Similarly

$$R_2 = \frac{1}{\sqrt{2\pi}} \cdot \lim_{z \to bi} \frac{(z - bi) z e^{ikz}}{(z - ai)(z + ai)(z - bi)(z + bi)}$$
$$= \frac{1}{2\sqrt{2\pi}} \cdot \frac{e^{-kb}}{a^2 - b^2}$$

Therefore

$$R_1 + R_2 = \frac{1}{2\sqrt{2\pi}} \cdot \frac{\exp(-\xi a) - \exp(-\xi b)}{b^2 - a^2}$$

On substitution we obtain

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(i\xi x) f(x) dx = \sqrt{\frac{1}{2}} \frac{\exp(-\xi a) - \exp(-\xi b)}{b^2 - a^2} i \qquad (3)$$

Computation of the integral

With $k = m$ in (3), we obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(imx) f(x) dx = \sqrt{\frac{1}{2}} \frac{\exp(-ma) - \exp(-mb)}{b^2 - a^2} i$$

On equating the imaginary parts on both sides, we have

$$\int_{-\infty}^{+\infty} \frac{x \sin mx}{(x^2 + a^2)(x^2 + b^2)} = \sqrt{\frac{\pi}{2}} \frac{\exp(-ma) - \exp(-mb)}{b^2 - a^2}$$

7.7.2 Exercises

1. Find the F.T. of each of the following functions.

(a) $f(x) = 1/(a^2 + x^2)$ (b) $f(x) = \cos bx/(a^2 + x^2)$

(c) $f(x) = \sin bx/(a^2 + x^2)$

2. Using results of the previous problem and Plancherel's identity, evaluate the following integrals

(a) $\int_0^\infty du/[(a^2 + u^2)^2]$ (b) $\int_0^\infty [u^2 du/(a^2 + u^2)^2]$

(c) $\int_0^\infty [(x \sin \pi x)/(1 - x^2)]\, dx$ (d) $\int_0^\infty [x \sin \pi x \cos \pi x)/(1 - x^2)]\, dx$

(e) $\int_0^\infty [(x \sin \pi x)/(1 - x^2)^2]\, dx$

(f) $\int_0^\infty [(sin x - x \cos x)/x^2]^2\, dx$

3. Determine the F.T. of

$$f(x) = \begin{cases} +1, & 0 < x \leq a \\ -1, & -a < x \leq 0 \\ 0, & x > |a| \end{cases}$$

where $a > 0$ and use it to evaluate the integral

$$\int_0^\infty \frac{(\cos ax - 1)}{x} \sin bx \, dx, \quad b > 0$$

4. Using the generalized Plancherel identity evaluate the integral

$$\int_0^\infty \frac{dx}{(a^2 + x^2)(b^2 + x^2)}, \quad a0, \quad b > 0$$

5. Let $f(x)$ be a complex-valued function of the real variable x, and let $F(k)$ be its F.T. If

(a) $F(\xi) = \begin{cases} 1 - \xi^2, & |\xi| \leq 1 \\ 0, & |\xi| > 1 \end{cases}$

(b) $F(\xi) = \begin{cases} 1 - \xi, & |\xi| \leq 1 \\ 0, & |\xi| > 1 \end{cases}$

find $f(x)$.

6. Calculate the F.T. of the following function (also called the *two-sided exponential pulse*, when x is interpreted as time t)

$$f(x) = \begin{cases} e^{ax}, & x \leq 0 \\ e^{-ax}, & x > 0 \end{cases}, \quad (a > 0).$$

7. Calculate the F.T. of the 'on-off' pulse shown in the figure below. 8. Sketch the graph of the function below, calculate its F.T.

$$f(x) = \begin{cases} A(+x/X + 1), & -X \leq x \leq 0 \\ A(-x/X + 1), & 0 < x \leq X \end{cases}.$$

What is the relationship between this pulse and that of the previous problem?

9. Calculate the F.T. of the following function.

$$f(x) = \begin{cases} 2c, & |x| \leq 2 \\ 0, & |x| > 2 \end{cases}.$$

and

$$g(x) = \begin{cases} c, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}.$$

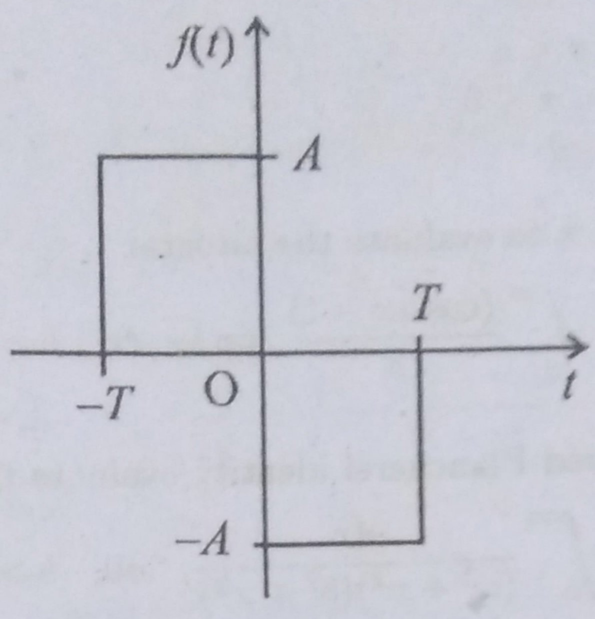

Figure 7.4: Graph of the on-off pulse.

Sketch he graph of the function $w(x) = f(x) - g(x)$ and calculate its F.T.

10. Calculate the F.T. of the off-on-off pulse represented by the function

$$f(x) = \begin{cases} 0, & x < -2 \\ -1, & -2 \leq x < -1 \\ +1, & -1 \leq x \leq 1 \\ -1, & 1 < x \leq 2 \\ 0, & x > 2 \end{cases}.$$

11. Show that the F.T. of the function

$$f(x) = \begin{cases} \sin ax, & |x| \leq \pi/a \\ 0, & |x| > \pi/a \end{cases}$$

is $12a \sin(\pi k/a)/(k^2 - a^2)$.

12. Find the the F.T. of the function

$f(x) = \exp(-ax) \, \sin \xi_0 x \; H(x)$.

13. Show that the Fourier sine and cosine transforms of the function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \leq x \leq a \\ 0, & x > a \end{cases}$$ are $(1 - \cos ax)/x$ and $(\sin ax)/x$ respectively.

14. Calculate the Fourier sine and cosine transforms of

$f(x) = \exp(-ax) \; H(x), \quad a > 0.$

23. Let $f(x)$ and $F(k)$ denote the F.T. pair, with the condition that $f(x)$ is continuous and absolutely integrable. Given that

$$F(\xi) + \int_{-\infty}^{+\infty} F(\xi - u)\exp(-|u|)du = \begin{cases} \xi^2, & 0 \leq u \leq 1 \\ 0, & \text{for } u < 0,\ u > 1 \end{cases}$$

find $f(x)$.

24. Let $f(x)$ and $F(\xi)$ denote the F.T. pair, as in the previous problem. If $F(\xi) = 0$, for all $k \geq |\xi_0|$, then show that for all $a > |\xi_0|$

$$f(x) \star \left(\frac{\sin ax}{\pi x}\right) = f(x)$$

25. Let $F(\xi)$ be the F.T. of the function $f(x)$, defined as follows

$$f(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{for all other values of } x \end{cases}$$

find the function $g(x)$ such that the F.T. $G(\xi)$ of $g(x)$ which satisfies $G(\xi) = |F(\xi)|^2$.

26. Use the formula

$$\mathcal{F}\{x^n f(x)\} = (-\imath)^n F^n(\xi)$$

to calculate $\mathcal{F}\{x\exp(-\alpha x^2)\}$, $\alpha > 0$.

(**Ans.**: $\mathcal{F}\{x\exp(-\alpha x^2)\} = (1/\sqrt{2\alpha})\exp[-\xi^2/(4\alpha)]$)

27. Find the the F.T. of the function

$$f(x) = \exp(-ax)\ \sin k_0 x\ H(x).$$

7.8 Use of Complex Fourier Transform in Solving B.V./I.V. Problems

When we have an I.V./B.V. problem in which the space coordinate x extends over the whole real line, we may use complex Fourier transform. For the validity of the formula for the derivatives it is required that both the unknown function u and its partial derivatives approach zero as x goes to $\pm\infty$.

7.8.1 Illustrative examples

Example 1

(a) Solve the problem by means of the Fourier transform method.

$$u_t = \kappa^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0 \tag{1}$$

$$u(x,\ 0) = f(x), \quad u(x,\ t),\ u_x(x,\ t) \ \rightarrow 0 \text{ as } x \rightarrow \pm\infty \tag{2}$$

(b) Obtain the solution of the problem in (a) when $u(x,\ 0) = \exp(-\alpha x^2)$.

Solution

The given PDE and the B.Cs. describe the conduction of heat through a rod or wire of infinite length. The initial temperature distribution is given by $f(x)$ and the temperature towards the end points gets smaller and smaller. The source of heat in the body is the initial temperature.

We assume that both u and $f(x)$ satisfy the conditions for the existence of their Fourier transforms. Taking the Fourier transforms of both sides of (1), and denoting the Fourier transform of $u(x,\ t)$ w.r.t. x by $U(\xi,\ t)$, we obtain the first order ODE

$$\frac{dU(\xi,\ t)}{dt} = -\kappa^2 \xi^2 U(\xi,\ t) \tag{3}$$

General solution of (3) is given by

$$U(\xi,\ t) = c \exp(-\kappa^2 \xi^2 t) \tag{4}$$

The I.C. in (2) can be transformed as $U(\xi,\ 0) = F(\xi)$. From this condition and (4), we obtain $c = F(\xi)$. Hence

$$U(\xi,\ t) = F(\xi) \exp(-\kappa^2 \xi^2 t)$$

Taking the inverse Fourier transform, we have

$$u(x,\ t) = \mathcal{F}^{-1}\{F(\xi) \ \exp(-\beta\xi^2)\}$$

where $\beta = \kappa^2 t$. Next using the convolution theorem, we can simplify the right side of the above relation as

$$u(x,\ t) = f(x) \star \mathcal{F}^{-1}\{\exp(-\beta\xi^2)\}$$

Now using the formula in equation (2) of example 3, *viz.*

$$\mathcal{F}^{-1}\{\exp(-\beta\xi^2)\} = \frac{1}{\sqrt{2\beta}} \exp\left(-\frac{x^2}{4\beta}\right)$$

we finally obtain

$$u(x,\ t) = \frac{1}{\sqrt{\pi t \kappa^2}} \int_{-\infty}^{+\infty} f(x') \exp[-(x - x')^2/(4\kappa^2 t)]$$

(b)

The initial temperature distribution in this case is the sharply peaked Gaussian function $f(x) = \exp(-\alpha x^2)$.

Physically it implies that if the rod is heated from the middle, the temperature gets smaller and smaller as one moves away from the middle of the rod.

On substituting for this value of $f(x)$ in (1), we have

$$\begin{aligned} u(x,\ t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\exp(-x^2/(4\beta))}{\sqrt{2\beta}}\, f(x - x')\, dx' \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\beta}} \int_{-\infty}^{+\infty} \exp(-x^2/(4\beta)) \exp[-\alpha(x^2 + x'^2 - 2xx')]\, dx' \\ &= \frac{\exp(-\alpha x^2)}{2\sqrt{\pi\beta}} \int_{-\infty}^{+\infty} \exp(-x'^2/4\beta) \exp[(\alpha + 1/\beta)x'^2 + 2\alpha x x']\, dx' \end{aligned}$$

Now

$$\begin{aligned} \alpha'\xi^2 + \imath\xi x &= \alpha'\left(\xi^2 + \frac{\imath x}{\alpha'}\xi\right) \\ &= \alpha'\left[\left(\xi + \frac{\imath x}{2\alpha'}\xi\right)^2 + \frac{x^2}{4\alpha'^2}\right] \end{aligned}$$

On substitution we obtain

$$\begin{aligned} u(x,\ t) &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{+\infty} \exp\left[-\alpha'\left(\xi + \frac{\imath x}{2\alpha'}\right)^2\right] \exp(-x^2/4\alpha')\, d\xi \\ &= \frac{1}{2\sqrt{\pi\alpha}} \exp(-x^2/4\alpha') \int_{-\infty}^{+\infty} \exp(-\alpha' K^2)\, dK \\ &= \frac{1}{2\sqrt{\pi\alpha}} \exp(-x^2/4\alpha') \sqrt{\frac{\pi}{\alpha'}} = \frac{\exp(-x^2/4\alpha')}{4\alpha t + 1} \end{aligned}$$

Example 2

Solve the problem by means of the Fourier transform method.

$$u_t = \alpha^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0 \tag{1}$$

$$u_x(x,\ 0) = f(x), \quad |u(x,\ 0)| < \infty, \quad -\infty < x < +\infty \tag{2}$$

Solution

We assume that both u and $u_x \to 0$ as $x \to \pm\infty$, and further that both u and $f(x)$ are absolutely integrable over $(-\infty, +\infty)$. Taking the Fourier transform of both sides of (1), and denoting the Fourier transform of $u(x,\ t)$ w.r.t. x by $U(\xi,\ t)$, we obtain the first order ODE

$$dU(\xi,\ t)/dt = -\alpha^2\,\xi^2\,U(\xi,\ t) \tag{3}$$

General solution of (3) is given by

$$U(\xi,\ t) = c\ \exp(-\alpha^2\,\xi^2\,t) \tag{4}$$

The I.C. in (2) can be transformed as $U(\xi,\ 0) = F(\xi)$. From this condition and (4), we obtain $c = F(\xi)$. Hence

$$U(\xi,\ t) = F(\xi)\ \exp(-\alpha^2\,\xi^2\,t)$$

Taking the inverse Fourier transform, we have

$$u(x,\ t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(-\imath\xi x)\,F(\xi)\,\exp(-\alpha^2\,\xi^2\,t)\,d\xi$$

or

$$\begin{aligned} u(x,\ t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(-\imath\xi x)\,\exp(-\alpha^2\,\xi^2\,t)\cdot \\ &\quad\times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(\imath\xi x')\,f(x')\,dx'\,d\xi \end{aligned}$$

or on changing order of integration

$$u(x,\ t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x')\left[\int_{-\infty}^{+\infty} \exp[-\imath\xi(x-x') - \alpha^2\,\xi^2\,t]\,d\xi\right]dx'$$

or

$$u(x,\ t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x')\,I(x,\ x')\,dx' \tag{5}$$

where

$$\begin{aligned} I(x,\ x') &= \int_{-\infty}^{\infty} \exp[-\imath\xi(x-x') - \alpha^2\,\xi^2\,t]\,d\xi \\ &= \int_{-\infty}^{+\infty} \exp(-\imath\xi u - \beta\,\xi^2)\,d\xi \end{aligned}$$

where $\beta = \alpha^2 t$ and $u = x - x'$. By completing the squares, we can express the Gaussian integral as

$$\begin{aligned} I(x,\ x') &= \exp(-u^2/4\beta)\int_{-\infty}^{+\infty} \exp[-\beta(\xi + \imath u/2\beta)^2\,d\xi \\ &= \exp(-u^2/4\beta)\cdot\sqrt{\frac{\pi}{\beta}} \end{aligned}$$

Hence on substituting in (5), we have

$$u(x,\ t) = \frac{1}{2\pi}\,\frac{1}{\sqrt{\pi\,t\,\alpha^2}}\int_{-\infty}^{+\infty} f(x')\,\exp\left(-\frac{(x-x')^2}{4\alpha^4t^2}\right)\,dx'$$

Example 3

Solve by the Fourier transform method

$$u_{xxxx} = (1/a^2)\, u_{tt} \tag{1}$$

where

$$u(x,\, 0) = f(x), \quad u_t(x,\, 0) = a g'(x) \tag{2}$$

(Suitable asymptotic behaviour for u and its derivatives, and for g is assumed); *i.e.* $g,\, u,\, u_x,\, u_{xx},\, u_{xxx} \to 0$ as $x \to \pm\infty$.

Solution

From (1) taking Fourier transform of both sides

$$\mathcal{F}\{u_{xxxx}\} = \mathcal{F}\{(1/a^2)\, u_{tt}\}$$

where $\mathcal{F}$ denotes F.T. operator w.r.t. x. On simplification the last equation becomes

$$(-i\xi)^4\, U(\xi,\, t) = \frac{1}{a^2} \frac{\partial^2}{\partial t^2} U(\xi,\, t) \tag{3}$$

where

$$U(\xi,\, t) = \mathcal{F}\{u(x,\, t)\} = (1/\sqrt{2\pi}) \int_{-\infty}^{+\infty} \exp(i\xi x)\, u(x,\, t)\, dx$$

From (3)

$$(d^2/dt^2)\, U(\xi,\, t) - a^2 \xi^4\, U(\xi,\, t) = 0$$

or

$$(D_t^{\,2} - a^2 \xi^4) U(\xi,\, t) = 0$$

Therefore

$$U(\xi,\, t) = A \exp(a\xi^2 t) + B \exp(-a\xi^2 t) \tag{4}$$

We now transform the initial conditions in terms of ξ by taking Fourier transform of both sides of conditions in (2) w.r.t. x.

From these equations

$$U(\xi,\, 0) = F(\xi) \tag{5a}$$

and

$$\left.\frac{d}{dt} U(\xi,\, t)\right|_{t=0} = a(-i\xi)\, G(\xi) = -i a \xi\, G(\xi) \tag{5b}$$

To find A, B in (4), we use I.Cs. (5). From these I.Cs. we obtain

$$U(\xi,\ 0) = F(\xi) = A + B \text{ or } A + B = F(\xi) \tag{6}$$

Again from (4)

$$\frac{d}{dt}U(\xi,\ t)\Big|_{t=0} = a\,\xi^2\,A\,\exp(a\xi^2 t) - a\,\xi^2\,B\,\exp(-a\xi^2 t)\big|_{t=0}$$

or

$-a\imath\xi\,G(\xi) = a\xi^2 A - a\xi^2 B$ which implies $\imath\,G(\xi) = \xi A - \xi B$, wherefrom

$$A - B = (-\imath/\xi)\,G(\xi) \tag{7}$$

From (6) and (7)

$$A = \frac{1}{2}\{F(\xi) - \frac{\imath}{\xi}G(\xi)\},\ \text{ and }\ B = \frac{1}{2}\{F(\xi) + \frac{\imath}{\xi}G(\xi)\}$$

Substituting in (4) and get

$$\begin{aligned} U(\xi,\ t) &= \frac{1}{2}\{F(\xi) - \frac{\imath}{\xi}G(\xi)\}\exp(a\xi^2 t) + \frac{1}{2}\{F(\xi) + \frac{\imath}{\xi}G(\xi)\}\exp(-a\xi^2 t) \\ &= \frac{1}{2}F(\xi)\{\exp(a\xi^2 t) + \exp(-a\xi^2 t)\} - \frac{1}{2}\frac{\imath}{\xi}G(\xi)\{\exp(a\xi^2 t) - \exp(-a\xi^2 t \end{aligned}$$

Taking inverse Fourier transform, we have

$$u(x,\ t) = \mathcal{F}^{-1}\{U(\xi,\ t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-\imath\xi\,x)\,U(\xi,\ t)\,d\xi$$

wherefrom we can obtain $u(x,\ t)$.

Example 4

Solve the following I.V/B.V problem by the F.T. method.

$$u_{xx}(x,\ t) = (1/c^2)\,u_{tt}(x,\ t) \tag{1}$$

$$u(x,\ 0) = f(x),\quad u_t(x,\ 0) = g(x) \tag{2}$$

and $u,\ u_x \rightarrow 0$, as $x \rightarrow \pm\infty$. (The problem describes the transverse vibrations of a string of infinite length.)

Solution

From (1) taking Fourier transform of both sides w.r.t x we have

$-\xi^2 U(\xi,\ t) - (1/c^2)\,(d^2/dt^2)\,U(\xi,\ t) = 0$

or

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{+\infty} \exp(\imath\xi x)f(x) + \int_{-\infty}^{0} \exp(\imath\xi x)f(-x)\right] dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} [\exp(\imath\xi x) + \exp(-\imath\xi x)]\ f(x)\ dx$$

or

$$F_c(\xi) = \sqrt{2/\pi}\ \int_0^\infty \cos\xi x\, f(x)\ dx$$

Similarly if we define the odd extension of $f(x)$ over the whole real line as

$$f_o(x) = \begin{cases} f(\ x), & \text{for } 0 \le x < \infty \\ -f(-x), & \text{for } \infty < x < 0 \end{cases}$$

and perform similar calculations, we arrive at the definition of the Fourier sine transform given above.

Linearity of Fourier sine and cosine transforms

As in case of exponential Fourier transform and its inverse, the cosine and sine transforms are linear operators and this result follows from their definitions.

7.9.3 Fourier sine and cosine transforms of derivatives

To calculate Fourier sine and cosine transforms of first order derivative, we assume that (i) $f(x)$ is real and (ii) $|f(x)| \to 0$ as $x \to \infty$. Then

$$\begin{aligned} \mathcal{F}_c\{f'(x)\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\ \cos\ \xi x\ dx \\ &= \sqrt{\frac{2}{\pi}}\left[\cos\xi x\, f(x)|_0^\infty + \xi\int_0^\infty f(x)\ \sin\xi x\ dx\right] \\ &= \sqrt{\frac{2}{\pi}}\times 0 - \sqrt{\frac{2}{\pi}}\, f(0) + \xi\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\xi x\ dx \end{aligned}$$

Therefore

$$\mathcal{F}_c\{f'(x)\} = -\sqrt{2/\pi}\, f(0) + \xi F_s(\xi) \qquad (7.7.5)$$

Similarly

$$\mathcal{F}_s\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\ \sin\xi x\ dx$$

$$= \sqrt{\frac{2}{\pi}} f(x) \sin \xi x \Big|_0^{\infty} - \xi \int_0^{\infty} f(x) \cos \xi x \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \times 0 - \sqrt{\frac{2}{\pi}} \times 0 - \xi \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \xi x \, dx$$

or

$$\mathcal{F}_s\{f'(x)\} = -\xi F_c(\xi) \tag{7.7.6}$$

To calculate Fourier sine and cosine transforms of second order derivatives, we assume further that (iii) $|f'(x)| \to 0$, as $x \to \infty$, then

$$\begin{aligned}
\mathcal{F}_s\{f''(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f''(x) \sin \xi x \, dx \\
&= \sqrt{\frac{2}{\pi}} \left[\sin \xi x \, f'(x) \Big|_0^{\infty} - k \int_0^{\infty} f'(x) \cos \xi x \, dx \right] \\
&= \sqrt{\frac{2}{\pi}} \left[0 - \xi \int_0^{\infty} f'(x) \cos \xi x \, dx \right] \\
&= -\xi \sqrt{\frac{2}{\pi}} \int_0^{\infty} f'(x) \cos \xi x \, dx \\
&= -\xi \mathcal{F}_c\{f'(x)\} \\
&= -\xi \left[-\sqrt{\frac{2}{\pi}} f(0) + \xi F_s(\xi) \right]
\end{aligned}$$

Therefore

$$\mathcal{F}_s\{f''(x)\} = \sqrt{2/\pi}\,\xi f(0) - \xi^2 F_s(\xi)$$

Similarly

$$\begin{aligned}
F_c\{f''(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f''(x) \cos \xi x \, dx \\
&= \sqrt{\frac{2}{\pi}} \left[f'(x) \cos \xi x \Big|_0^{\infty} + \xi \int_0^{\infty} f'(x) \sin \xi x dx \right] \\
&= 0 - \sqrt{\frac{2}{\pi}} f'(0) + \xi \sqrt{\frac{2}{\pi}} \int_0^{\infty} f'(x) \sin \xi x \, dx \\
&= -\sqrt{\frac{2}{\pi}} f'(0) + \xi \mathcal{F}_s\{f'(x)\} \\
&= -\sqrt{\frac{2}{\pi}} f'(0) + k[-\xi F_c(\xi)] \\
&= -\sqrt{\frac{2}{\pi}} f'(0) - \xi^2 F_c(\xi)
\end{aligned}$$

7.9.4 Illustrative examples on Fourier sine and cosine transforms

Example 1

Calculate Fourier sine transform of the function $f(x) = \exp(-x)\cos x$.

Solution

By definition

$\mathcal{F}_s\{\exp(-x)\cos x\} = \sqrt{2/\pi}\int_0^\infty \exp(-x)\cos x \sin \xi x dx$

Using the result $\sin A\cos B = (1/2)[\sin(A+B) + \sin(A-B)]$, we have

$$\sin \xi x \cos x = (1/2)\,[\sin(\xi+1)x + \sin(\xi-1)x]$$

Therefore

$$\mathcal{F}_s\{\exp(-x)\cos x\} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \exp(-x)\,\{\sin(\xi+1)x + \sin(\xi-1)x\}\,dx$$

Now we use the formula

$$\int e^{ax}\sin bx\,dx = \frac{\exp(ax)}{a^2+b^2}\{a\sin bx - b\cos bx\}$$

and obtain

$$\begin{aligned}\int_0^\infty \exp(-x)\sin(\xi+1)x\,dx &= \frac{\exp(-x)}{1+(\xi+1)^2}\times \\ &\times \{[-\sin(\xi+1)x - (\xi+1)\cos(\xi+1)x\}|_0^\infty \\ &= 0 - \frac{1}{\xi^2+2\xi+2}(0-(\xi+1)) \\ &= \frac{\xi+1}{\xi^2+2\xi+2}\end{aligned}$$

Similarly

$$\begin{aligned}\int_0^\infty e^{-x}\sin(\xi-1)x\,dx &= \frac{\exp(-x)}{1+(\xi-1)^2}\times \\ &\times \{-\sin(\xi-1)x - (k-1)\cos(\xi-1)\}|_0^\infty \\ &= 0 - \frac{1}{\xi^2-2\xi+2}(0-(\xi-1)) \\ &= \frac{\xi-1}{\xi^2-2\xi+2}\end{aligned}$$

Therefore

$$\mathcal{F}_s\{\exp(-x)\cos x\} = \frac{1}{\sqrt{2\pi}}\left[\frac{\xi+1}{\xi^2+2\xi+2} + \frac{\xi-1}{\xi^2-2\xi+2}\right]$$