}{dt} U(\xi,\ t)\right|_{t=0} = a\,\xi^2\, A\, \exp(a\xi^2 t) - a\,\xi^2\, B\, \exp(-a\xi^2 t)\big|_{t=0}$$

or

$-a\imath\xi\, G(\xi) = a\xi^2 A - a\xi^2 B$ which implies $\imath G(\xi) = \xi A - \xi B$, wherefrom

$$A - B = (-\imath/\xi)\, G(\xi) \tag{7}$$

From (6) and (7)

$$A = \frac{1}{2}\{F(\xi) - \frac{\imath}{\xi} G(\xi)\}, \text{ and } B = \frac{1}{2}\{F(\xi) + \frac{\imath}{\xi} G(\xi)\}$$

Substituting in (4) and get

$$\begin{aligned} U(\xi,\ t) &= \frac{1}{2}\{F(\xi) - \frac{\imath}{\xi} G(\xi)\} \exp(a\xi^2 t) + \frac{1}{2}\{F(\xi) + \frac{\imath}{\xi} G(\xi)\} \exp(-a\xi^2 t) \\ &= \frac{1}{2} F(\xi)\, \{\exp(a\xi^2 t) + \exp(-a\xi^2 t)\} - \frac{1}{2}\frac{\imath}{\xi}\, G(\xi)\, \{\exp(a\xi^2 t) - \exp(-a\xi^2 t \end{aligned}$$

Taking inverse Fourier transform, we have

$$u(x,\ t) = \mathcal{F}^{-1}\{U(\xi,\ t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-\imath\xi\, x)\, U(\xi,\ t)\, d\xi$$

wherefrom we can obtain $u(x,\ t)$.

Example 4

Solve the following I.V/B.V problem by the F.T. method.

$$u_{xx}(x,\ t) = (1/c^2)\, u_{tt}(x,\ t) \tag{1}$$

$$u(x,\ 0) = f(x), \quad u_t(x,\ 0) = g(x) \tag{2}$$

and u, $u_x \to 0$, as $x \to \pm\infty$. (The problem describes the transverse vibrations of a string of infinite length.)

Solution

From (1) taking Fourier transform of both sides w.r.t x we have

$-\xi^2 U(\xi,\ t) - (1/c^2)\,(d^2/dt^2)\, U(\xi,\ t) = 0$

or

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^{+\infty} \exp(\imath\xi x) f(x) + \int_{-\infty}^{0} \exp(\imath\xi x) f(-x) \right] dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} [\exp(\imath\xi x) + \exp(-\imath\xi x)]\, f(x)\, dx$$

or

$$F_c(\xi) = \sqrt{2/\pi}\, \int_0^\infty \cos \xi x\, f(x)\, dx$$

Similarly if we define the odd extension of $f(x)$ over the whole real line as

$$f_o(x) = \begin{cases} f(\ x), & \text{for } 0 \le x < \infty \\ -f(-x), & \text{for } \infty < x < 0 \end{cases}$$

and perform similar calculations, we arrive at the definition of the Fourier sine transform given above.

Linearity of Fourier sine and cosine transforms

As in case of exponential Fourier transform and its inverse, the cosine and sine transforms are linear operators and this result follows from their definitions.

7.9.3 Fourier sine and cosine transforms of derivatives

To calculate Fourier sine and cosine transforms of first order derivative, we assume that (i) $f(x)$ is real and (ii) $|f(x)| \to 0$ as $x \to \infty$. Then

$$\begin{aligned} \mathcal{F}_c\{f'(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos \xi x\, dx \\ &= \sqrt{\frac{2}{\pi}} \left[\cos \xi x\, f(x)|_0^\infty + \xi \int_0^\infty f(x) \sin \xi x\, dx \right] \\ &= \sqrt{\frac{2}{\pi}} \times 0 - \sqrt{\frac{2}{\pi}}\, f(0) + \xi \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \xi x\, dx \end{aligned}$$

Therefore

$$\mathcal{F}_c\{f'(x)\} = -\sqrt{2/\pi}\, f(0) + \xi F_s(\xi) \qquad (7.7.5)$$

Similarly

$$\mathcal{F}_s\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin \xi x\, dx$$

$$= \sqrt{\frac{2}{\pi}} f(x) \sin \xi x \Big|_0^\infty - \xi \int_0^\infty f(x) \cos \xi x \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \times 0 - \sqrt{\frac{2}{\pi}} \times 0 - \xi \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \xi x \, dx$$

or

$$\mathcal{F}_s\{f'(x)\} = -\xi F_c(\xi) \tag{7.7.6}$$

To calculate Fourier sine and cosine transforms of second order derivatives, we assume further that (iii) $|f'(x)| \to 0$, as $x \to \infty$, then

$$\begin{aligned}
\mathcal{F}_s\{f''(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f''(x) \sin \xi x \, dx \\
&= \sqrt{\frac{2}{\pi}} \left[\sin \xi x \, f'(x) \Big|_0^\infty - k \int_0^\infty f'(x) \cos \xi x \, dx \right] \\
&= \sqrt{\frac{2}{\pi}} \left[0 - \xi \int_0^\infty f'(x) \cos \xi x \, dx \right] \\
&= -\xi \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos \xi x \, dx \\
&= -\xi \mathcal{F}_c\{f'(x)\} \\
&= -\xi \left[-\sqrt{\frac{2}{\pi}} f(0) + \xi F_s(\xi) \right]
\end{aligned}$$

Therefore

$$\mathcal{F}_s\{f''(x)\} = \sqrt{2/\pi}\, \xi f(0) - \xi^2 F_s(\xi)$$

Similarly

$$\begin{aligned}
F_c\{f''(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f''(x) \cos \xi x \, dx \\
&= \sqrt{\frac{2}{\pi}} \left[f'(x) \cos \xi x \Big|_0^\infty + \xi \int_0^\infty f'(x) \sin \xi x dx \right] \\
&= 0 - \sqrt{\frac{2}{\pi}} f'(0) + \xi \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin \xi x \, dx \\
&= -\sqrt{\frac{2}{\pi}} f'(0) + \xi \mathcal{F}_s\{f'(x)\} \\
&= -\sqrt{\frac{2}{\pi}} f'(0) + k[-\xi F_c(\xi)] \\
&= -\sqrt{\frac{2}{\pi}} f'(0) - \xi^2 F_c(\xi)
\end{aligned}$$

7.9.4 Illustrative examples on Fourier sine and cosine transforms

Example 1

Calculate Fourier sine transform of the function $f(x) = \exp(-x)\cos x$.

Solution

By definition

$\mathcal{F}_s\{\exp(-x)\cos x\} = \sqrt{2/\pi}\int_0^\infty \exp(-x)\cos x \sin\xi x dx$

Using the result $\sin A\cos B = (1/2)[\sin(A+B) + \sin(A-B)]$, we have

$$\sin\xi x\cos x = (1/2)[\sin(\xi+1)x + \sin(\xi-1)x]$$

Therefore

$$\mathcal{F}_s\{\exp(-x)\cos x\} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \exp(-x)\{\sin(\xi+1)x + \sin(\xi-1)x\}\,dx$$

Now we use the formula

$$\int e^{ax}\sin bx\,dx = \frac{\exp(ax)}{a^2+b^2}\{a\sin bx - b\cos bx\}$$

and obtain

$$\begin{aligned}\int_0^\infty \exp(-x)\sin(\xi+1)x\,dx &= \frac{\exp(-x)}{1+(\xi+1)^2}\times\\ &\times\ \{[-\sin(\xi+1)x - (\xi+1)\cos(\xi+1)x\}|_0^\infty\\ &= 0 - \frac{1}{\xi^2+2\xi+2}(0-(\xi+1))\\ &= \frac{\xi+1}{\xi^2+2\xi+2}\end{aligned}$$

Similarly

$$\begin{aligned}\int_0^\infty e^{-x}\sin(\xi-1)x\,dx &= \frac{\exp(-x)}{1+(\xi-1)^2}\times\\ &\times\ \{-\sin(\xi-1)x - (k-1)\cos(\xi-1)\}|_0^\infty\\ &= 0 - \frac{1}{\xi^2-2\xi+2}(0-(\xi-1))\\ &= \frac{\xi-1}{\xi^2-2\xi+2}\end{aligned}$$

Therefore

$$\mathcal{F}_s\{\exp(-x)\cos x\} = \frac{1}{\sqrt{2\pi}}\left[\frac{\xi+1}{\xi^2+2\xi+2} + \frac{\xi-1}{\xi^2-2\xi+2}\right]$$

$$= \sqrt{\frac{1}{\sqrt{2\pi}}}\frac{2\xi^2}{\xi^2+4} = \sqrt{\frac{2}{\pi}}\frac{\xi^2}{\xi^2+4}$$

Example 2

Calculate Fourier sine transform of the function

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \pi, \\ 0, & x > \pi \end{cases}$$

Solution

$$\begin{aligned} \mathcal{F}_s(f(x)) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin \xi x\, dx \\ &= \sqrt{\frac{2}{\pi}}\int_0^\pi f(x)\sin \xi x\, dx + \sqrt{\frac{2}{\pi}}\int_\pi^\infty f(x)\sin \xi x\, dx \\ &= \sqrt{\frac{2}{\pi}}\int_0^\pi \sin x \sin \xi x\, dx + \sqrt{\frac{2}{\pi}}\int_\pi^\infty 0 \sin \xi x\, dx \end{aligned}$$

Continuing further, we have

$$\begin{aligned} \mathcal{F}_s(f(x)) &= \sqrt{\frac{2}{\pi}}\int_0^\pi \sin x \sin \xi x\, dx \\ &= -\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\pi [\cos(\xi+1)x - \cos(\xi-1)x]\, dx \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\pi [-\cos(\xi+1)x + \cos(\xi-1)x]\, dx \\ &= \frac{1}{\sqrt{2\pi}}\left\{-\frac{\sin(\xi+1)x}{\xi+1} + \frac{\sin(\xi-1)x}{\xi-1}\right\}\Bigg|_0^\pi \\ &= \frac{1}{\sqrt{2\pi}}\left[-\frac{\sin(\xi+1)\pi}{\xi+1} + \frac{\sin(\xi-1)\pi}{\xi-1}\right] \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\xi^2-1}[-(k-1)\sin(\xi+1)\pi + (\xi+1)\sin(\xi-1)\pi] \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\xi^2-1}[(k-1)\sin \xi\pi - (\xi+1)\sin \xi\pi] \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\xi^2-1}(-2\sin \xi\pi) \\ &= \sqrt{\frac{2}{\pi}}\frac{\sin \xi\pi}{1-\xi^2} \end{aligned}$$

Example 3

Show that

$$\text{(a)}\quad \mathcal{F}_s(x\exp(-ax)) = \sqrt{\frac{2}{\pi}}\frac{2a\xi}{(a^2+\xi^2)^2}$$

$$\text{(b)}\quad \mathcal{F}_c(x\exp(-ax)) = \sqrt{\frac{2}{\pi}}\frac{a^2-\xi^2}{(a^2+\xi^2)^2}$$

where $\mathcal{F}_s$ and $\mathcal{F}_c$ denote Fourier sine and cosine transforms.

Solution

We will use the results

$$\int \exp(ax)\sin bx\,dx = \frac{\exp(ax)}{a^2+b^2}(a\sin bx - b\cos bx)$$

and

$$\int \exp(ax)\cos bx\,dx = \frac{\exp(ax)}{a^2+b^2}(a\cos bx + b\sin bx)$$

(a)

$$\begin{aligned}
\mathcal{F}_s(x\exp(-ax)) &= \sqrt{\frac{2}{\pi}}\int_0^\infty x(\exp(-ax)\sin(\xi x)\,dx \\
&= \sqrt{\frac{2}{\pi}}\left[x\frac{\exp(-ax)}{a^2+\xi^2}(-a\sin\xi x - \xi\cos(\xi x))\right]_0^\infty \\
&+ \frac{1}{a^2+\xi^2}\int_0^\infty \exp(-ax)(a\sin\xi x + \xi\cos\xi x)\,dx\Big] \\
&= \sqrt{\frac{2}{\pi}}(0-0) + \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\xi^2}\int_0^\infty \exp(-ax)\sin\xi x \\
&+ \sqrt{\frac{2}{\pi}}\frac{\xi}{a^2+\xi^2}\int_0^\infty \exp(-ax)\cos\xi x\,dx \\
&= \left[\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\xi^2}\frac{\exp(-ax)}{a^2+\xi^2}(-a\sin\xi x - \xi\cos\xi x)\right]_0^\infty \\
&+ \left[\sqrt{\frac{2}{\pi}}\frac{\xi}{a^2+\xi^2}\frac{\exp(-ax)}{a^2 n\,\xi^2}(-a\cos\xi x + \xi\sin(\xi x))\right]_0^\infty
\end{aligned}$$

On simplification

$$\begin{aligned}
\mathcal{F}_s(x\exp(-ax)) &= \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\xi^2}\left(0-\frac{\xi}{a^2+\xi^2}\right) \\
&+ \sqrt{\frac{2}{\pi}}\frac{\xi}{a^2+\xi^2}\left(0-\frac{-a}{a^2+\xi^2}\right)
\end{aligned}$$

$$= \sqrt{\frac{2}{\pi}}\frac{a\xi}{(a^2+\xi^2)^2} + \sqrt{\frac{2}{\pi}}\frac{a\xi}{(a^2+\xi^2)^2}$$

or finally $\mathcal{F}_s\{x\exp(-ax)\} = \sqrt{2/\pi}\,(2a\xi/(a^2+\xi^2)^2$

(b)

Now we will prove that

$$\mathcal{F}_c\{x\exp(-ax)\} = \sqrt{\frac{2}{\pi}}\frac{a^2-\xi^2}{(a^2+\xi^2)^2}, \quad a>0$$

By definition

$$\mathcal{F}_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty x\exp(-ax)\cos\xi x\,dx$$

On integration by parts and using the results (1) and (2), we obtain

$$\begin{aligned}
\mathcal{F}_c\{x\exp(-ax)\} &= \sqrt{\frac{2}{\pi}}\left[x\frac{\exp(-ax)}{a^2+\xi^2}(-a\cos\xi x+\xi\sin\xi x)\right]\Bigg|_0^\infty \\
&- \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\exp(-ax)}{a^2+\xi^2}(-a\cos\xi x+\xi\sin\xi x)\,dx \\
&= \sqrt{\frac{2}{\pi}}(0-0) + \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\xi^2}\int_0^\infty \exp(-ax)\cos\xi x\,dx \\
&\quad \sqrt{\frac{2}{\pi}}\frac{\xi}{a^2+\xi^2}\int_0^\infty \exp(-ax)\sin\xi x\,dx \\
&= \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\xi^2}\frac{\exp(-ax)}{a^2+\xi^2(-a\cos\xi x+\xi\sin\xi x)}\Bigg|_0^\infty \\
&- \sqrt{\frac{2}{\pi}}\frac{\xi}{a^2+\xi^2}\frac{\exp(-ax)}{a^2+\xi^2}(-a\sin\xi x-\xi\cos\xi x)\Bigg|_0^\infty \\
&= \sqrt{\frac{2}{\pi}}\frac{a}{(a^2+\xi^2)^2}(0+a) + \sqrt{\frac{2}{\pi}}\frac{\xi}{(a^2+\xi^2)^2}(0-\xi) \\
&= \sqrt{\frac{2}{\pi}}\frac{a^2}{(a^2+\xi^2)^2}\sqrt{\frac{2}{\pi}}\frac{\xi^2}{(a^2+\xi^2)^2} \\
&= \sqrt{\frac{2}{\pi}}\frac{a^2-\xi^2}{(a^2+\xi^2)^2}
\end{aligned}$$

Example 4

Determine $\mathcal{F}_c\{x^{\alpha-1}\}$ and $\mathcal{F}_s\{x^{\alpha-1}\}$.

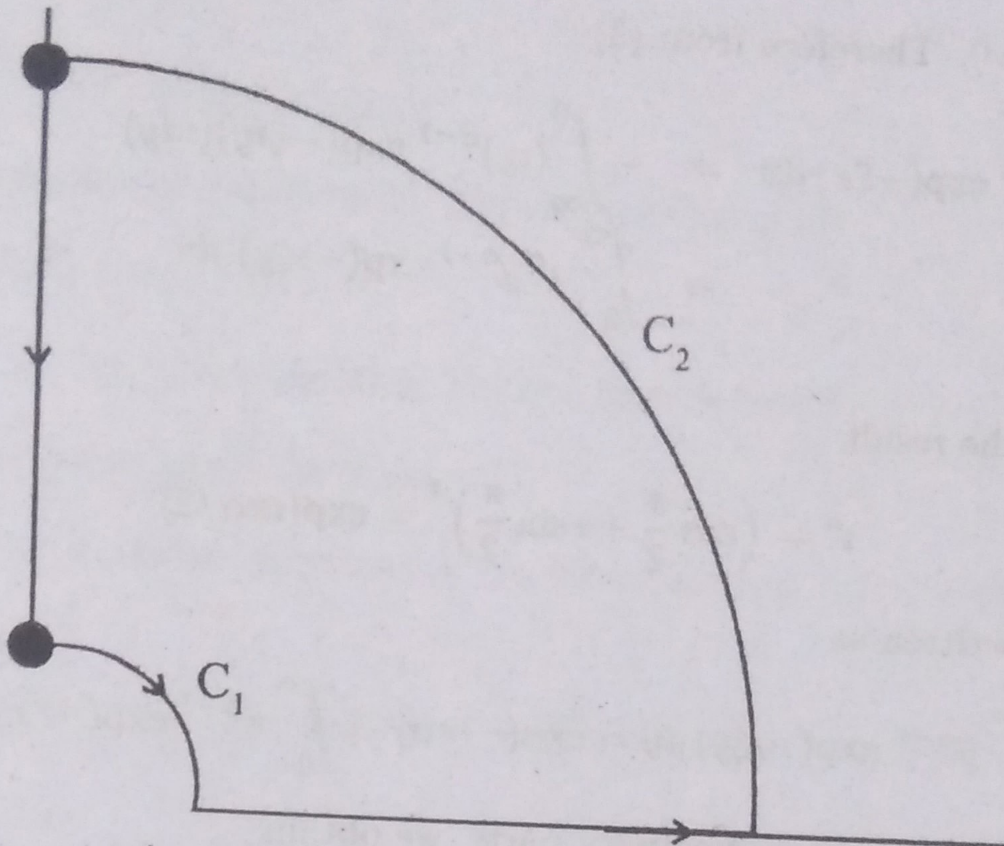

Figure 7.5: Contours C_1, C_2 in example 4.

Solution

From definition

$$\mathcal{F}_c\{x^{\alpha-1}\} = \sqrt{2/\pi}\int_0^\infty x^{\alpha-1}\cos\xi x\,dx$$

and

$$\mathcal{F}_s\{x^{\alpha-1}\} = \sqrt{2/\pi}\int_0^\infty x^{\alpha-1}\sin\xi x\,dx$$

*hspace*1.0*in*(2)

To calculate the integrals on R.H.S. of equations (1) and (2), we define

$$f(z) = z^{\alpha-1}\exp(-\xi z), \quad 0 < \alpha < 1$$

If this is analytic in a contour C, then by Cauchy's theorem $\oint_C f(z)\,dz =$ (
We choose the contour C as shown in the figure. Therefore

$$\int_{C_1} f(z)\,z + \int_\epsilon^R x^{\alpha-1}\exp(-\xi x)\,dx + \int_{C_2} f(z)\,dz$$
$$+\int_R^\epsilon (\imath y)^{\alpha-1}\exp(-\xi\imath y)\,(\imath\,dy) = 0 \qquad (3)$$

In the limit $\epsilon \to 0$, $R \to \infty$ we can prove that $\int_{C_1} f(z)\,dz = 0$ and

$\int_{C_2} f(z)\, dz = 0$. Therefore from (3)

$$\int_0^\infty x^{\alpha-1}\exp(-\xi x)\,dx = -\int_\infty^0 (\imath y)^{\alpha-1}\exp(-\xi \imath y)(\imath dy)$$
$$= \int_0^\infty \imath^\alpha y^{\alpha-1}\exp(-\imath\xi y)\,dy \qquad (4)$$

Now using the result

$$\imath^\alpha = \left(\cos\frac{\pi}{2} + \imath\sin\frac{\pi}{2}\right)^\alpha = \exp(\imath\pi\alpha/2)$$

(4) can be written as

$$\int_0^\infty y^{\alpha-1}\exp(-\imath\xi y)\,dy = \exp(-\imath\pi\alpha/2)\int_0^\alpha x^{\alpha-1}\exp(-\xi x)\,dx$$

Separating the real and imaginary parts, we obtain

$$\int_0^\infty y^{\alpha-1}\cos\xi y\,dy = \cos\frac{\pi\alpha}{2}\int_0^\infty x^{\alpha-1}\exp(-\xi x)\,dx \qquad (5)$$

and

$$\int_0^\infty y^{\alpha-1}\sin\xi y\,dy = \sin\frac{\pi\alpha}{2}\int_0^\infty x^{\alpha-1}\exp(-\xi x)\,dx \qquad (6)$$

On making substitutions in (1) and (2)

$$\mathcal{F}_c\{x^{\alpha-1}\} = \sqrt{\frac{2}{\pi}}\cos(\pi\alpha/2)\int_0^\infty t^{\alpha-1}\exp(-\xi t)\,dt$$

Putting $\xi t = t'$, we get

$$\mathcal{F}_c\{x^{\alpha-1}\} = \sqrt{\frac{2}{\pi}}\cos\frac{\pi\alpha}{2}\int_0^\infty \left(\frac{t'}{\xi}\right)^{\alpha-1}\exp(-t')\frac{dt'}{\xi}$$
$$= \sqrt{\frac{2}{\pi}}\cos\frac{\pi\alpha}{2}\frac{1}{\xi^\alpha}\int_0^\infty t'^{\alpha-1}\exp(-t')\,dt'$$
$$= \sqrt{\frac{2}{\pi}}\cos(\pi\alpha/2)\frac{\Gamma(\alpha)}{\xi^\alpha}, \qquad \text{(Q.E.D.)}$$

Similarly we can prove (2).

7.9.5 Exercises

1. Show that

$$\mathcal{F}_c\{x^{\alpha-1}\} = \sqrt{\frac{2}{\pi}}\Gamma(\alpha)\frac{\cos(\pi\alpha/2)}{\xi^\alpha}$$

2. Show that

$\mathcal{F}_s\{x^{\alpha-1}\} = \sqrt{2/\pi}\Gamma(\alpha)\sin[(\pi\alpha/2)/\xi^{\alpha}$.

3. Find the (complex or exponential) Fourier transform of

$f(x) = \exp(-\lambda x^2)\cos\beta x, \ (\lambda > 0)$.

4. Find the (complex or exponential) Fourier transforms of

$\exp(-\lambda x^2)\cos\beta x$ and $\exp(-\lambda x^2)\sin\beta x$.

[Hint: Use the modulation property stated in section 7.3 and obtain the results

$$\mathcal{F}\{e^{-\lambda x^2}\cos\beta x\} = \frac{1}{2}\frac{1}{\sqrt{2\lambda}}\left[\exp\left(-(\xi+\beta)^2/4\lambda\right) + \exp\left(-(\xi-\beta)^2/4\lambda\right)\right]$$

and

$$\mathcal{F}\{e^{-\lambda x^2}\sin\beta x\} = \frac{1}{2\iota}\frac{1}{\sqrt{2\lambda}}\left[\exp\left(-(\xi+\beta)^2/4\lambda\right) - \exp\left(-(\xi-\beta)^2/4\lambda\right)\right]$$

5. Find the (complex or exponential) Fourier transform of

$F(x) = \cos\beta x/(a^4 + x^4)$.

6. Find the (complex or exponential) Fourier transform of

$$f(x) = \begin{cases} \cos k_0 x, & |x| < N\pi/k_0 \\ 0, & |x| > N\pi/k_0 \end{cases}$$

7. Calculate the following:

(a) $\mathcal{F}_s\{f''(x)\}$, (b) $\mathcal{F}_s\{f^{(\text{iv})}(x)\}$, (c) $\mathcal{F}_c\{f^{(\text{iv})}(x)\}$

8. Show that

(a) $\mathcal{F}_c\{\exp(-ax)\} = \sqrt{2/\pi}\ a/(a^2+\xi^2), \ (a > 0)$

(b) $\mathcal{F}_s\{\exp(-ax\} = \sqrt{2/\pi}\ \xi/(a^2+\xi^2), \ (a > 0)$

(c) $\mathcal{F}\{\exp(-|a|x)\} = \sqrt{2/\pi}\ a/(a^2+\xi^2), \ (a > 0)$

9. Use the result of subsection 7.4.2 to calculate $\mathcal{F}\{xe^{-\alpha x^2}, \alpha > 0$.

(Hint: Use the formula $\mathcal{F}\{xe^{-\alpha x^2}\} = (1/\sqrt{2\alpha})\exp[-\xi^2/(4\alpha)]$).

7.10 Use of Fourier Sine and Cosine Transforms in B.V./I.V. Problems

The B.V/I.V problems usually consist of second order linear PDEs along with I.Cs and B.Cs. If the x coordinate extends from 0 to ∞, as happens in heat conduction problems involving conducting rod of infinite length, or in vibration problems involving strings of infinite length, then we may use sine or cosine transforms in the solution of such problems. The choice of sine or cosine transform is dictated by the type of boundary condition. From the formulas

$$\mathcal{F}_s\{u''(x)\} = \sqrt{2/\pi}\,\xi\,u(0) - k^2 U_s(\xi)$$

and

$$\mathcal{F}_c\{u''(x)\} = -\sqrt{2/\pi}\,u'(0) - \xi^2\,U_c(\xi)$$

where $\mathcal{F}_s$ and $\mathcal{F}_c$ denote Fourier sine and cosine transform operators.

We can see that if the problem contains B.C. of the form $u(0,\ t) = c$ where c is a constant, then we use sine transform. On the other hand if the problem contains a B.C. of the form $u_x(0,\ t) =$ constant, then we use cosine transform. These ideas will be further illustrated by means of the solved examples.

7.10.1 Illustrative examples

Example 1

Solve the potential equation for the potential $u(x,\ y)$ in the semi-infinite strip $0 < x < c,\ y > 0$ that satisfies the following conditions

$$u(0,\ y) = 0,\ u_y(x,\ 0) = 0,\ u_x(c,\ y) = f(y) \tag{1}$$

Solution

The potential equation is given by

$$u_{xx} + u_{yy} = 0, \quad 0 < x < c, \quad y > 0 \tag{2}$$

The second B.C. in (1) suggests that we take Fourier cosine transform w.r.t. y. Therefore

$$\mathcal{F}_c\{u_{xx}\} + \mathcal{F}_c\{u_{yy}\} = 0$$

or on using formulas for Fourier transforms of derivatives ...

$$(d^2/dx^2)U_c(x,\ \xi) - \sqrt{2/\pi}\, u_y(x,\ 0) - \xi^2 U_c(x,\ \xi) = 0$$

or using the second B.C. in (1), we have

$$(d^2/dx^2 - \xi^2)\ U_c(x,\ \xi) = 0$$

whose general solution can be written as

$$U_c(x,\ \xi) = c_1(y)\exp(\xi x) + c_2(y)\exp(-\xi x) \tag{3}$$

Now we transform the first and third B.Cs in (1) and obtain

$$U_c(0,\ \xi) = 0, \quad (d/dx)U(c,\ \xi) = F_c(\xi) \tag{4}$$

Using the B.Cs. in (4), we obtain from (3)

$$c_1 + c_2 = 0,\ \ c_1\,\xi\,\exp(c\xi) - c_2\,\xi\,\exp(-c\xi) = F_c(\xi) \tag{5}$$

On solving equations in (5), we have

$$c_1 = -c_2 = \frac{F_c(\xi)}{\xi(\exp(c\xi) + \exp(-c\xi))} = \frac{F_c(\xi)}{2\xi\cosh c\xi}$$

Therefore on substitution in (3)

$$\begin{aligned} U_c(x,\ \xi) &= \frac{F_c(\xi)e^{\xi x}}{2\xi\cosh\,c\xi} - \frac{F_c(\xi)\exp(-\xi x)}{2\xi\cosh\,c\xi} \\ &= \frac{F_c(\xi)}{2\cosh\,c\xi}(\exp(\xi x) - \exp(-\xi x)) \\ &= F_c(\xi)\frac{\sinh\,\xi x}{\xi\cosh\,c\xi} \end{aligned}$$

On taking inverse cosine Fourier transform, we obtain

$$\begin{aligned} u(x,\ y) &= \mathcal{F}_c^{-1}\{U_c(x,\ \xi)\} \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{F_c(\xi)\sinh\xi x\ \cos\xi y}{\xi\cosh c\xi}\,d\xi \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sinh\xi x\ \cos\xi y}{\xi\cosh c\xi}\cdot\sqrt{\frac{2}{\pi}}\int_0^\infty f(y')\cos\xi y'\,dy'\,d\xi \\ &= \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{\sinh\xi x\ \cos\xi y\ \cos\xi y'}{\xi\cosh c\xi}\,f(y')\,dy'\,d\xi \end{aligned}$$

Example 2

Solve the problem using Fourier transform method.

$$u_t = u_{xx},\ \ u(0,\ t) = u_0,\ \ u(x,\ 0) = 0,\ \ x > 0,\ \ t > 0,\ \ u_0 > 0$$

ution

e B.C. suggests that we take the Fourier sine transform w.r.t. x. Using the ation $\mathcal{F}_s\{u(x,\ t)\} = U_s(\xi,\ t)$. The given DE becomes $F_s\{u_t\} = F_s\{u_{xx}\}$.

$$(d/dt)U_s(\xi,\ t) = \sqrt{2/\pi}\,\xi u(0,\ t) - \xi^2 U_s(\xi,\ t)$$

$$(d/dt)\,U_s(\xi,\ t) = \sqrt{2/\pi}\,\xi u_0 - \xi^2 U_s(\xi,\ t)$$

$$(d/dt)\,U_s + \xi^2 U_s = \sqrt{2/\pi}\,\xi u_0$$

ch is a first order linear non-homogeneous ODE, whose integrating factor xp($\xi^2 t$), and whose solution is given by

$$\begin{aligned}\exp(\xi^2 t)\,U_s &= \int \exp(\xi^2 t)\sqrt{/\pi}\,\xi u_0 dt + \text{constant}\\ &= \sqrt{\frac{2}{\pi}}\frac{\xi u_0 e^{\xi^2 t}}{\xi^2} + c = \sqrt{\frac{2}{\pi}}\frac{u_0}{\xi}\exp(\xi^2 t) + c\end{aligned}$$

$$U_s(\xi,\ t) = \sqrt{2/\pi}\,(u_0/\xi) + c\exp(-\xi^2 t) \qquad (1)$$

v the I.C. $u(x,\ 0) = 0$ becomes $U_s(\xi,\ 0) = 0$. From this result and (1), we ain $c = -\sqrt{2/\pi}\,u_0/\xi$. Hence

$$U_s(\xi,\ t) = \sqrt{\frac{2}{\pi}}\frac{u_0}{\xi}(1 - \exp(-\xi^2 t)) \qquad (2)$$

ing the inverse Fourier sine transform, we have

$$\begin{aligned}u(x,\ t) &= \frac{2}{\pi}\int_0^\infty \frac{u_0}{\xi}(1 - \exp(-\xi^2 t)\ \sin\xi x\, d\xi\\ &= \frac{2}{\pi}u_0\left[\int_0^\infty [(\sin\xi x)/\xi]\, d\xi - \int_0^\infty \frac{\exp(-\xi^2 t)}{\xi}\sin\xi x\, d\xi\right]\\ &= (2/\pi)\,u_0\,(\pi/2 - I_1)\end{aligned}$$

ere we have used the result $\int_0^\infty [(\sin u)/u]\,du = \pi/2$.

evaluate the integral I_1, we note that

Therefore

$$I_1 = \int_0^\infty \exp(-\xi^2 t) \int_0^x \cos \xi x' \, dx' \, d\xi$$

On changing the order of integration

$$I_1 = \int_0^x \left[\int_0^\infty \exp(-\xi^2 t) \, \cos \xi x' \, d\xi\right] \, dx'$$

Now we will show that

$$I_2 = \int_0^\infty \exp(-\xi^2 t) \cos \xi x' \, d\xi = \sqrt{\pi/t} \exp(-x'^2/4t)$$

To prove this result we proceed as follows.

$$\begin{aligned} I_2 &= \int_0^\infty \exp(-\xi^2 t) \, \exp(\imath \xi x') \, d\xi \\ &= \int_0^\infty \exp\left[-t(\xi^2 - \imath \xi x'/t)\right] \, d\xi \\ &= \int_0^\infty \exp\left[-t(\xi - \imath x'/2t)^2\right] \, \exp(-x'^2/4t) \, d\xi \\ &= \exp(-x'^2/4t) \int_0^\infty \exp(-t\xi'^2) \, d\xi' \\ &= \exp(-x'^2/4t) \, \sqrt{\pi/t} \end{aligned}$$

Therefore on substitution

$$I_1 = \int_0^x \sqrt{\pi/t} \, \exp(-x'^2 t/4) \, dx'$$

and

$$\begin{aligned} u(x,\, t) &= \frac{2}{\pi} u_0 \left[\frac{\pi}{2} - \int_0^x \sqrt{\frac{\pi}{t}} \, \exp(-x'^2/4t) \, dx'\right] \\ &= u_0 - \sqrt{\frac{2}{t}} \, u_0 \int_0^x \exp(-x'^2/4t) \, dx' \\ &= u_0 \left[1 - \frac{2}{\sqrt{\pi t}} \int_0^x \exp(-x'^2/4t) \, dx'\right] \\ &= u_0 \left[1 - \text{erf} \left(\frac{x}{2\sqrt{t}}\right)\right] = u_0 \, \text{erfc} \left(\frac{x}{2\sqrt{t}}\right) \end{aligned}$$

where

$\text{erf}\,(x) = \int_0^x \exp(-t^2) dt$, $\text{erfc}\,(x) = 1 - \text{erf}\,(x)$

Example 3

Solve the problem, using Fourier transform method:

$$u_t = u_{xx}, \quad x > 0, \quad x > 0, \quad t > 0$$

$$u_x(0,\ t) = f(t), \quad u(x,\ 0) = 0, \quad x > 0, \quad t > 0.$$

Solution

The B.C. suggests that we take cosine transform w.r.t. x.

The given PDE will then become $\mathcal{F}_c\{u_t\} = \mathcal{F}_c\{u_{xx}\}$, which can also be written as

$$\begin{aligned}\frac{d}{dt} U_c(\xi,\ t) &= -\sqrt{\frac{2}{\pi}} u_x(0,\ t) - \xi^2 U_c(\xi,\ t) \\ &= -\sqrt{\frac{2}{\pi}} f(t) - \xi^2 U_c(\xi,\ t)\end{aligned}$$

This is a linear inhomogeneous D.E. of order 1 with integrating factor is $e^{\xi^2 t}$. Therefore its solution is given by

$$\exp(\xi^2 t)\, U_c = -\sqrt{2/\pi} \int f(t) \exp(\xi^2 t)\, dt + c$$

or

$$U_c(\xi,\ t) = -\sqrt{\frac{2}{\pi}} \exp(-\xi^2 t) \int_0^t f(t') \exp(-\xi^2 t')\, dt' + c \exp(-\xi^2 t)$$

The I.C. $u(x,\ 0) = 0$ gives $U_c(\xi,\ 0) = 0$. Applying this condition we obtain $c = 0$. Hence

$$U_c(\xi,\ t) = -\exp(\xi^2 t) \sqrt{\frac{2}{\pi}} \int_0^t f(t') \exp(\xi^2 t')\, dt' = \sqrt{\frac{2}{\pi}} \int_0^t f(t') \exp[(t'-t)\xi^2]\, dt'$$

Therefore

$$u(x,\ t) = \mathcal{F}_c^{-1}\{U_c(\xi,\ t)\} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \cos \xi x \int_0^t f(t') \exp[-\xi^2(t-t')]\, dt'\, d\xi$$

Changing the order of integration

$$u(x,\ t) = -\sqrt{\frac{2}{\pi}} \int_0^t f(t') \left[\sqrt{\frac{2}{\pi}} \int_0^\infty \cos \xi x \exp[-\xi^2(t-t')]\, d\xi \right] dt'$$

where the integral in the square brackets is the cosine Fourier transform of the function $exp(-\alpha\xi^2)$ where $\alpha = t - t' > 0$ which equals $(1/\alpha) \exp(-x^2/4\alpha)$.

Example 4

Solve the problem by means of the F...

$$u_x(0,\ t) = 0, \quad u(x,\ 0) = f(x), \quad 0 < x < \infty, \quad t > 0 \tag{2}$$

ution

e $0 < x < \infty$, and not $-\infty < x < +\infty$, we may use Fourier cosine or transform. The B.C. $u_x(0,\ t) = 0$ suggests that we should use the cosine sform.

we note that the the condition $u_x(0,\ t) = 0$, shows that there is perfect lation at the end-point $x = 0$, which means that there is no heat flux ss this end-point.

assume that both u and $u_x \to 0$ as $x \to \infty$, and further that both u $f(x)$ are absolutely integrable over $(0,\ \infty)$. Denoting the Fourier cosine nsform of $u(x,\ t)$ w.r.t. x by $U_c(k,\ t)$, we have from (1)

$$\mathcal{F}_c\{u_t\} = \mathcal{F}_c\{u_{xx}\}$$

using the formula for transform of derivatives, we have

$$(d/dt)\, U_c(\xi,\ t) = -\xi\sqrt{2/\pi}\, u_x(0,\ t) + \xi^2 U_c(\xi,\ t)$$

naking use of (2), we have

$$(d/dt)\, U_c(\xi,\ t) = -\xi^2\, U_c(\xi,\ t)$$

ose solution is given by

$$U_c(\xi,\ t) = A\exp(-\xi^2\, t) \tag{3}$$

v the I.C. (3) can be transformed into the condition

$$U_c(\xi,\ 0) = F(\xi) \tag{4}$$

m (3) and (4), we obtain $A = F(\xi)$. Therefore

$$U_c(\xi,\ t) = F(\xi)\exp(-\xi^2\, t) \tag{5}$$

obtain the required solution we take the inverse cosine transform. From we have

$$u(x,\ t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\xi)\, e^{-\xi^2 t} \cos \xi x\, d\xi$$

pressing $F(\xi)$ in terms of $f(x)$, we obtain

$$u(x,\ t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\int_0^\infty f(x') \cos \xi x'\, dx'\right] \exp(-\xi^2 t) \cos \xi x\, d\xi$$