Chapter 7

The Fourier Transform and its Applications

In this chapter we discuss another well-known integral transform which goes by the name of *Fourier transform*. After discussing its theory we will turn to its applications.

7.1 Definition and Basic Properties

Given an integrable function $f(x)$ for $-\infty < x < \infty$. We can associate with it another function $F(\xi)$ of variable ξ, $(-\infty < \xi < +\infty)$, by the relation

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(\imath\xi x)\, f(x)\, dx \tag{7.1.1}$$

The function $F(\xi)$ is called the *Fourier transform* of $f(x)$, and $f(x)$ is called the inverse Fourier transform of $F(\xi)$. It can be shown that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-\imath\xi x)\, F(\xi)\, d\xi \tag{7.1.2}$$

Notation and Convention

If we write

$$F(\xi) = c_1 \int_{-\infty}^{+\infty} \exp(\imath\,\xi\, x)\, f(x)\, dx$$

and

$$f(x) = c_2 \int_{-\infty}^{+\infty} \exp(-\imath\xi x) F(\xi)\, d\xi$$

then the following forms of coefficients are mutually consistent.

(i) $c_1 = (1/\sqrt{2\pi}),\ \ c_2 = 1/\sqrt{2\pi}$

(ii) $c_1 = 1,\ \ c_2 = 1/2\pi$

(iii) $c_1 = 1/2\pi,\ \ c_2 = 1$

Also we will use the notation $F(\xi) = \mathcal{F}\{f(x)\}$.

where the operator $\mathcal{F}$ is called the *Fourier transform operator.*

It is also possible to define the Fourier transform and its inverse in such a way that the coefficients c_1, c_2 in each are unity. In this definition they are given by the relations

$$F(\xi) = \int_{-\infty}^{+\infty} \exp(2\pi\, \imath\xi x)\, f(x)\, dx$$

and

$$f(x) = \int_{-\infty}^{+\infty} \exp(-2\pi\, \imath\xi x)\, F(\xi)\, d\xi$$

These relations can be obtained from (7.1.1) and (7.1.2) by making the transformations $x' = \sqrt{2\pi}\, x$ and $\xi' = \sqrt{2\pi}\, \xi$ and then reverting to the unprimed symbols.

Various choices of pairs of variables such as $(x,\ p)$, $(t,\ \omega)$ are used by different authors. In order to indicate the associated variable the following notation is also used for the Fourier transform:

$$\mathcal{F}\{f(x),\ x \to \xi\},\ \mathcal{F}\{f(t),\ t \to \omega\}$$

for the Fourier transforms $F(\xi)$, $F(\omega)$ of $f(x)$, $f(x)$ and $f(t)$ respectively.

7.1.1 The Fourier transform and its inverse

If the function $f(x)$ or $F(\xi)$ is continuous or piecewise continuous over $(-\infty, +\infty)$ and bounded then Fourier transform and inverse Fourier transform exist.

If the function $f(x)$ is absolutely integrable i.e. the integral $\int_{-\infty}^{+\infty} |f(x)|\, dx$ exists, then the Fourier transform exists. This is a sufficient condition. Similarly for the inverse Fourier transform.

Linearity of $\mathcal{F}$ and $\mathcal{F}^{-1}$ Operators

The operators $\mathcal{F}$ and $\mathcal{F}^{-1}$ are linear i.e.

$$\mathcal{F}\{c_1 f_1(x) + c_2 f_2(x)\} = c_1 \mathcal{F}\{f_1(x)\} + c_2 \mathcal{F}\{f_2(x)\}$$

and

$$\mathcal{F}^{-1}\{c_1 F_1(\xi) + c_2 F_2(\xi)\} = c_1 \mathcal{F}^{-1}\{F_1(\xi)\} + c_2 \mathcal{F}^{-1}\{F_2(\xi)\}$$

7.1.2 Fourier series and Fourier transform

The Fourier series representation of a periodic piecewise smooth function over the interval $(-l,\ l)$ leads to the integral representation of the same function as $l \to \infty$ and the index n in the Fourier series $\to \infty$. The condition of periodicity is replaced by the condition of absolute integrability for the function $f(x)$ over $(-\infty,\ \infty)$. This can be seen as follows.

We start with the complex from of the Fourier series representation for the function $f(x)$, as explained in chapter 1.

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \exp(\imath n\pi x/\ell), \quad -\ell < x < \ell \tag{7.1.3}$$

where the complex Fourier coefficients c_n are given by

$$c_n = \frac{1}{2\ell} \int_{-\ell}^{+\ell} f(x) \exp(\imath n\pi x/\ell)\, dx \tag{7.1.4}$$

Now we consider the situation in which $\ell \to \infty$. Let $n\pi/\ell = \xi$ then $n = \ell\xi/\pi$ and the increment Δn in n will be given by $\ell\,\Delta\xi/\pi$ i.e.

$\Delta n = \ell\,\Delta\xi/\pi$ or $\Delta\xi = \pi/\ell$, where $\Delta n = 1$.

In the limit $\ell \to \infty$, $\Delta\xi \to 0$. In view of this we can rewrite (7.1.3) as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \Delta n \exp(\imath n\pi x/\ell) = \sum_j c_l(j) \frac{\ell}{\pi} \Delta\xi \exp(\imath\xi x) \tag{7.1.5}$$

where we have put $c_n = c(\ell\xi/\pi) = c_\ell(\xi)$ to show the dependence of the coefficients c_n on ℓ and ξ.

Similarly (7.1.4) can be written as

$$c(\ell\xi/\pi) \equiv c_\ell(\xi) = \frac{1}{2\ell} \int_{-\ell}^{+\ell} f(x) \exp(\imath\xi x)\, dx$$

or

$$\frac{\ell}{\pi} c_\ell(\xi) = \frac{1}{2\pi} \int_{-\ell}^{+\ell} f(x) \exp(-\imath\xi x)\, dx \tag{7.1.6}$$

Equations (7.1.5) and (7.1.6) correspond to each other in the same way equations (7.1.3) and (7.1.4) do. Now we let $\ell \to \infty$ so that ξ now income continuous variable, and assuming that the sum goes over into the Riema integral, we have from (7.1.5)

$$f(x) = \int_{-\infty}^{+\infty} c(\xi) \exp(\imath\xi x) \, d\xi \qquad (7.1$$

where $c(\xi) = \lim_{\ell\to\infty}(l/\pi)\, c(\ell\xi/\pi)$. Also from (7.1.6)

$$c(\xi) = \frac{1}{2\pi} \int_{-\ell}^{+\ell} f(x) \exp(-\imath\xi x) \, dx \qquad (7.1$$

To conform to the notation followed in this book we further set $c(\xi) = (1/\sqrt{2}$ then (7.1.7) and (7.1.8) become

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \exp(\imath\xi x) \, dx$$

and

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\xi) \exp(-\imath\xi x) \, d\xi$$

7.2 Properties of Fourier Transformation

1. **Linearity property**

It is a linear transformation; both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are linear.

2. **Conjugation property**

If $f(x)$ is real, then $F(-\xi) = \overline{F(\xi)}$, (where the bar symbol denotes the compl conjugate).

Proof

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\imath\xi x} f(x) \, dx$$

and therefore

$$\overline{F(\xi)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-\imath\xi x) f(x) \, dx$$

Also

$$F(-\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-\imath\xi x) f(x) \, dx$$

which proves that $F(-\xi) = \overline{F(\xi)}$.

3. **Real and Complex Values of the F.T.**

(a) If $f(x)$ is real and even, $F(k)$ is real.

(b) If $f(x)$ is real and odd, $F(k)$ is pure imaginary.

(c) If $f(x)$ is complex, then

$$\mathcal{F}\{\overline{f(-x)}\} = \overline{F(\xi)}$$

Proof of (3) a

We have to prove that if $f(x)$ is even, then $F(\xi)$ is real.

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(\imath\xi x)\, f(x)\, dx$$

When f(x) is even, *i.e.* $f(x) = f(-x)$, then

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(\imath\xi x)\, f(-x)\, dx$$

Let $-x = x'$ or $dx = -dx'$, therefore

$$\begin{aligned} F(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \exp(-\imath\xi x')\, f(x')\, (-dx') \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-\imath\xi x')\, f(x')\, (-dx') = F(-\xi) \end{aligned}$$

Hence $F(\xi) = \overline{F(\xi)}$, which shows that $F(\xi)$ is real.

Proof of (3 b)

$$F(\xi) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} \exp(-\imath\xi x)\, f(x)\, dx$$

When $f(x)$ is odd i.e. $f(x) = -f(-x)$, we have

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-\imath\xi x)\, [-f(-x)]\, dx$$

Let $x' = -x$, then $dx' = -dx$, and

$$F(\xi) = \frac{-1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} \exp(-\imath x'\xi)\, f(x')\, (-dx')$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-\imath\xi x')\, f(x')\, dx'$$

or $F(\xi) = -F(-\xi) = -\overline{F(\xi)}$, which shows that $F(\xi)$ is pure imaginary.

Proof of 3(c)

$$\begin{aligned}\mathcal{F}\{\bar{f}(-x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(\imath\xi x)\, \bar{f}(-x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-\imath\xi x')\, \bar{f}(x')\, dx', \quad (x' = -x)\end{aligned}$$

$$\begin{aligned}&= \text{complex conjugate of} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(\imath\xi x')\, f(x')\, dx' \\ &= \text{complex conjugate of } F(\xi) = \overline{F(\xi)}\end{aligned}$$

4. Attenuation property

$$\mathcal{F}\{\exp(ax)\, f(x)\} = F(\xi - a\imath)$$

It can be proved directly from the definition.

$$\begin{aligned}\mathcal{F}\{e^{ax}\, f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(\imath\xi x)\, \exp(ax)\, f(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[(\imath\xi + a)x]\, f(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[\imath(\xi - \imath a)x]\, f(x)\, dx \\ &= F(\xi - a\imath)\end{aligned}$$

5. Shifting property

(i) $\mathcal{F}\{f(x - a)\} = \exp(\imath\xi a)\, F(\xi)$

(ii) $\mathcal{F}\{\exp(\imath\lambda x)\, f(x)\} = F(\xi + \lambda)$

Proof of (i)

$$\mathcal{F}\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(\imath\xi x)\; f(x-a)\, dx$$

Put $x - a = x'$, or $dx = -dx'$, then

$$\begin{aligned}\mathcal{F}\{f(x-a)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[\imath\xi(x'+a)]\, f(x')\, dx' \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(\imath\xi a) \exp(\imath\xi x') f(x')\, dx' \\ &= \exp(\imath\xi a) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(\imath\xi x') f(x')\, dx' \\ &= \exp(\imath\xi a)\, F(\xi)\end{aligned}$$

Proof of (ii)

$$\begin{aligned}\mathcal{F}\{\exp(\imath a x)\, f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(\imath a x) \exp(\imath\xi x)\, f(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp[\imath(\xi + a)x]\, f(x)\, dx \\ &= F(\xi + a)\end{aligned}$$

6. Scaling property If c is a non-zero constant, then $\mathcal{F}\{f(cx)\} = (1/|c|)\, F(\xi/c)$

Proof

Let $c > 0$, then

$$\begin{aligned}\mathcal{F}\{f(cx)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(\imath\xi x)\, f(cx)\, dx, \quad x' = cx \\ &= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} \exp(\imath\xi x'/c)\, f(x')\, dx'/c \\ &= \frac{1}{c} \frac{1}{\sqrt{2\pi}} \int_{-infty}^{+\infty} \exp[\imath\xi x/c]\, f(x)\, dx \\ &= \frac{1}{c} F(\xi/c), \quad c > 0\end{aligned}$$

If $c < 0$, then we can show that

$$\mathcal{F}\{f(cx)\} = -(1/c)\, F(\xi/c)$$

Combining the two results, we have

$$\mathcal{F}\{f(cx)\} = (1/|c|)\, F(\xi/c)$$

7. Modulation property of Fourier transform

$\lim_{|\xi|\to\infty}|F(\xi)| \leq \lim_{|\xi|\to\infty}(1/\xi|)\ (1/\sqrt{2\pi})$, a finite positive number.

or $\lim_{|\xi|\to\infty}|F(\xi)| \leq 0$. Hence the theorem.

7.3 Fourier Transforms of some Simple Functions

In this section we present examples to evaluate the Fourier transforms of some simple functions.

7.4 Examples and Exercises

7.4.1 Illustrative examples

Example 1

Find the Fourier transform of the box function $f(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}$

where $a > 0$.

Solution

$$\begin{aligned}
\mathcal{F}\{f(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(\imath\xi x) f(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} 0 \times \exp(\imath\xi x)\, dx \\
&+ \int_{-a}^{+a} 1 \times \exp(\imath\xi x)\, dx + \int_{+a}^{+\infty} 0 \times \exp(\imath\xi x)\, dx \\
&= \frac{1}{\sqrt{2\pi}}\left(0 + \frac{\exp(\imath\xi x)}{\imath\xi}\right)\Bigg|_{-a}^{+a} = \frac{1}{\sqrt{2\pi}}\frac{\exp(\imath a\xi) - \exp(\imath\xi a)}{\imath\xi} \\
&= \frac{1}{\sqrt{2\pi}}\frac{\exp(\imath a\xi) - \exp(-\imath a\xi)}{\imath\xi} = \frac{2}{\sqrt{2\pi}}\frac{\exp(\imath a\xi) - \exp(-\imath a\xi)}{2\imath\xi} \\
&= \sqrt{2/\pi}\,\frac{\sin\xi a}{\xi}
\end{aligned}$$

Example 2

Evaluate the Fourier transform of the function

$$f(x) = \begin{cases} 0, & x < 0 \\ \exp(-\alpha x), & x \geq 0,\ \alpha > 0 \end{cases}$$

Solution

The given function is piecewise continuous and it is absolutely integrable, i.e. the integral $\int_{-\infty}^{+\infty} |f(x)|dx < \infty$. Hence the Fourier transform of the given function exists.

By definition

$$\begin{aligned}\mathcal{F}\{f(x)\} \equiv F(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\imath\xi x} f(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{\imath\xi x} 0\,dx \\ &+ \int_{0}^{+\infty} e^{\imath\xi x} e^{-\alpha x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty} e^{(\imath\xi-\alpha)x}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\left.\frac{e^{(\imath\xi-\alpha)x}}{\imath\xi-\alpha}\right|_0^{\infty} = \frac{1}{\sqrt{2\pi}}\frac{1}{\alpha-\imath\xi} = \frac{1}{\sqrt{2\pi}}\frac{\alpha+\imath\xi}{\alpha^2+\xi^2}\end{aligned}$$

Example 3

Discuss if the function $f(x)$, defined below, satisfies the conditions for the existence of its Fourier transform and calculate its complex (exponential) Fourier transform.

$$f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

Further show that

$$\int_0^{\infty} \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\frac{x}{2}\,dx = \frac{3\pi}{16}$$

Solution

The given function is bounded and the integral $\int_{-\infty}^{+\infty} f(x)dx$ is absolutely integrable. Hence the Fourier transform exists.

$$\begin{aligned}F(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(\imath\xi x) f(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{+1} \exp(\imath\xi x)(1-x^2)\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-1}^{+1} (1-x^2)\left(\cos\xi x + \imath\sin\xi x\right)dx \\ &= \sqrt{\frac{2}{\pi}}\int_0^1 (1-x^2)\cos\xi x\,dx\end{aligned}$$

To simplify further, we evaluate the integral on the right side of the last equation by Kronecker's rule, and obtain

$$F(\xi) = \sqrt{\frac{2}{\pi}}\cdot\left[(1-x^2)\frac{\sin\xi x}{\xi} - \left(\frac{-\cos\xi x}{\xi^2}\right)\cdot(-2x) + \left(\frac{-\sin\xi x}{\xi^3}\right)\cdot(-2)\right]\Bigg|_0^1$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\exp[(1+\imath\xi)\pi\imath/2}{2\exp(\pi\imath)}$$
$$= \frac{-\imath}{2\sqrt{2\pi}} \exp(-\xi\pi/2)$$

Therefore on substitution and simplification

$$\frac{1}{\sqrt{2\pi}} \oint_{\Gamma_r} z \frac{\exp(\imath\xi z)dz}{1+\exp(2z)} = \sqrt{\frac{\pi}{2}} \exp(-\xi\pi/2)$$

or rewriting the contour integral on the left in terms of integrals along the component paths, we obtain

$$\begin{aligned} F(\xi) &= \frac{1}{\sqrt{2\pi}} \left[\int_{-r}^{+r} \frac{\exp(\imath\xi x)dx}{\exp(x)+\exp(-x)} + \int_0^{\pi} \frac{\exp(\imath\xi(r+\imath y)}{\exp(r+\imath y)+\exp(-r-\imath y)} \imath dy \right. \\ &+ \left. \int_{+r}^{-r} \frac{\exp[\imath\xi(x+\pi\imath)]dx}{\exp(x+\pi\imath)+\exp(-x-\pi\imath)} + \int_{\pi}^{0} \frac{\exp(\imath\xi(-r+\imath y)}{\exp(-r+\imath y)+\exp(r-\imath y)} \right] \\ &= \sqrt{\frac{\pi}{2}} \exp(-\xi\pi/2) \end{aligned}$$

7.4.2 Exercises

1. Compute the F.T. of $f(x)$ defined by

$$f(x) = \begin{cases} 1+x, & -1 < x \le 0 \\ 1-x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

2. Compute the F.T. of $f(x)$ defined by

$$f(x) = \begin{cases} c, & 0 < x < b, \quad (c > 0) \\ 0, & |x| > b \text{ or } \end{cases}$$

3. Discuss if the given function $f(x)$ satisfies the conditions for the existence of its Fourier transform and calculate its complex (exponential) Fourier transform.

$$f(x) = \begin{cases} x_0, & -a < x < +a \\ 0, & |x| > a, \ a > 0 \end{cases}$$

(**Ans.** $F(k) = 2x_0(\sin ka)/k$).

4. Discuss if the given function $f(x)$ satisfies the conditions for the existence of its Fourier transform and calculate its complex (exponential) Fourier transform.

$$f(x) = \begin{cases} x_0, & -a < x < 0 \\ x_1, & 0 < x < a \\ 0, & \text{otherwise}, \quad a > 0 \end{cases}$$

(**Ans.** $F(\xi) = -\imath(x_1 - x_0)/\xi + x_0 \exp(\imath\omega a) - x_1 \exp(-\imath\omega a)$).

5. Discuss if the given function $f(x)$ satisfies the conditions for the existence of its Fourier transform and calculate its complex (exponential) Fourier transform.

$$f(x) = \begin{cases} \exp(-x), & x > 0 \\ 0, & x \le 0 \end{cases}$$

6. Discuss if the given function $f(x)$ satisfies the conditions for the existence of its Fourier transform and calculate its complex (exponential) Fourier transform.

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

7. Find the (complex or exponential) Fourier transform of

$$f(x) = \begin{cases} \cos k_0 x, & |x| < N\pi/k_0 \\ 0, & |x| > N\pi/k_0 \end{cases}$$

8. Calculate the Fourier transform of $f(x)$ where

$$f(x) = \begin{cases} \exp(-x), & x \ge 0 \\ 0, & x < 0 \end{cases}$$

9. Calculate the F.T. of the following function (also called the *two-sided exponential pulse*, when x is interpreted as time t)

$$f(x) = \begin{cases} \exp(ax), & x \le 0 \\ \exp(-ax), & x > 0 \end{cases}, \quad (a > 0).$$

10. Calculate the F.T. of the 'on-off' pulse shown in the figure below. 11. Calculate the F.T. of the function below.

$$f(x) = \begin{cases} A(+x/X + 1), & -X \le x \le 0 \\ A(-x/X + 1), & 0 < x \le X \end{cases}$$

What is the relationship between this pulse and that of the previous problem?

12. Calculate the F.T. of the off-on-off pulse represented by the function

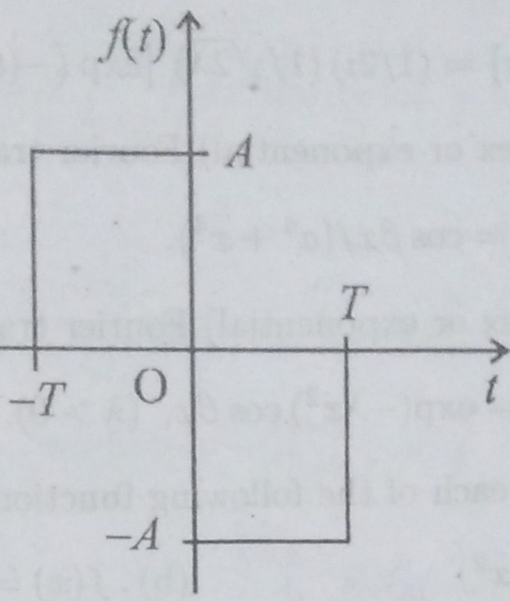

Figure 7.3: Graph of the on-off pulse.

$$f(x) = \begin{cases} 0, & x < -2 \\ -1, & -2 \le x < -1 \\ +1, & -1 \le x \le 1 \\ -1, & 1, x \le 2 \\ 0, & x > 2 \end{cases}.$$

13. Calculate the F.T. of the following functions.

$$f(x) = \begin{cases} 2c, & |x| \le 2 \\ 0, & |x| > 2 \end{cases}, \text{ and } g(x) = \begin{cases} c, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}.$$

14. Show that the F.T. of the function

$$f(x) = \begin{cases} \sin ax, & |x| \le \pi/a \\ 0, & |x| > \pi/a \end{cases}.$$

is $12a \sin(\pi\xi/a)/(\xi^2 - a^2)$.

15. Find the (complex or exponential) Fourier transforms of $\exp(-\lambda x^2) \cos \beta x$ and $\exp(-\lambda x^2) \sin \beta x$.

[Hint: We will make use of the following modulation properties

$$\mathcal{F}\ \{\cos \alpha x\ f(x)\} = (1/2)\ \ (F(\xi + \alpha) + F(\xi - \alpha))$$

$$\mathcal{F}\ \{\sin \alpha x\ f(x)\} = (1/2)\ \ (F(\xi + \alpha) - F(\xi - \alpha))$$

and obtain the results

$$\mathcal{F}\{\exp(-\lambda x^2) \cos \beta x\} = (1/2)\,(1/\sqrt{2\lambda})\ \left[\exp\left(-(\xi + \beta)^2/4\lambda\right) + \exp\left(-(\xi - \beta)^2/4\lambda\right)\right]$$

and

$$\mathcal{F}\{\exp(-\lambda x^2)\sin\beta x\} = (1/2\imath)\,(1/\sqrt{2\lambda})\,\left[\exp\left(-(\xi+\beta)^2/4\lambda\right) - \exp\left(-(\xi-\beta)^2/4\lambda\right)\right]$$

16. Find the (complex or exponential) Fourier transform of

$$F(x) = \cos\beta x/(a^4 + x^4).$$

17. Find the (complex or exponential) Fourier transform of

$$f(x) = \exp(-\lambda x^2)\cos\beta x, \ (\lambda > 0).$$

18. Find the F.T. of each of the following functions.

(a) $f(x) = 1/(a^2 + x^2)$ (b) $f(x) = \cos bx/(a^2 + x^2)$

(c) $f(x) = \sin bx/(a^2 + x^2)$

7.5 Fourier Transforms of Derivatives and other Functions

7.5.1 Fourier transforms of derivatives

The Fourier transforms of derivatives of a function $f(x)$ whose Fourier transform exists are given by

$$\mathcal{F}\{f'(x)\} = (-\imath\xi)F(\xi) \tag{7.5.1}$$

where $f(x)$ is supposed to tend to zero as $x \to \pm\infty$.

$$\mathcal{F}\{f''(x)\} = (-\imath\xi)^2 F(\xi) \tag{7.5.2}$$

where $f(x)$, $f'(x)$ are supposed to tend to 0 as $x \to \pm\infty$. and

$$\mathcal{F}\{f^n(x)\} = (-\imath\xi)^n F(\xi) \tag{7.5.3}$$

where $f(x)$, $f'(x)$, $\cdots$ $f^{n-1}(x) \to 0$ as $x \to \pm\infty$.

Proof

For (7.5.1) we have

$$\begin{aligned}\mathcal{F}\{f'(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(\imath\xi x)f'(x)\,dx \\ &= \frac{1}{\sqrt{2\pi}}\left[\exp(\imath\xi x)f(x)\big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} f(x)(\imath\xi)\exp(\imath\xi x)\,dx\right]\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 + (-\imath\xi)\int_{-\infty}^{\infty} f(x)\,\exp(\iota\xi x)\,dx\right]$$
$$= (-\imath\xi)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\exp(\imath\xi x)dx = (-\imath\xi)\,F(\xi)$$

For (7.5.2)

$$\begin{aligned}\mathcal{F}\{f''(x)\} &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty}\exp(\imath\xi x)f''(x)\,dx\right]\\ &= \frac{1}{\sqrt{2\pi}}\left[\exp(\imath\xi x)\,f'(x)|_{-\infty}^{+\infty} - \imath\xi\int_{-\infty}^{+\infty} e^{\iota\xi x}f'(x)\,dx\right]\end{aligned}$$

Continuing further, we have

$$\begin{aligned}&= (-\imath\xi)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(\imath\xi x)f'(x)\,dx\\ &= (-\imath\xi)\mathcal{F}\{f'(x)\} = (-\imath\xi)(-\imath\xi)F(\xi)\end{aligned}$$

or finally

$$\mathcal{F}\{f''(x)\} = (-\imath\xi)^2\,F(\xi) = -\,\xi^2 F(\xi)$$

For (7.5.3) we have

$$\begin{aligned}\mathcal{F}\{f^n(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(\imath\xi x)f^n(x)\,dx\\ &= \frac{1}{\sqrt{2\pi}}\left\{\exp(\imath\xi x)f^{n-1}(x)\right]_{-\infty}^{+\infty} - (\imath\xi)\int_{-\infty}^{+\infty}\exp(\imath\xi x)\,f^{n-1}(x)\,dx\Big\}\\ &= \frac{(-\imath\xi)}{\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty} e^{\iota\xi x}f^{n-1}(x)\,dx\right]\\ &= \frac{(-\imath\xi)}{\sqrt{2\pi}}\left\{e^{\iota\xi x}f^{n-2}(x)\right]_{-\infty}^{+\infty} - (\imath\xi)\int_{-\infty}^{+\infty}\exp(\imath\xi x)f^{n-2}(x)\,dx\Big\}\\ &= \frac{(-\iota\xi)^2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(\imath\xi x)f^{n-2}(x)dx\end{aligned}$$

Continuing this process we get

$$\mathcal{F}\{f^n(x)\} = (-\iota\xi)^n\int_{-\infty}^{+\infty} f(x)\exp(\imath\xi x)\,dx = \frac{(-\imath\xi)^n}{\sqrt{2\pi}} = (-\imath\xi)^n F(\xi)\;F(\xi)$$

7.5.2 Fourier transform of functions of the form $x^n f(x)$

Let n be a positive integer and $f(x)$ a piecewise continuous function on the interval $[-\ell, +\ell]$ for every positive ℓ. Suppose that $\int_{-\infty}^{+\infty} |x^n f(x)|dx$ converges. Then

$$\mathcal{F}\{x^n f(x)\} = [-\imath\,(d/d\xi)]^n\; F(\xi)$$

Proof

Using the definition

$$F(\xi) = (1/\sqrt{2\pi})\int_{-\infty}^{+\infty} \exp(\imath\xi x)\, f(x)\, dx$$

and differentiating successively w.r.t. ξ we obtain

$$F'(\xi) = \imath\mathcal{F}\{xf(x)\}, \quad F''(\xi) = \imath^2\mathcal{F}\{x^2 f(x)\}$$

and
$$F^{(n)}(\xi) = \imath^n\mathcal{F}\{x^n f(x)\}$$

From the last relation we obtain

$$\mathcal{F}\{x^n f(x)\} = \frac{1}{\imath^n} F^{(n)}(\xi) = \imath^{-n} F^{(n)}(\xi)$$

7.5.3 Fourier transform of an integral

We suppose that $f(x)$ is piecewise continuous on $(-\infty,+\infty)$ and that $\int_{-\infty}^{+\infty}|f(x)|dx <$ ∞. Also let $F(0)=0$, with $F(\xi)=\mathcal{F}\{f(x)\}$. Then

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(x')dx'\right\} = \frac{-1}{\imath\xi} F(\xi)$$

Proof

Let

$$g(x) = \int_{-\infty}^{x} f(x')dx'$$

Then by the Leibnitz rule (see appendix B), $g'(x) = f(x)$, whenever $f(x)$ is continuous. From the defining relation for $g(x)$, $g(-\infty)=0$

Now using the definitions

$$F(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(\imath\xi x)\, f(x)\, dx, \quad \text{and} \quad F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\, dx$$

we have

$$\lim_{x\to\infty} g(x) = \int_{-\infty}^{+\infty} f(x)dx = \sqrt{2\pi}F(0)$$

Now using the result

$$\mathcal{F}\{g'(x)\} = -\imath\xi\mathcal{F}\{g(x)\}$$

we have

$$F(\xi) = \mathcal{F}\{f(x)\} = \mathcal{F}\{g'(x)\} = \imath\xi\mathcal{F}\{g(x)\}, dx$$
$$= -\imath\xi\mathcal{F}\left\{\int_{-\infty}^{x} f(x')\right\} dx'$$

which gives

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(x')dx'\right\} = \frac{\imath}{\xi} F(\xi)$$

7.6 Further Discussion on Fourier Transforms

7.6.1 Convolution, Parseval's theorems and other results

Definition

If $f(x)$ and $g(x)$ are functions of x defined over the interval $(-\infty, +\infty)$ then their convolution, denoted by $f \star g$ is defined as follows

$$f \star g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\eta)g(x-\eta)\, d\eta$$

We can show that $g \star f = f \star g$.

Now consider

$$f \star g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\eta)\, g(x - \eta)\, d\eta$$
$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-x')\, g(x')\, dx', \text{ where } x - \eta = x'$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x')\, f(x-x')\, dx' = g \star f$$

In a similar manner we can prove the following properties of the convolution.

$$f \star (g \star h) = (f \star g) \star h, \text{ and}$$
$$f \star (g + h) = f \star g + f \star h$$

7.6.2 Convolution theorem

If $F(\xi)$ and $G(k)$ are Fourier transforms of $f(x)$ and $g(x)$, then

$$\mathcal{F}\{f \star g\} = F(\xi)\, G(\xi)$$

or

$$\mathcal{F}^{-1}\{F(\xi)\,G(\xi)\} = f(x) \star g(x)$$

Proof

$$\begin{aligned}\mathcal{F}^{-1}\{F(\xi)G(\xi)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(-\imath\xi x)\,F(\xi)\,G(\xi)\,d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(-\imath\xi x)\,F(\xi)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(\imath\xi x')g(x')\,dx'\right\}d\xi\end{aligned}$$

where we have used the definition of the Fourier transform of $g(x)$. Changing the order of integration we have

$$\mathcal{F}^{-1}\{F(\xi)G(\xi)\} = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} \exp[-\imath\xi(x-x')]\,F(\xi)\,d\xi\right\}g(x')\,dx'$$

Now by definition

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp[-\imath\xi(x-x')]\,F(\xi)\,d\xi = f(x-x')$$

We find that

$$\mathcal{F}^{-1}\{F(\xi)\,G(\xi)\} = (1/\sqrt{2\pi})\,\int_{-\infty}^{+\infty} f(x-x')\,g(x')\,dx' = f \star g$$

Hence $\mathcal{F}^{-1}\{F(\xi)\,G(\xi)\} = f(x) \star g(x)$

which is equivalent to $F(\xi)G(\xi) = \mathcal{F}\{f \star g\}$.

7.6.3 Parseval's theorems

These theorems are named after the French mathematician Marc Anotoine des Chenes Parseval (1755-1836). Note that there are similar theorems in the theory of Fourier series. These theorems are also referred to as Parseval's identities. These identities are also called *Plancherel's identities*, named after the Swiss mathematician Michel Plancherel(1885–1967), who introduced them in 1910.

The Parseval first and the second theorems may be stated as follows:

$$\int_{-\infty}^{+\infty} F(\xi)\,\overline{F(\xi)}\,d\xi = \int_{-\infty}^{+\infty} f(x)\,\overline{f(x)}\,dx$$

or

$$\int_{-\infty}^{\infty} |F(\xi)|^2\,d\xi = \int_{-\infty}^{+\infty} |f(x)|^2\,dx$$

or

$$\int_{-\infty}^{+\infty} F(\xi)\,G(\xi)\,d\xi = \int_{-\infty}^{+\infty} f(u)g(-u)\,du = \int_{-\infty}^{+\infty} f(x)g(-x)$$

Proof

We prove the second theorem. The first follows from it as a special case. convolution theorem

$$\mathcal{F}^{-1}\{F(\xi)\,G(k)\} = f(x) * g(x)$$

or

$$(1/\sqrt{2\pi})\int_{-\infty}^{+\infty} \exp(-\imath\xi x)F(\xi)G(\xi)d\xi = (1/\sqrt{2\pi})\int_{-\infty}^{+\infty} f(u)\,g(x-u)\,du$$

Putting $x = 0$ on both sides, we have

$$\int_{-\infty}^{+\infty} F(\xi)\,G(\xi)\,d\xi = \int_{-\infty}^{+\infty} f(u)g(-u)\,du = \int_{-\infty}^{+\infty} f(x)\,g(-x)\,dx$$

which is Parseval's second theorem. To derive the first theorem from it, take $g(-x) = \overline{f(x)} i.e. g(x) = \overline{f(-x)}$.

Therefore $\mathcal{F}\{g(x)\} = \mathcal{F}\{\overline{f(-x)}\}$ or $G(\xi) = \overline{F(\xi)}$

Hence

$$\int_{-\infty}^{+\infty} F(\xi)\,\overline{F(\xi)}\,d\xi = \int_{-\infty}^{\infty} f(x)\,\overline{f(x)}\,dx$$

or

$$\int_{-\infty}^{\infty} |F(\xi)|^2\,d\xi = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

which can equivalently be written as $\|F\| = \|f\|$.

7.6.4 The Fourier integral theorem

If $f(x)$ is a real function defined over $(-\infty, +\infty)$, and the integral $\int_{-\infty}^{+\infty} f(x)\,d$ is absolutely convergent, then

$$f(x) = \frac{1}{\pi}\int_0^{\infty} d\xi \int_{-\infty}^{+\infty} \cos\xi(\xi - x)f(\xi)\,d\xi$$

Proof

Since the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is absolutely convergent, its Fourier transform and the inverse both exist. Therefore

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\imath\xi x}F(\xi)\,d\xi$$

Splitting the infinite integral in parts and using the fact that $f(x)$ is real, we have

$$f(x) = \frac{1}{\sqrt{2\pi}}\left[\int_0^\infty \exp(-\imath\xi x)F(\xi)\,d\xi + \int_{-\infty}^0 \exp(-\imath\xi x)F(\xi)\,d\xi\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\imath\xi x}F(\xi)\,d\xi + \int_\infty^0 \exp(\imath\xi' x)F(-\xi')\,(-d\xi'),\quad \xi' = -$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^\infty \exp(-\imath\xi x)F(\xi)\,d\xi + \int_0^\infty \exp(\imath\xi' x)\,F(-\xi')\,d\xi'\right]$$

Since $f(x)$ is real, by conjugation property, we must have $F(-\xi) = \bar{F}$
Therefore we can write

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \left[\exp(-\imath\xi x)F(\xi) + \exp(\imath\xi x)\bar{F}(\xi)\right]\,d\xi$$

Now

$$F(\xi) = (1/\sqrt{2\pi})\int_{-\infty}^{+\infty} \exp(\imath\xi x')f(x')\,dx'$$

Therefore

$$\exp(-\imath\xi x)\,F(\xi) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} \exp[-\imath\xi(x-x')]\,f(x')\,dx'$$

and taking complex conjugate of both sides

$$\exp(\imath\xi x)\bar{F}(\xi) = (1/\sqrt{2\pi})\int_{-\infty}^{+\infty} \exp[\imath\xi(x-x')]\,f(x')\,dx'\ .$$

Adding the last two equations, and denoting the sum by S, we have

$$S = \exp[-\imath\xi x]\,F(\xi) + \exp[\imath\xi x]\bar{F}(\xi)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x')\ \left[\exp[\imath\xi(x-x')] + \exp[-\imath\xi(x-x')]\right]\,dx'$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')\,2\cos\xi(x-x')\,dx'$$

Substituting in the equation for $f(x)$, we obtain

$$f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{+\infty} f(x')\,\cos\xi(x-x')\,dx'\,d\xi$$

7.6.5 Fourier transforms of Dirac delta and the Heaviside unit step functions

First we derive the relationship between $H'(x-x_0)$ and $\delta(x-x_0)$. We proceed as follows.

$$< H'(x-x_0),\ f(x) > = \int_{-\infty}^{+\infty} H'(x-x_0)\,f(x)\,dx$$

$$
\begin{aligned}
&= f(x)H(x-x_0)|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} H(x-x_0)\, f'(x)\, dx \\
&= -\int_{-\infty}^{+\infty} H(x-x_0)\, f'(x)\, dx = -\int_{x_0}^{+\infty} f'(x)\, dx \\
&= -f(x)|_{x_0}^{\infty} = f(x_0) = \int_{x_0}^{+\infty} \delta(x-x_0)\, f(x)\, dx
\end{aligned}
$$

where we have used the result $\lim_{x\to\pm\infty} f(x) = 0$.

On comparison $H'(x-x_0) = \delta(x-x_0)$.

Fourier transforms of Dirac delta function

$$\mathcal{F}\{\delta(x-x_0)\} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \exp(\imath\xi x)\, \delta(x-x_0)\, dx = \frac{1}{\sqrt{2\pi}} \exp(\imath\xi x_0)$$

Fourier transforms of Heaviside unit step function

Using the results $\mathcal{F}\{f'(x)\} = \imath\xi F(\xi)$ and $H'(x-x_0) = \delta(x-x_0)$, we have

$$\mathcal{F}\{H'(x-x_0)\} = \imath\xi\, \mathcal{F}\{\delta(x-x_0)\} \tag{7.6.1}$$

In (7.6.1) we use the result $H'(x-x_0) = \delta(x-x_0)$ and obtain

$$\mathcal{F}\{\delta(x-x_0)\} = \imath\xi\, \mathcal{F}\{H(x-x_0)\} \tag{7.6.2}$$

or $\mathcal{F}\{H(x-x_0)\} = (-\imath/\sqrt{2\pi})\xi\ \exp(\imath\xi\, x_0)$

7.7 Examples and Exercises

7.7.1 Illustrative examples

In this subsection we will discuss the examples illustrating the applications of the convolution theorem and Parseval's identities (also called Plancherel's identities). In this way we will be able to evaluate certain integrals.

Example 1

Use Plancherel's identity for the function $f(x) = \exp(-|x|)$ to evaluate the integral $\int_{-\infty}^{+\infty} dx/(1+x^2)^2$.

Solution

By Plancheral's identity $\int_{-\infty}^{+\infty}|f(x)|^2dx = \int_{-\infty}^{+\infty}|F(\xi)|^2d\xi$, where $\{f(x), F(\xi)\}$ are a F.T. pair. When $f(x) = \exp(-|x|)$. Then

$$\begin{aligned}
\int_{-\infty}^{+\infty}|f(x)|^2dx &= \int_{-\infty}^{+\infty}\exp(-2|x|)\,dx \\
&= 2\int_0^{\infty}\exp(-2x)\,dx = 2\cdot\left.\frac{\exp(-2x)}{-2}\right|_0^{\infty} = 1
\end{aligned}$$

Next we calculate $\mathcal{F}\{\exp(-|x|)\}$. Using the definition

$$\begin{aligned}
\mathcal{F}\{\exp(-|x|)\} &= F(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(\imath\xi x)\exp(-|x|)\,dx \\
&= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}e^{\imath kx}e^{x} + \int_0^{\infty}\exp(\imath\xi x)\exp(-x)\right]dx \\
&= \frac{1}{\sqrt{2\pi}}\left(\int_0^{\infty}e^{(1+\imath k)x} + \int_0^{\infty}\exp(1-\imath\xi x)\right)dx \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{\exp((1+\imath\xi x)}{-(1+\imath\xi)} + \frac{\exp(-(1-\imath\xi)x)}{-(1-\imath\xi)}\right]\Bigg|_0^{\infty} \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+\imath\xi} + \frac{1}{1-\imath\xi}\right) = \sqrt{\frac{2}{\pi}}\,\frac{1}{1+\xi^2}
\end{aligned}$$

Now by Plancherel's identity $\int_{-\infty}^{+\infty}|f(x)|^2dx = \int_{-\infty}^{+\infty}|F(\xi)|^2d\xi$. Therefore on substituting for $f(x)$ and $F(\xi)$, we obtain

$$\int_{-\infty}^{+\infty}|f(x)|^2dx = 1 = \int_{-\infty}^{+\infty}\frac{2}{\pi}\,\frac{dk}{(1+\xi^2)^2}$$

which gives

$$\int_{-\infty}^{+\infty}\frac{dk}{(1+\xi^2)^2} = \frac{\pi}{2}$$

Example.2

Let a function $f(x)$ be defined as follows.

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

compute the convolutions $f\star f$ and $f\star f\star f\star f$ and using the convolution theorem evaluate the integrals

$$\int_{-\infty}^{+\infty}\frac{\sin^2\xi}{\xi^2}d\xi \quad\text{and}\quad \int_{-\infty}^{+\infty}\frac{\sin^4\xi}{\xi^4}d\xi$$

Solution

By definition

$$\begin{aligned} f(x) \star f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x')\, f(x - x')dx' \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{-1} 0 + \int_{-1}^{+1} 1 + \int_{+1}^{+\infty} 0 \right) dx' \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} 1 dx' = \frac{1}{\sqrt{2\pi}}\, x' |_{-1}^{+1} = \sqrt{\frac{2}{\pi}} \end{aligned}$$

By convolution theorem $\mathcal{F}\{f \star f\} = F(\xi)\, F(\xi) = F^2(\xi)$, where

$$\begin{aligned} F(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, \exp(\imath\xi x) dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} \exp(\imath\xi x) dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{\exp(\imath\xi x)}{\imath\xi} \Big|_{-1}^{+1} = \frac{1}{\sqrt{2\pi}} \frac{\exp(\imath\xi) - \exp(-\imath\xi)}{\imath\xi} = \sqrt{\frac{2}{\pi}} \frac{\sin\xi}{\xi} \end{aligned}$$

Hence using the result $\mathcal{F}\{f \star f\} = F^2(\xi)$, we obtain

$$\mathcal{F}^{-1}\{F^2(\xi)\} = f(x) \star f(x) \quad \text{or} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{2}{\pi} \exp(-\imath\xi x) \frac{\sin^2\xi}{\xi^2} d\xi = 2$$

or

$$\int_{-\infty}^{+\infty} \exp(-\imath\xi x) \frac{\sin^2\xi}{\xi^2} d\xi = \left(\frac{2}{\pi}\right)^{3/2}$$

To solve the second part of the problem, let

$$\begin{aligned} g(x) &= f(x) \star f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x') f(x - x') dx' \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} 1 dx' = \sqrt{\frac{2}{\pi}} \end{aligned}$$

Therefore

$$g(x') = \sqrt{2/\pi}, \quad g(x - x') = \sqrt{2/\pi}.$$

Hence

$$g \star g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x')\, g(x - x') dx' = \frac{4}{(2\pi)^{3/2}}$$

Now

$$G(\xi) = F(\xi)\, F(\xi) = F(\xi)^2 = (2\sin^2\xi)/(\pi\xi^2)$$

Hence using the convolution theorem

$$\mathcal{F}\{g(x) \star g(x)\} = \mathcal{F}\{f \star f \star f \star f\} = G(\xi)\, G(\xi) = (4\sin^4\xi/\pi^2\xi^4)$$

Therefore

$$\mathcal{F}^{-1}\left\{\frac{4\sin^4\xi}{\pi^2\xi^4}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{4\exp(-\imath\xi x)\sin^4\xi}{\pi^2\xi^4}d\xi = 2\pi^3$$

wherefrom on simplification we obtain $[(\sin^4\xi)/\xi^4]\,d\xi = 2\pi^3$.

Example 3

Let the functions $f_a(x)$, $f_b(x)$ be defined as follows.

$$f_a(x) = \begin{cases} 1, & |x| \leq a, \quad a > 0 \\ 0, & |x| > a \end{cases}$$

$$f_b(x) = \begin{cases} 1, & |x| \leq b, \quad b > 0 \\ 0, & |x| > b \end{cases}$$

Show that

$$F_a(\xi) = \frac{2}{\sqrt{2\pi}}\frac{\sin a\xi}{\xi}, \quad F_b(\xi) = \frac{2}{\sqrt{2\pi}}\frac{\sin b\xi}{\xi}$$

Using these results and Plancherel's second identity, show that

$$\int_{-\infty}^{+\infty}\frac{\sin a\xi\,\sin b\xi}{\xi^2}d\xi = \pi\min(a,b)$$

Solution

By definition

$$\begin{aligned} F_a(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(\imath\xi x)f_a(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{+a}\exp(\imath\xi x)f_a(x)dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-a}^{+a}\exp(\imath\xi x)dx = \frac{1}{\sqrt{2\pi}}\left.\frac{\exp(\imath\xi x)}{\imath\xi}\right|_{-a}^{+a} \\ &= \frac{1}{\sqrt{2\pi}}\frac{\exp(\imath\xi a)-\exp(-\imath\xi a)}{\imath\xi} = \sqrt{\frac{2}{\pi}}\frac{\sin\xi a}{\xi} \end{aligned}$$

Similarly

$$F_b(\xi) = \sqrt{\frac{2}{\pi}}\frac{\sin\xi b}{\xi}$$

By direct calculation, we find that

$$\int_{-\infty}^{+\infty}|f_a(x)|^2dx = \int_{-a}^{+a}1dx = 2a, \quad \int_{-\infty}^{+\infty}|f_b(x)|^2dx = \int_{-b}^{+b}1dx = 2b$$

From Plancherel's second identity, viz:

$$\int_{-\infty}^{+\infty}f(x)g(x)dx = \int_{-\infty}^{+\infty}F(\xi)G(\xi)d\xi$$

with $f(x)$ and $g(x)$ replaced by $f_a(x)$ and $f_b(x)$, we obtain

$$\int_{-\infty}^{+\infty} F_a(\xi)F_b(\xi)d\xi = \int_{-\infty}^{+\infty} f_a(x)f_b(x)dx$$

or on substituting the values for $f_a(x), F_a(\xi)$ *etc.*, we have

$$\begin{aligned}\frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{\sin a\xi \ \sin b\xi}{\xi^2}d\xi &= \int_{-\infty}^{+\infty} f_a(x)f_b(x)dx \\ &= \int_{-c}^{+c} 1dx = 2c\end{aligned}$$

where c is the smaller of the two numbers a, b, i.e. $c = \min(a, b)$. Hence

$$\int_{-\infty}^{+\infty}\frac{\sin a\xi \ \sin b\xi}{\xi^2}dk = \pi \ \min(a, b)$$

Example 4

Find the F.T. of the function $f(x) = x/[(x^2+a^2)(x^2+b^2)]$, and hence calculate the integral

$$\int_{-\infty}^{+\infty}\frac{x\sin mx dx}{(x^2+a^2)(x^2+b^2)}, \quad (a, b \text{ real}), \ a > 0, \ b > 0, \ b > a$$

Solution

Let $f(z) = z/x/[(z^2+a^2)(z^2+b^2)]$. By definition

$$\mathcal{F}\{f(x)\} \equiv F(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(\imath\xi x)\frac{xdx}{(x^2+a^2)(x^2+b^2)} \tag{1}$$

To compute the integral in (1), we introduce the contour integral $\oint_{\Gamma_r} f(z)\exp(\imath\xi z)dz$, where Γ_r is the contour of figure (10.2). Then by the residue theorem

$$\oint_{\Gamma_r} f(z)\exp(\imath\xi z)dz = 2\pi\imath\sum_j R_j \tag{2}$$

The poles of $f(z)$ are given by $z = \pm\imath a$ and $z = \pm\imath b$. Only the poles at $z_1 = \imath a$ and $z_2 = \imath b$ lie inside the contour Γ_r. The residues R_1, R_2 of these poles are given by

$$\begin{aligned} R_1 &= \frac{1}{\sqrt{2\pi}}\cdot\lim_{z\to a\imath}\frac{(z-a\imath)z\exp(\imath\xi z)}{(z-a\imath)(z+a\imath)(z-b\imath)(z+b\imath)} \\ &= \frac{1}{2\sqrt{2\pi}}\cdot\frac{\exp(-\xi a)}{b^2-a^2}\end{aligned}$$

Similarly

$$R_2 = \frac{1}{\sqrt{2\pi}} \cdot \lim_{z \to bi} \frac{(z - bi)ze^{ikz}}{(z - ai)(z + ai)(z - bi)(z + bi)}$$
$$= \frac{1}{2\sqrt{2\pi}} \cdot \frac{e^{-kb}}{a^2 - b^2}$$

Therefore

$$R_1 + R_2 = \frac{1}{2\sqrt{2\pi}} \cdot \frac{\exp(-\xi a) - \exp(-\xi b)}{b^2 - a^2}$$

On substitution we obtain

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(i\xi x) f(x) dx = \sqrt{\frac{1}{2}} \frac{\exp(-\xi a) - \exp(-\xi b)}{b^2 - a^2} i \qquad (3)$$

Computation of the integral

With $k = m$ in (3), we obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(imx) f(x) dx = \sqrt{\frac{1}{2}} \frac{\exp(-ma) - \exp(-mb)}{b^2 - a^2} i$$

On equating the imaginary parts on both sides, we have

$$\int_{-\infty}^{+\infty} \frac{x \sin mx}{(x^2 + a^2)(x^2 + b^2)} = \sqrt{\frac{\pi}{2}} \frac{\exp(-ma) - \exp(-mb)}{b^2 - a^2}$$

7.7.2 Exercises

1. Find the F.T. of each of the following functions.

(a) $f(x) = 1/(a^2 + x^2)$

(b) $f(x) = \cos bx/(a^2 + x^2)$

(c) $f(x) = \sin bx/(a^2 + x^2)$

2. Using results of the previous problem and Plancherel's identity, evaluate the following integrals

(a) $\int_0^\infty du/[(a^2 + u^2)^2]$

(b) $\int_0^\infty [u^2 du/(a^2 + u^2)^2]$

(c) $\int_0^\infty [(x \sin \pi x)/(1 - x^2)]\, dx$

(d) $\int_0^\infty [x \sin \pi x \cos \pi x)/(1 - x^2)]\, dx$

(e) $\int_0^\infty [(x \sin \pi x)/(1 - x^2)^2]\, dx$

(f) $\int_0^\infty [(\sin x - x \cos x)/x^2]^2\, dx$

3. Determine the F.T. of

$$f(x) = \begin{cases} +1, & 0 < x \leq a \\ -1, & -a < x \leq 0 \\ 0, & x > |a| \end{cases}$$

where $a > 0$ and use it to evaluate the integral

$$\int_0^\infty \frac{(\cos ax - 1)}{x} \sin bx \, dx, \quad b > 0$$

4. Using the generalized Plancherel identity evaluate the integral

$$\int_0^\infty \frac{dx}{(a^2 + x^2)(b^2 + x^2)}, \quad a0, \quad b > 0$$

5. Let $f(x)$ be a complex-valued function of the real variable x, and let $F(k)$ be its F.T. If

(a) $F(\xi) = \begin{cases} 1 - \xi^2, & |\xi| \leq 1 \\ 0, & |\xi| > 1 \end{cases}$

(b) $F(\xi) = \begin{cases} 1 - \xi, & |\xi| \leq 1 \\ 0, & |\xi| > 1 \end{cases}$

find $f(x)$.

6. Calculate the F.T. of the following function (also called the *two-sided exponential pulse*, when x is interpreted as time t)

$$f(x) = \begin{cases} e^{ax}, & x \leq 0 \\ e^{-ax}, & x > 0 \end{cases}, \quad (a > 0).$$

7. Calculate the F.T. of the 'on-off' pulse shown in the figure below. 8. Sketch the graph of the function below, calculate its F.T.

$$f(x) = \begin{cases} A(+x/X + 1), & -X \leq x \leq 0 \\ A(-x/X + 1), & 0 < x \leq X \end{cases}.$$

What is the relationship between this pulse and that of the previous problem?

9. Calculate the F.T. of the following function.

$$f(x) = \begin{cases} 2c, & |x| \leq 2 \\ 0, & |x| > 2 \end{cases}.$$

and

$$g(x) = \begin{cases} c, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}.$$

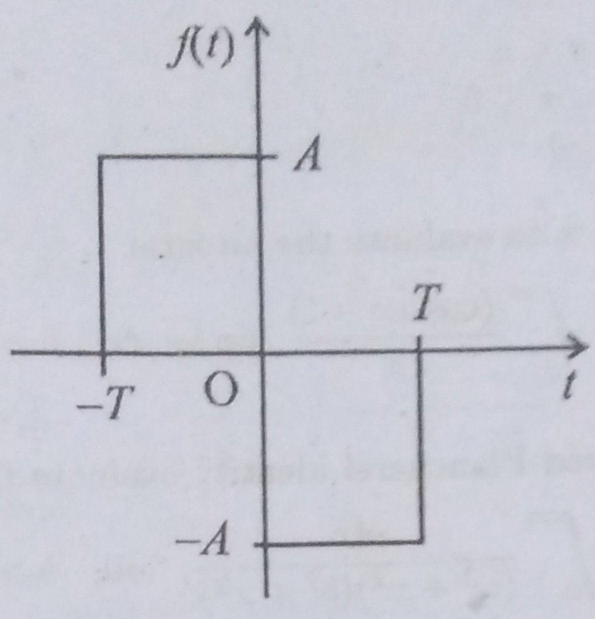

Figure 7.4: Graph of the on-off pulse.

Sketch he graph of the function $w(x) = f(x) - g(x)$ and calculate its F.T.

10. Calculate the F.T. of the off-on-off pulse represented by the function

$$f(x) = \begin{cases} 0, & x < -2 \\ -1, & -2 \leq x < -1 \\ +1, & -1 \leq x \leq 1 \\ -1, & 1 < x \leq 2 \\ 0, & x > 2 \end{cases}.$$

11. Show that the F.T. of the function

$$f(x) = \begin{cases} \sin ax, & |x| \leq \pi/a \\ 0, & |x| > \pi/a \end{cases}.$$

is $12a \sin(\pi k/a)/(k^2 - a^2)$.

12. Find the the F.T. of the function

$f(x) = \exp(-ax)\ \sin \xi_0 x\ H(x)$.

13. Show that the Fourier sine and cosine transforms of the function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \leq x \leq a \\ 0, & x > a \end{cases}$$ are $(1 - \cos ax)/x$ and $(\sin ax)/x$ respectively.

14. Calculate the Fourier sine and cosine transforms of

$f(x) = \exp(-ax)\ H(x), \quad a > 0$.

23. Let $f(x)$ and $F(k)$ denote the F.T. pair, with the condition that $f(x)$ is continuous and absolutely integrable. Given that

$$F(\xi) + \int_{-\infty}^{+\infty} F(\xi - u)\exp(-|u|)du = \begin{cases} \xi^2, & 0 \le u \le 1 \\ 0, & \text{for } u < 0,\ u > 1 \end{cases}$$

find $f(x)$.

24. Let $f(x)$ and $F(\xi)$ denote the F.T. pair, as in the previous problem. If $F(\xi) = 0$, for all $k \ge |\xi_0|$, then show that for all $a > |\xi_0|$

$$f(x) \star \left(\frac{\sin ax}{\pi x}\right) = f(x)$$

25. Let $F(\xi)$ be the F.T. of the function $f(x)$, defined as follows

$$f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{for all other values of } x \end{cases}$$

find the function $g(x)$ such that the F.T. $G(\xi)$ of $g(x)$ which satisfies $G(\xi) = |F(\xi)|^2$.

26. Use the formula

$$\mathcal{F}\{x^n f(x)\} = (-\imath)^n F^n(\xi)$$

to calculate $\mathcal{F}\{x\exp(-\alpha x^2)\}$, $\alpha > 0$.

(**Ans.**: $\mathcal{F}\{x\exp(-\alpha x^2)\} = (1/\sqrt{2\alpha})\exp[-\xi^2/(4\alpha)]$)

27. Find the the F.T. of the function

$$f(x) = \exp(-ax)\ \sin k_0 x\ H(x).$$

7.8 Use of Complex Fourier Transform in Solving B.V./I.V. Problems

When we have an I.V./B.V. problem in which the space coordinate x extends over the whole real line, we may use complex Fourier transform. For the validity of the formula for the derivatives it is required that both the unknown function u and its partial derivatives approach zero as x goes to $\pm\infty$.

7.8.1 Illustrative examples

Example 1

(a) Solve the problem by means of the Fourier transform method.

$$u_t = \kappa^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0 \tag{1}$$

$$u(x,\ 0) = f(x), \quad u(x,\ t),\ u_x(x,\ t) \to 0 \text{ as } x \to \pm\infty \tag{2}$$

(b) Obtain the solution of the problem in (a) when $u(x,\ 0) = \exp(-\alpha x^2)$.

Solution

The given PDE and the B.Cs. describe the conduction of heat through a rod or wire of infinite length. The initial temperature distribution is given by $f(x)$ and the temperature towards the end points gets smaller and smaller. The source of heat in the body is the initial temperature.

We assume that both u and $f(x)$ satisfy the conditions for the existence of their Fourier transforms. Taking the Fourier transforms of both sides of (1), and denoting the Fourier transform of $u(x,\ t)$ w.r.t. x by $U(\xi,\ t)$, we obtain the first order ODE

$$\frac{dU(\xi,\ t)}{dt} = -\kappa^2 \xi^2 U(\xi,\ t) \tag{3}$$

General solution of (3) is given by

$$U(\xi,\ t) = c \exp(-\kappa^2 \xi^2 t) \tag{4}$$

The I.C. in (2) can be transformed as $U(\xi,\ 0) = F(\xi)$. From this condition and (4), we obtain $c = F(\xi)$. Hence

$$U(\xi,\ t) = F(\xi) \exp(-\kappa^2 \xi^2 t)$$

Taking the inverse Fourier transform, we have

$$u(x,\ t) = \mathcal{F}^{-1}\{F(\xi)\ \exp(-\beta\xi^2)\}$$

where $\beta = \kappa^2 t$. Next using the convolution theorem, we can simplify the right side of the above relation as

$$u(x,\ t) = f(x) \star \mathcal{F}^{-1}\{\exp(-\beta\xi^2)\}$$

Now using the formula in equation (2) of example 3, *viz.*

$$\mathcal{F}^{-1}\{\exp(-\beta\xi^2)\} = \frac{1}{\sqrt{2\beta}} \exp\left(-\frac{x^2}{4\beta}\right)$$

we finally obtain

$$u(x,\ t) = \frac{1}{\sqrt{\pi t \kappa^2}} \int_{-\infty}^{+\infty} f(x') \exp[-(x - x')^2/(4\kappa^2 t)]$$