(e) $u'' + 4u = \sin t$

11. Solve the I.V.P.

$$y''(t) + ty'(t) - y(t) = 0, \quad y(0) = 0, \quad y'(0) = 1$$

→ Formula to study on page 243.

(**Hint:** On applying L.T. and using I.Cs. we obtain

$Y'(s) + (2 - s^2)/s\, Y(s) = -1/s.$

The integrating factor for this DE is $s^2 \exp(-s^2/2)$, and the solution is given by $Y(s) = 1/s^2 + c \exp(s^2/2)$).

6.7 Laplace and Inverse Laplace Transforms of some other Functions

6.7.1 Some useful results

The following results will be used in the sequel.

The Gaussian Integral

$$\int_{-\infty}^{+\infty} \exp(-x^2)\, dx = \sqrt{\pi}$$

Proof

Let

$I = \int_{-\infty}^{+\infty} \exp(-x^2)\, dx = \int_{-\infty}^{+\infty} \exp(-y^2)\, dy$

Therefore

$$\begin{aligned} I^2 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp(x^2)\, \exp(-y^2)\, dx\, dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp[-(x^2 + y^2)\, dx\, dy \end{aligned}$$

Reverting to the polar coordinates (r, θ), we have

$$\begin{aligned} I^2 &= \int_{r=0}^{r=\infty} \int_{\theta=0}^{\theta=2\pi} \exp(-r^2)\, r\, dr\, d\theta \\ &= 2\pi \int_{0}^{\infty} \exp(-r^2)\, r\, dr \end{aligned}$$

$$= \pi \int_0^\infty \exp(-p)\,dp, \quad (p = r^2)$$

Hence

$I = \int_{-\infty}^{+\infty} \exp(-x^2)\,dx = \sqrt{\pi}$ and $\int_0^\infty \exp(-x^2)\,dx = \sqrt{\pi}/2$

The Gamma Function

The gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^\infty \mathrm{e}^{-t}\, t^{x-1}\, dt$$

From this definition it follows from integration by parts that

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty \mathrm{e}^{-t}\, t^x\, dt \\
&= t^x\,(-\mathrm{e}^{-t})]_0^\infty + \int_0^\infty \mathrm{e}^{-t}\, x\, t^{x-1}\, dt \\
&= 0 + x\int_0^\infty \mathrm{e}^{-t}\, t^{x-1}\, dt = x\,\Gamma(x)
\end{aligned}$$

Hence $\Gamma(x+1) = x\,\Gamma(x)$.

Also $\Gamma(1) = \int_0^\infty \mathrm{e}^{-t} t^{1-1}\, dt = \int_0^\infty \mathrm{e}^{-t}\, dt = 1$.

Therefore with $x = n$, n a positive integer, using the above relation, we obtain

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) = n(n-1)(n-2)\cdots 2\cdot 1\cdot \Gamma(1) \\
&= n(n-1)(n-2)\cdots 2 \cdot 1 \cdot 1 = n!
\end{aligned}$$

where we have used the result $\Gamma(1) = 1$.

From this result it is clear that the gamma function can be regarded as the generalization of the factorial function.

6.7.2 Laplace transform of the step function

By definition

$$\begin{aligned}
L\{H(t-t_0)\} &= \int_0^\infty \mathrm{e}^{-st}\, H(t-t_0)\, dt \\
&= \int_0^{t_0} \mathrm{e}^{-st}\, H(t-t_0)\, dt + \int_{t_0}^\infty \mathrm{e}^{-st}\, H(t-t_0)\, dt
\end{aligned}$$

$$\begin{aligned} &= 0 + \int_{t_0}^{\infty} e^{-st}\, dt, \quad (H(t-t_0) = 1) \\ &= \left.\frac{e^{-st}}{-s}\right|_{t_0}^{\infty} = \frac{e^{-st_0}}{s} \end{aligned}$$

Hence $L\{H(t-t_0)\} = e^{-st_0}/s$.

In particular when $t_0 = 0$, $L\{H(t)\} = 1/s$.

6.7.3 Laplace transform of the logarithmic function

To calculate $L\{\ln t\}$, proceed as follows.

$$L\{\ln t\} = \int_0^{\infty} e^{-st} \ln t\, dt$$

Now we put $st = u$, in the integral on the right side, and obtain

$$\begin{aligned} L\{\ln t\} &= \int_0^{\infty} e^{-u} (\ln u - \ln s) \frac{du}{s} \\ &= \frac{1}{s}\int_0^{\infty} e^{-u} \ln u\, du - \frac{\ln s}{s}\int_0^{\infty} e^{-u}\, du \\ &= \frac{1}{s}\int_0^{\infty} e^{-u} \ln u\, du - \frac{\ln s}{s} \times 1 \end{aligned}$$

By definition $\Gamma(x+1) = \int_0^{\infty} e^{-u} u^x\, du$. On differentiating

$$\Gamma'(x+1) = \int_0^{\infty} e^{-u} u^x \ln u\, du$$

or on putting $x = 0$, we have

$$\Gamma'(1) = \int_0^{\infty} e^{-u} \ln u\, du$$

Therefore on substitution

$$L\{\ln t\} = (\Gamma'(1) - \ln s)/s$$

where $\Gamma'(1)$ is a constant called *Euler's constant*, whose value is approximately 0.577215665.

6.7.4 Laplace transform of functions of the form $t^n f(t)$

Theorem

If $f(t)$ is a function of exponential order c, then

$$L\{t^n\ f(t)\} \ = \ (-1)^n \left(\frac{d}{ds}\right)^n \ F(s), \quad \text{for } s > a$$

Proof

We know that

$$F(s) = \int_0^\infty \exp(-st)\ f(t)\ dt$$

Differentiating both sides w.r.t. s, we have

$$\begin{aligned} F'(s) &= \int_0^\infty (d/ds)\exp(-st)\,f(t)\,dt \\ &= \int_0^\infty (-t)\exp(-st)\,f(t)\,dt = (-1)\,L\{t\,f(t)\} \end{aligned}$$

which is equivalent to the statement

$$L\{t\ f(t)\} = -\,F'(s) = \left(-\frac{d}{ds}\right)\ F(s)$$

Repeating the same process, we have

$$L\{t^2\ f(t)\} \ = \ \left(-\frac{d}{ds}\right)^2\ F(s)$$

and finally

$$L\{t^n\ f(t)\} = (-\frac{d}{ds})^n\,F(s) = (-1)^n\ \left(\frac{d}{ds}\right)^n\ F(s)$$

which is equivalent to the statement

$$L\{(-t)^n\ f(t)\} = \left(-\frac{d}{ds}\right)^n\ F(s) = \left(\frac{d}{ds}\right)^n\ F(s)$$

Corollary

The following result can be deduced as a corollary from the above.

$$L\{p(t)\ f(t)\} \ = \ p(-D)\ F(s)$$

where $p(t)$ is a polynomial in t, $D = d/ds$ and $F(s) = L\{f(t)\}$.

Let

$$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n = \sum_i a_i\,t^i$$

Then

$$\begin{aligned}
L\{p(t)\ f(t)\} &= L\{\sum_i (a_i t^i)\ f(t)\} \\
&= \sum_i a_i\ L\{t^i\ f(t)\} = \sum_i a_i \left(-\frac{d}{ds}\right)^i\ L\{f(t)\} \\
&= \sum_i\ a_i\ (-D)^i\ F(s) = p(-D)\ F(s)
\end{aligned}$$

6.7.5 Laplace transforms of Bessel functions

In many physical problems, we have to calculate Laplace transforms of Bessel functions. The calculation for different Bessel functions of the first kind of order 0 i.e. $J_0(t)$ is illustrated in examples 1 and 2 below.

6.7.6 The second shifting/translation theorem

Just as the first shifting theorem enables us to calculate Laplace transforms of products of functions of the type $e^{kt}\ f(t)$, the second shifting theorem in a similar fashion enables us to compute inverse Laplace transforms of functions of the form $e^{-as}\ F(s)$. It can be stated as

$$L^{-1}\{e^{-as}\ F(s)\} = H(t-a)\ f(t-a), \quad (a > 0)$$

Proof

By definition

$L\{H(t-a)\,f(t-a)\} = \int_0^\infty \exp(-st)\,H(t-a)\,f(t-a)\,dt$

If we let $t - a = t'$, then the limits of integration on the right side will vary from $-a$ to ∞. Therefore

$$\begin{aligned}
L\{H(t-a)\ f(t-a)\} &= \int_{-a}^{\infty} \exp[-s(t'+a)]\ H(t')\ f(t')\ dt' \\
&+ \int_0^\infty \exp[-s(t'+a)]\ H(t')\ f(t')\ dt' \\
&= 0 + \exp(-as) \int_0^\infty \exp(-st')\ H(t')\ f(t')\ dt' \\
&= \exp(-as)\ L\{f\} = \exp(-as)\ F(s)
\end{aligned}$$

where we have used the defining properties of the step function.

6.7.7 Illustrative examples

Example 1

Find the Laplace transform of $J_0(t)$ where

$J_0(t) = (1/\pi)\int_0^\pi \cos(t\sin\theta)\,d\theta$

Solution

$$L\{J_0(t)\} = \int_0^\infty \exp(-st)\,\frac{1}{\pi}\left\{\int_0^\pi \cos(t\ \sin\theta)\,d\theta\right\}\,dt$$

Reversing the order of integration, we have

$$L\{J_0(t)\} = \frac{1}{\pi}\int_0^\pi \left\{\int_0^\infty e^{-st}\cos(t\ \sin\theta)\,dt\right\}\,d\theta$$

Now let I denote the integral in braces. Then using the formula

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\,(a\cos bx + b\sin bx)$$

we have

$$\begin{aligned} I &= \int_0^\infty \exp(-st)\cos(t\ \sin\theta)\,dt \\ &= \frac{\exp(-st)}{s^2+\sin^2\theta}\,[-s\cos(t\ \sin\theta)+\sin\theta\cdot\sin\,(t\sin\theta)]\Big|_0^\infty \\ &= 0 - \frac{1}{s^2+\sin^2\theta}\cdot[-s] = \frac{s}{s^2+\sin^2\theta} \end{aligned}$$

On substitution, we obtain

$$\begin{aligned} L\{J_0(t)\} &= \frac{s}{\pi}\int_0^\pi \frac{d\theta}{s^2+\sin^2\theta} \\ &= \frac{s}{\pi}\int_{-\pi/2}^{+\pi/2} \frac{d\theta'}{s^2\ +\ \cos^2\theta'}, \quad (\text{where } \theta = \theta' + \pi/2) \\ &= \frac{2s}{\pi}\int_0^{\pi/2} \frac{d\theta}{s^2+\cos^2\theta} \\ &= \frac{2s}{\pi}\int_0^{\pi/2} \frac{\sec^2\theta\ d\theta}{1+s^2\ \sec^2\theta} \\ &= \frac{2s}{\pi}\int_0^{\pi/2} \frac{\sec^2\theta\ d\theta}{1+s^2\,(1+\tan^2\theta)} \end{aligned}$$

Letting $u = \tan\theta$, $\sec^2\theta \, d\theta = du$, we obtain

$$\begin{aligned}
L\{J_0(t)\} &= \frac{2s}{\pi}\int_0^\infty \frac{du}{1+s^2(1+u^2)} \\
&= \frac{2s}{\pi}\int_0^\infty \frac{du}{(1+s^2)+s^2u^2} \\
&= \frac{2}{\pi s}\int_0^\infty \frac{du}{(1+s^2)/s^2+u^2} \\
&= \frac{2}{\pi s}\int_0^\infty \frac{du}{a^2+u^2}, \quad a^2 = \frac{1+s^2}{s^2} \\
&= \frac{2}{\pi s}\cdot\frac{1}{a}\cdot\tan^{-1}\frac{u}{a}\Big]_0^\infty \\
&= \frac{2}{\pi}\cdot\frac{1}{\sqrt{1+s^2}}\cdot\frac{\pi}{2} \\
&= \frac{1}{\sqrt{1+s^2}}
\end{aligned}$$

Example 2

Given that Bessel functions of the first kind and positive integral order satisfy the recurrence relations

$$J_1 = -J_0', \quad J_{n+1} = J_{n-1} - 2J_n', \quad n \geq 1$$

with $J_0(0) = 1$, $J_n(0) = 0$, $n > 0$, show that

$$L\{J_n(t)\} = \frac{(\sqrt{s^2+1}-s)^n}{\sqrt{s^2+1}}$$

Also find the Laplace transform for $J_0(at)$, $a > 0$.

Solution

From the first recurrence relation, using the formula for Laplace transform of a derivative, we obtain

$$\begin{aligned}
F_1(s) &= L\{J_1(t)\} = -L\{J_0'(t)\} \\
&= -\{sL\{J_0(t)\} - J_0(0)\} \\
&= -sL\{J_0(t)\} + J_0(0), \quad \text{(where } J_0(0) = 1\text{)} \\
&= -sF_0(s) + 1 \qquad (1)
\end{aligned}$$

But from example 1

$$F_0(s) = L\{J_0(t)\} = 1/\sqrt{s^2+1}$$

Therefore the given formula for $L\{J_n(t)\}$ is true for $n = 0$. Also

$$\begin{aligned} F_1(s) &= L\{J_1(t)\} = -sL\{J_0(t)\} + 1 \\ &= \frac{-s}{\sqrt{s^2+1}} + 1 = \frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}} \end{aligned}$$

which shows that the given formula is true for $n = 1$. Next we use the second recurrence relation, *viz.* $J_{n+1} = J_{n-1} - 2J'_n$, which gives $J_2 = J_0 - 2J'_1$. Therefore

$$\begin{aligned} F_2(s) &= L\{J_2(t)\} = L\{J_0(t)\} - 2L\{J'_1(t)\} \\ &= F_0(s) - 2\{sF_1(s) - J_1(0)\} \\ &= F_0 - 2sF_1(s) + J_1(0) \\ &= F_0(s) - 2sF_1(s), \quad \text{because } J_1(0) = 0 \\ &= \frac{1}{\sqrt{s^2+1}} - \frac{2s(\sqrt{s^2+1}-s)}{\sqrt{s^2+1}} \\ &= \frac{1 - 2s\sqrt{s^2+1} + 2s^2}{\sqrt{s^2+1}} \\ &= \frac{(\sqrt{s^2+1}-s)^2}{\sqrt{s^2+1}} \end{aligned}$$

which shows that the formula is true for $n = 2$. Now we use mathematical induction to establish the formula for the general index $n+1$, assuming that it is true for n and $n-1$.

From the second recurrence relation

$$\begin{aligned} L\{J_{n+i}(t)\} &= L\{J_{n-1}(t)\} - 2L\{J'_n(t)\} \\ &= L\{J_{n-1}(t)\} - 2[sL\{J_n(t)\} - J_n(0)] \\ &= L\{J_{n-1}(t)\} - 2sL\{J_n(t)\}, \quad J_n(0) = 0,\ n \geq 1 \\ &= \frac{(\sqrt{s^2+1}-s)^{n-1}}{\sqrt{s^2+1}} - \frac{2s(\sqrt{s^2+1}-s)^n}{\sqrt{s^2+1}} \\ &= \frac{(\sqrt{s^2+1}-s)^{n-1} \cdot [1 - 2s\sqrt{s^2+1} + 2s^2)]}{\sqrt{s^2+1}} \\ &= \frac{(\sqrt{s^2+1}-s)^{n-1} \cdot (\sqrt{s^2+1}-s)^2}{\sqrt{s^2+1}} \\ &= \frac{(\sqrt{s^2+1}-s)^{n+1}}{\sqrt{s^2+1}} \end{aligned}$$

Hence the proof. To obtain the same formula when the argument is at, we use the rule of scales, *viz.* $L\{f(at)\} = (1/a)\,F(s/a), a > 0$ and obtain

$$L\{J_n(at)\} = \frac{(\sqrt{s^2+a^2}-s)^n}{a^n\sqrt{s^2+a^2}}$$

Example 3 H.w

Evaluate $L\{\exp(at) - (\cos bt)/t\}$ and deduce that

$$L\left\{\frac{\sin^2 t}{t}\right\} = (1/2)\ \ln\left(\frac{\sqrt{s^2+4}}{s}\right), \quad s > 1$$

Solution

Here we will use the result

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s')\, ds' \tag{1}$$

provided the limit of $(f(t)/t)$ as $t \to 0$ exists. In this problem

$$\lim_{t\to 0}\frac{f(t)}{t} = \lim_{t\to 0}\frac{\exp(at) - \cos(bt)}{t} = \lim_{t\to 0}\frac{ae^{at} + b\sin(bt)}{1} = a$$

Also

$$F(s) = L\{f(t)\} = L\{e^{at} - \cos(bt)\} = \frac{1}{s-a} - \frac{s}{s^2+b^2}, \quad (s > a)$$

Hence from (1)

$$\begin{aligned}
L\left\{\frac{f(t)}{t}\right\} &= L\left\{\frac{e^{at} - \cos bt}{t}\right\} \\
&= \int_s^\infty \left[\frac{1}{s'-a} - \frac{s'}{s'^2+b^2}\right] ds' \\
&= \left\{\ln(s'-a) - \frac{1}{2}\ln(s'^2+b^2)\right\}\Big|_s^\infty \\
&= \ln\frac{s'-a}{(s'^2+b^2)^{1/2}}\Bigg]_s^\infty = \ln\frac{(1-0)}{(1+0)} - \ln\frac{s-a}{\sqrt{s^2+b^2}} \\
&= 0 - \ln\frac{s-a}{\sqrt{s^2+b^2}} = \ln\frac{\sqrt{s^2+b^2}}{s-a}, \quad s > a
\end{aligned}$$

To deduce the second part, we put $a = 0,\ b = 2,$ so that $1 - \cos 2t = 2\sin^2 t.$ Hence

$$L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{2}\ln\frac{\sqrt{s^2+4}}{s}, \quad s > 0$$

6.7.8 Exercises

1. Use the result $L\{\exp(-kt)\ f(t)\} = F(s)$, where $f(t)$ is a real function of the real variable t to show that

$$L\{\exp(-\alpha t)\ \cos\ \beta t\ f(t)\} = \operatorname{Re} F(s + \alpha + \imath\beta)$$

and

$$L\{\exp(-\alpha t)\ \sin\ \beta t\ f(t)\} = -\operatorname{Im} F(s + \alpha + \imath\beta)$$

(**Hint:** (Take $k = \alpha + \imath\beta$, and equate real and imaginary parts on both sides)

2. Evaluate $L\{\exp(kt)\, t^n\}$ when n is an integer, and deduce the expressions for $L\{t^n \cos kt\}$ and $L\{t^n \sin kt\}$.

Hint: We have

$$\{L\{\exp(kt)\ t^n\} = \left.\frac{n!}{s^{n+1}}\right|_{s \to s-k} = \frac{n!}{(s-k)^{n+1}},\quad \operatorname{Re}(s-k) > 0$$

Now let $k = \iota\alpha$, where α is real and positive, then

$$\begin{aligned} L\{\exp(\imath\alpha t)\, t^n\} &= L\{(\cos\alpha t + \imath\sin\alpha t)\, t^n\} \\ &= \frac{n!}{(s-\imath\alpha)^{n+1}},\quad \operatorname{Re} s > 0 \end{aligned}$$

Now let $s = r\cos\theta$, $\alpha = r\sin\theta$, so that

$$r = \sqrt{s^2+\alpha^2},\ \tan\theta = \alpha/s,\ 0 \le \theta < \pi/2$$

Since α and s are positive, $s - \iota\alpha = r(\cos\theta - \iota\sin\theta)$, and therefore

$$\begin{aligned} \frac{1}{(s-\imath\alpha)^{n+1}} &= \frac{1}{r^{-n-1}}\,(\cos\theta + \imath\sin\theta)^{-n-1} \\ &= \frac{1}{r^{n+1}}\,[\cos(n+1)\theta + \imath\sin(n+1)\theta] \end{aligned}$$

Hence

$$L\{\exp(\imath\alpha t)\, t^n\} = \frac{n!}{r^{n+1}}\,[\cos(n+1)\theta + \imath\sin(n+1)\theta]$$

Equating real and imaginary parts.

$$L\{\cos\alpha t\, t^n\} = n!\,(s^2+\alpha^2)^{-(n+1)/2}\ \cos(n+1)\theta$$

and

$$L\{\sin\alpha t\, t^n\} = n!\,(s^2+\alpha^2)^{-(n+1)/2}\ \sin(n+1)\theta$$

where $\theta = \tan^{-1}\ \alpha/s$.)

3. Show that

$$L\left\{\frac{f(t)}{t+a}\right\} = \exp(as)\int_s^\infty \exp(-st')\, F(t)$$

(**Hint:** Let $I = L\{f(t)/(t+a)\}$, then with $D = d/ds$

$$\begin{aligned}(D-a)I &= (D-a)\int_0^\infty \exp(-st)\,\frac{f(t)}{t+a}\,dt \\ &= \int_0^\infty (D-a)\exp(-st)\,\frac{f(t)}{t+a}\,dt \\ &= \int_0^\infty (-t-a)\exp(-st)\,\frac{f(t)}{t+a}\,dt \\ &= -\int_0^\infty \exp(-st)\,f(t).dt = -F(s)\end{aligned}$$

Also

$$dI/ds - aI = -F(s)$$

which is a nonhomogeneous linear DE with integrating factor $\mu = \exp(-as)$. The solution of this equation is given by

$$\mu I = -\int \mu F(s)\,ds + \text{constant}$$

or

$\exp(-as)\,I = -\int \exp(-as)\,F(s)\,ds + c_1$

To calculate the constant c_1, we take the limit as $s \to \infty$,

$0 = -\int_{s0}^\infty \exp(-as)\ F(s)\ ds + c_1$

which gives

$c_1 = \int_{s0}^\infty \exp(-as)\ F(s)\ ds$

Therefore on substitution

$exp(-as)\,I = \int_s^\infty \exp(-as)\ F(s)\ ds$

or finally

$I = \exp(as)\int_s^\infty \exp(-as')\ F(s')\ ds'$

4. Prove that

$$L\left\{\exp(-at)\int_0^t f(t')\exp(at')\,dt'\right\} = \frac{F(s)}{s+a}$$

(**Hint:** Note that this is general form of a previous result.

$$\text{L.H.S:} = \int_0^\infty \exp[-(s+a)t]\left\{\int_0^t f(t')\ \exp(at')\ dt'\right\}\ dt$$

Integrating by parts w.r.t. t, we have

$$\text{L.H.S.} = \frac{\exp[-(s+a)t]}{-(s+a)} \cdot \int_0^t f(t') \exp(at')\, dt' \Big|_0^\infty$$

$$+\frac{1}{(s+a)} \int_0^\infty \exp[(s+a)t] \cdot e^{at} f(t)\, dt$$

$$= 0 + \frac{1}{(s+a)} \int_0^\infty \exp(-st) \cdot f(t)\, dt = \frac{F(s)}{s+a}\ )$$

5. Prove that

$$L\{\text{erfc}\,(t^{1/2})\} = \frac{1}{s} - \frac{1}{\sqrt{s+1}}, \text{ where erfc } t = \frac{2}{\sqrt{\pi}} \int_t^\infty \exp(-x^2)\, dx$$

(**Hint**:

$$\text{erfc}\,(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty \exp(-x^2)\, dx$$

$$= \frac{2}{\sqrt{\pi}} \left[\int_0^\infty \exp(-x^2)\, dx - \int_0^t \exp(-x^2)\, dx\right]$$

or

$$\text{erfc}\,(t) = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}} \int_0^t \exp(-x^2)\, dx = 1 - \text{erf}\,(t)$$

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$$

$$\text{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx$$

where

$\text{erf}\,(t) = (2/\sqrt{\pi}) \int_0^t \exp(-x^2)\, dx$

Therefore

$$L\left\{\text{erfc}\,(\sqrt{t})\right\} = L\{1\} - L\left\{\text{erf}\,(\sqrt{t})\right\}$$

$$= \frac{1}{s} - \frac{2}{\sqrt{\pi}} L\left\{\int_0^{\sqrt{t}} \exp(-x^2)\, dx\right\}$$

$$= \frac{1}{s} - \frac{2}{\sqrt{\pi}} L\left\{\int_0^t \exp(-v) \frac{dv}{2\sqrt{v}}\right\}, \quad (x^2 = v)$$

$$= \frac{1}{s} - \frac{1}{\sqrt{\pi}} L\left\{\int_0^t \frac{\exp(-t')}{\sqrt{t'}}\, dt'\right\} \ *$$

$$\because L\left(\frac{1}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}}$$

$$= \frac{1}{s} - \frac{1}{s\sqrt{\pi}} L\left\{\frac{1}{\sqrt{t}}\right\}\Big|_{s \to s+1}$$

$$= \frac{1}{s} - \frac{1}{s\sqrt{s+1}} \longrightarrow \text{Q3/P 233}$$

Therefore

$$L\{\text{erfc}\,\sqrt{t}\} = \frac{1}{s}\left(1 - \frac{1}{\sqrt{s+1}}\right)$$

using $L\{\int_0^t f(u)du\} =$

$$* \Rightarrow L\left\{\int_0^t \frac{e^{-t'}}{\sqrt{t'}} dt'\right\} = \frac{1}{s} L\left(\frac{e^{-t}}{\sqrt{t}}\right)$$

w that

(a) $L\left\{\frac{\exp(-t)}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s+1}}$

(b) $L\left\{\int_0^t \exp(-v)\frac{1}{\sqrt{v}}\,dv\right\} = \frac{1}{s}\sqrt{\frac{\pi}{s+1}}$

(c) $L\left\{\exp(t)\operatorname{erfc}\sqrt{t}\right\} = \frac{1}{s+\sqrt{s}}$

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Further Discussion of Inverse Laplace Transform

1 Uniqueness of the inverse Laplace transform

h's theorem guarantees the uniqueness of inverse Laplace transform up to rtain additive function called the *null function*. A function $N(x)$ is called function if $\int_0^\infty N(x)\,dx = 0$.

The theorem asserts that if $f(t)$ and $g(t)$ are inverse Laplace transforms F(s), then $f(t) - g(t) = N(t)$ where $N(t)$ is a null function. If $f(t)$ is tinuous, then it will be unique inverse Laplace transform of $F(s)$.

3.2 Heaviside expansion theorem

If $M(s)$ and $N(s)$ are polynomials of degree m and n respectively with $< n$, and $N(s)$ has n distinct zeros α_i, $i = 1, 2, 3, \cdots, n$, none of which zero of $M(s)$, then

$$L^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=1}^{n} \frac{M(\alpha_i)}{N'(\alpha_i)} \exp(\alpha_i t)$$

) If $N(s)$ has a repeated root α_1 of multiplicity r while other roots at $\alpha_3, \cdots \alpha_n$ are not repeated, the corresponding formula is given by

$$L^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=2}^{n} \frac{M(\alpha_i)}{N'(\alpha_i)} \exp(\alpha_i t) + \sum_{j=1}^{r} \frac{1}{(j-1)!}\left\{\frac{d^{j-1}}{ds^{j-1}}(s-\alpha_1)^j F(s)\right\} \exp(\alpha_1 t)\Big|_{s=\alpha_1}$$

$\frac{1}{s} L\left(\frac{1}{\sqrt{t}}\right)_{s \to s+1}$ and then $L\left(\frac{1}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}}$

$$
\begin{aligned}
c_2 &= \lim_{s \to s_2} (s - s_2) F(s) = \lim \frac{}{s^2(s - s_1)} \\
&= \frac{1}{s_2^2(s_2 - s_1)}
\end{aligned}
$$

Next we express c_1, c_2 in terms of α and β.

$$s_2 = -\alpha + \iota\beta = \sqrt{\alpha^2 + \beta^2}\, e^{\iota\theta}$$

where $\tan\theta = -\beta/\alpha$.

$$s_1 = -\alpha - \iota\beta = \sqrt{\alpha^2 + \beta^2}\, \exp(-\imath\theta)$$

Therefore

$$s_1^2 = (\alpha^2 + \beta^2)\exp(2\imath\theta) = b^2 \exp(2\imath\theta)$$

and

$$s_2^2 = (\alpha^2 + \beta^2)\exp(-2\imath\theta) = b^2 \exp(-2\imath\theta)$$

Also $s_1 - s_2 = 2\imath\beta$. Hence we can write

$$c_1 = \frac{1}{b^2(2\iota\beta)} e^{-2\iota\theta}, \quad c_2 = \frac{1}{b^2(-2\iota\beta)} e^{2\iota\theta}$$

Finally

$$f(t) = \frac{t}{b^2} - \frac{2\alpha}{b^4} + c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

where the last two terms (T_{34}) can be further simplified as follows.

$$
\begin{aligned}
T_{34} &= \frac{1}{2\iota\beta b^2} e^{-2\iota\theta}\, e^{(\alpha + \iota\beta)t} + \frac{1}{-2\iota\beta b^2} e^{2\iota\theta}\, e^{(-\alpha - \iota\beta)t} \\
&= \frac{1}{2\iota\beta b^2} e^{-\alpha t} \left[e^{\iota(2\theta + \beta)} - e^{-\iota(2\theta + \beta t)} \right] \\
&= \frac{e^{-\alpha t}}{2\iota\beta b^2} 2\iota \sin(\beta - 2\theta) \\
&= \frac{e^{-\alpha t}}{\beta b^2} \sin(\beta + 2\theta)
\end{aligned}
$$

Example 3

Find the general solution of the differential equation

$$y''(t) + k^2 y(t) = f(t)$$

Solution

Taking the Laplace transform of both sides

$L\{y''(t)\} + k^2 L\{y(t)\} = L\{f(t)\}$

Now we take Laplace transform of derivatives, assuming that all functions are of class A) and obtain

$s^2 Y(s) - sy(0) - y'(0) + k^2 Y(s) = F(s)$

or

$(s^2 + k^2) Y(s) = F(s) + s\,y(0) + y'(0)$

or

$Y(s) = [c_1 + c_2 s + F(s)]/(s^2 + k^2)$

where $c_1 = y'(0)$ and $c_2 = y(0)$ are constants. Therefore

$$\begin{aligned} y(t) &= L^{-1}\{Y(s)\} \\ &= L^{-1}\left\{\frac{c_1}{s^2+k^2}\right\} + L^{-1}\left\{\frac{c_2 s}{s^2+k^2}\right\} \\ &+ L^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\} \\ &= \frac{c_1}{k}\sin kt + c_2\cos kt + L^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\} \\ &= \frac{c_1}{k}\sin kt + c_2\cos kt + \frac{1}{k}\sin kt \star f(t) \end{aligned}$$

where we have used the convolution theorem. Finally we have

$$y(t) = \frac{c_1}{k}\sin kt + c_2(\cos kt) + \frac{1}{k}\int_0^t \sin(t-\tau)\, f(\tau)\, d\tau$$

Example 4

Solve the I.V.P.

$y''(t) + ty'(t) - y(t) = 0, \quad y(0) = 0, \quad y'(0) = 1$

Solution

Taking the Laplace transform of both the sides

$L\{y''\} + L\{ty'(t)\} - L\{y(t)\} = 0$

or

$$s^2Y(s) - sy(0) - y'(0) + \left(-\frac{d}{ds}\right) L\{y'(t)\} - Y(s) = 0$$

or on substituting for the initial values, we have

$$s^2Y(s) - 1 - \frac{d}{ds}\{sY(s) - y(0)\} - Y(s) = 0$$

or

$$s^2Y(s) - 1 - \{sY'(s) + Y(s)\} - Y(s) = 0$$

or

$$(s^2 - 1)\,Y(s) - sY'(s) - Y(s) = 1$$

or

$$-s\,Y'(s) + (s^2 - 2)\,Y(s) = 1$$

or

$$Y'(s) + \frac{2 - s^2}{s}\,Y(s) = \frac{-1}{s} \qquad (1)$$

which is of the form $Y' + p(x)\,Y = q(x)$, (i.e. it is first order linear nonhomogeneous DE). Its integrating factor $\mu(x)$ is given by

$$\mu(x) = \exp\int\left(\frac{2}{s} - s\right)\,ds = \exp(2\ln s - s^2/2) = s^2\exp(-s^2/2)$$

Therefore general solution of (1) is given by

$$\begin{aligned} Y(s)s^2e^{-s^2/2} &= \int\left(\frac{-1}{s}\right)\,s^2\exp(-s^2/2)\,ds + \text{constant} \\ &= \int\{-s\,\exp(-s^2/2)\}ds + \text{constant} \\ &= \exp(-s^2/2) + c \end{aligned}$$

where c is a constant. Therefore $Y(s) = 1/s^2 + c\,\exp(s^2/2)$

Now taking limit as $s \to \infty$, we have $c = 0$. Therefore $Y(s) = 1/s^2$.

Taking inverse Laplace transform, we obtain $y(t) = t$, as the required solution.

Example 5

Use Laplace transform to solve the problem.

$y'' - ay = f(t),\; y(0) = y_0,\; y'(0) = y_1$

Solution

Applying the Laplace transform operator on both sides, we get

$$L\{y''\} - aL\{y(t)\} = L\{f(t)\}$$

or

$$s^2Y(s) - sY(0) - Y'(0) - ay(s) = F(s)$$

or using the initial conditions

$$(s^2 - a)Y(s) - sy_0 - y_1 = F(s)$$

or

$$(s^2 - a)Y(s) = F(s) + y_0 s + y_1$$

or

$$Y(s) = \frac{F(s)}{s^2 - a} + \frac{y_1}{s^2 - a} + \frac{y_0 s}{s^2 - a}$$

Taking the inverse Laplace transform of both sides, we have

$$\begin{aligned}
y(t) &= L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{F(s)}{s^2 - a}\right\} \\
&+ L^{-1}\left\{\frac{y_1}{s^2 - a}\right\} + L^{-1}\left\{\frac{y_0 s}{s^2 - a}\right\} \\
&= \frac{1}{\sqrt{a}} L^{-1}\left\{\frac{\sqrt{a}}{s^2 - a} F(s)\right\} + \frac{y_1}{\sqrt{a}} \sinh \sqrt{a}\, t + u_0 \cosh \sqrt{a}\, t \\
&= \frac{1}{\sqrt{a}} \sinh \sqrt{a}\, t \star f(t) + \frac{y_1}{\sqrt{a}} \sinh \sqrt{a}\, t + y_0 \cosh \sqrt{a}\, t
\end{aligned}$$

where we have used the convolution theorem for the first term and the fact that $L^{-1}\{F(s)\} = f(t)$.

Example 6

Use Laplace transform to solve the problem

$$y^{(\text{iv})} + 5y''' + 4y' + 4y = \left\}\begin{array}{ll} \sin t, & 0 < t \leq 2\pi \\ \sin 2t, & t > 2\pi \end{array}\right.$$

with the end-point conditions

$$y(0) = y''(0) = y'''(0) = 0, \; y'(0) = 1 \qquad (1)$$

Solution

inverse Laplace t

the function $F(s) = 1/(s+1)(s^2+1)]$.

(**Ans.** $f(t) = [\sin t - \cos t + e^{-t}]/2$).

4. **Evaluate the inverse Laplace transform of the function** $F(s) = 1/s^3(s^2+1)$.

(**Ans.** $f(t) = t^2/2 + \cos t - 1$).

5. **Evaluate the inverse Laplace transform of the function** $F(s) = (s^2-1)1/(s^2+1)^2]$.

(**Ans.** $f(t) = t\cos t$).

6. **Evaluate the inverse Laplace transform of the function** $F(s) = s/(s^2+1)^4$.

Ans. $f(t) = [3t^2\cos t + (t^3 - 3t)\sin t]/48$).

Evaluate the inverse Laplace transform of the function $(s) = (3s-1)/[s(s-1)^2(s+1)]$.

Evaluate the inverse Laplace transform of the function $) = 1/[(s^4+1)]$

ns. $f(t) = [\sin t\cosh t - \cos t\sinh t]/4$).

0 Applications to Partial Differential Eq

.1 Introductory remarks

ne partial differential equation involve the unknown variabl we'll take the Laplace transform of $u(x,\ t)$ w.r.t. the variable

$L\{u(x,\ t)\} = U(x,\ s)$

For example

$L\{\exp(-at)\ \sin \pi x\} = (\sin \pi x)/(s+a)$

and

$$\begin{aligned} L\{\sin(x\ +\ t)\} &= L\{\sin x\ \cos t\ +\ \cos x\ \sin t\} \\ &= L\{\sin x \cos t\}\ +\ L\{\cos x\ \sin t\} \\ &= \frac{s\ \sin x}{s^2\ +\ 1} + \frac{\cos x}{s^2\ +\ 1} \end{aligned}$$

Similarly

$$\begin{aligned} L\left\{\frac{\partial u}{\partial x}\right\} &= \int_0^\infty \exp(-st)\frac{\partial u}{\partial x}\ dt \\ &= \frac{\partial}{\partial x}\int_0^\infty \exp(-st)u(x,\ t)\ dt \\ &= \frac{\partial}{\partial x}L\{u(x,\ t)\} = \frac{\partial}{\partial x}\ U(x,\ s) \end{aligned}$$

and

$$L\left\{\frac{\partial u}{\partial t}\right\} = s\ L\{u\} - u(x,\ t)|_{t=0} = sU(x,\ s) - u(x,\ 0)$$

where we have used the derivative rule.

6.10.2 Illustrative examples

In this subsection we discuss some applications of the Laplace transform formalism to the solution of B.V.Ps. associated with partial differential equations.

Example 1

Use Laplace transform method to solve the problem

$$u_{xx} = u_t, \quad 0 < x < a, \quad 0 \le t < \infty \tag{1}$$

$$u(0,\ t) = 1, \ \ u(1,\ t) = 1, \ \ t > 0 \tag{2}$$

$$u(x,\ 0) = 1 + \sin \pi x, \ \ 0 < x < 1 \tag{3}$$

Solution

It is a (one dimensional) heat problem describing conduction of heat through a rod of unit length, whose end-points are maintained at zero temperature, and whose initial temperature profile is prescribed.

We denote the Laplace transform of $u(x, t)$ by $U(x, s)$, i.e. $L\{u(x, t)\} = U(x, s)$. From (1)

$$(\partial^2/\partial x^2)\,U(x, s) = L\{(\partial/\partial t)\,u(x, t)\}$$

or

$$(\partial^2/\partial x^2)\,U(x, s) = sU(x, s) - u(x, 0) = sU(x, s) - 1 - \sin \pi x$$

where we have used (3). Simplifying further, we have

$$(\partial^2/\partial x^2)\,U(x, s) - sU(x, s) = -(1 + \sin \pi x)$$

which is a non-homogeneous linear second order differential equation, whose solution is given by $U(x, s) = U_c + U_p$.

where

$$U_c = c_1 \exp(\sqrt{s}x) + c_2 \exp(-\sqrt{s}x)$$

and

$$\begin{aligned} U_p &= \frac{-1}{D^2 - s}\,(1 + \sin \pi x) \\ &= \frac{-1}{D^2 - s}\,(1) - \frac{1}{D^2 - s}\,\sin \pi x \end{aligned}$$

or continuing further

$$\begin{aligned} U_p &= \frac{-1}{D^2 - s}\,\exp(0x) - \frac{1}{D^2 - s}\,\sin \pi x \\ &= \frac{1}{s} - \frac{1}{-\pi^2 - s}\,\sin \pi x \\ &= \frac{1}{s} + \frac{1}{\pi^2 + s}\,\sin \pi x \end{aligned}$$

Therefore

$$\begin{aligned} U(x, s) &= U_c + U_p \\ &= c_1 \exp(\sqrt{s}x) + c_2 \exp(-\sqrt{s}x) + \frac{1}{s} + \frac{1}{\pi^2 + s}\,\sin \pi x \end{aligned} \qquad (4)$$

Conditions (2) when translated in terms of the variable s become

$$U(0, s) = (1/s), \quad U(1, s) = (1/s) \qquad (5)$$

From (4) and (5) $c_1 + c_2 + 1/s + 0 = 1/s$ wherefrom $c_1 + c_2 = 0$.

and

$c_1 \exp(\sqrt{s}) + c_2 \exp(-\sqrt{s}) + (1/s) + 0 = (1/s)$

which gives

$$c_1 \exp(\sqrt{s}) + c_2 \exp(-\sqrt{s}) = 0 \tag{6}$$

Because of the result in (6), we obtain from (4)

$$U(x,\ s) = 1/s + (\sin \pi x)/(\pi^2 + s)$$

Hence the required solution of the given problem is

$$\begin{aligned} u(x,\ t) &= L^{-1}\{U(x,\ s)\} \\ &= L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{\sin \pi x}{\pi^2 + s}\right\} \\ &= 1 + \sin \pi x\ \mathrm{e}^{-\pi^2 t} \end{aligned}$$

Example 2

Solve the problem by Laplace transform method

$$u_{tt}(x,\ t) = a^2 u_{xx}(x,\ t), \quad (t > 0,\ x > 0) \tag{1}$$

$$u(x,\ 0) = u_t(x,\ 0) = 0 \tag{2}$$

$$u(0,\ t) = f(t), \quad \lim_{x \to \infty} u(x,\ t) = 0 \tag{3}$$

Solution

we define

$$U(x,\ s) = \int_0^\infty u(x,\ t) \exp(-st)\ dt$$

Taking the Laplace transform of both sides of (1) w.r.t t and using the formulas

$L\{u_t(x,\ t)\} \ = \ sU(x,\ s) - u(x,\ 0)$

and

$L\{u_{tt}(x,\ t)\} \ = \ s^2 U(x,\ s) - u(x,\ 0) - u_t(x,\ 0)$

we have

$$s^2 U(x,\ s) - su(x,\ 0) - u_t(x,\ 0) = a^2(\partial^2/\partial x^2)\ U(x,\ s)$$

or using the initial conditions (2)

$s^2 U(x,\ s) = a^2\,(\partial^2 U(x,\ s)/\partial x^2)$

or

$$(\partial^2/\partial x^2)U(x,\ s)-(s^2/a^2)U(x,\ s)=0$$

whose solution is given by

$$U(x,\ s)\ =\ c_1\exp(sx/a)+c_2\exp(-sx/a)$$ (4

where c_1, c_2 may depend on s, but are constant w.r.t. x. From (3)

$U(0,\ s)=F(s)$ and $\lim_{x\to+\infty}U(x,\ s)=0$.

From the last relation and (4) noting that $s>0$, we must have $c_1 =$ (
Therefore

$$U(x,\ s)=c_2\exp(-xs/a)$$

This will satisfy (3′) if $c_2=F(s)$. Therefore $U(x,\ s)=F(s)\exp(-x/as)$.

Finally the solution is given by

$u(x,\ t)=L^{-1}\{U(x,\ s)\}\ =\ L^{-1}\{\exp(-xs/a)F(s)\}=H(t-x/a)\ f(t-x/a$

where

$$H(t\ -\ x/a)\,f(t\ -\ x/a)=\begin{cases}f(t\ -\ x/a), & t>x/a\\ 0, & t<x/a\end{cases}$$

Example 3

Solve the problem using the Laplace transform formalism

$$u_{tt}(x,\ t)=a^2\ u_{xx}(x,\ t)\ -\ g$$ (

$$u(x,\ 0)=u_t(x,\ 0)=0$$ (2)

$$u(0,\ t)=0,\quad \lim_{x\to+\infty}u_x(x,\ t)=0$$ (3)

Solution

We define

$$U(x,\ s)\ =\ \int_0^{\infty}u(x,\ t)\exp(-st)\ dt$$ (4)

Also

$$L\{u_t(x,\ t)\}=sU(x,\ s)\ -\ u(x,\ 0)$$ (5

and

$$L\{u_{tt}(x,\ t)\}=s^2U(x,\ s)-su(x,\ 0)-u_t(x,\ 0)$$ (

Taking Laplace transform of both sides of (1), we have

$$L\{u_t t\} = a^2 L\{u_{xx}(x,\ t)\} - L\{g\}$$

or making substitutions from (5), (6) and (7), we have

$$s^2 U(x,\ s) - s u(x,\ 0) - u_t(x,\ 0) = a^2\,(d^2/dx^2)\,U(x,\ s) - g/s$$

Again using the initial conditions (2)

$$s^2\,U(x,\ s) = a^2(d^2/dx^2)\ U(x,\ s) - g/s$$

which on rearrangement becomes

$$(d^2/dx^2)\,U(x,\ s) - (s^2/a^2)\,U(x,\ s) = (g/a^2 s) \tag{7}$$

Equation (7) is a linear nonhomogeneous second order DE. Therefore its solution can be written as $U(x,\ s) = U_c + U_p$.

For complimentary function U_c, we consider the DE

$$(d^2/dx^2)\,U(x,\ s) - (s^2/a^2)\,U(x,\ s) = 0$$

whose auxiliary equation $m^2 - s^2/a^2 = 0$ gives $m = \pm(s/a)$.

Therefore complementary function is given by

$$U_c = c_1 \exp(sx/a) + c_2 \exp(-sx/a)$$

and the particular integral U_p of (8) is given by $(D = d/dx)$

$$\begin{aligned} U_p &= 1/(D^2 - s^2/a^2)\,(g/a^2 s) \\ &= g/(a^2 s)\,1/(D^2 - s^2/a^2)\ \exp(0x) \\ &= g/(a^2 s)\,1/(0 - s^2/a^2) = -g/s^3 \end{aligned}$$

Hence

$$U(x,\ s) = U_c + U_p = c_1\ \exp(sx/a) + c_2\ \exp(-sx/a) - g/s^3 \tag{8}$$

Next the first B.C. of (3) can be transformed into $U(0,\ s) = 0$.

From this B.C. and (8), we obtain

$$c_1 + c_2 - (g/s^3) = 0 \tag{9}$$

Before applying the second B.C. of (3), we find that under L.T. it is transformed into $\lim_{x\to\infty} U_x(x,\ s) = 0$, where

$$U_x(x,\ s) = (s/a)\ (c_1 \exp(sx/a) - c_2 \exp(-sx/a))$$

The B.C. $\lim_{x\to\infty} U_x(x,\ s) = 0$, gives $c_1 = 0$.

Therefore from (9), we have $c_2 = -g/s^3$. Hence

$$U(x,\ s) = -(g/s^3)\exp(-sx/a) - g/s^3$$

Taking inverse Laplace transform of both sides we get

$$\begin{aligned} L^{-1}\{U(x,\ s)\} &= L^{-1}\{g\exp(-sx/a)/s^3\} - L^{-1}\{g/s^3\} \\ u(x,\ t) &= gL^{-1}\{\exp(-sx/a)/s^3\} - gL^{-1}\{1/s^3\} \qquad (10) \end{aligned}$$

We know that

$$L^{-1}\{\exp(-ks)\ F(s)\} = H(t-k)\ f(t-k)$$

In (9), let $F(s) = 1/s^3$ and $k = x/a$. Then

$$f(t) = L^{-1}\{F(s)\} = t^2/2$$

Therefore

$$f(t-k) = (1/2)\,(t - x/a)^2 = (at-x)^2/2a^2 \text{ and } H(t-k) = H(t - x/a)$$

Writing

$$I = L^{-1}\{\exp(-ks)/s^3\} = (1/2)\ (t - x/a)^2\ H(t - x/a)$$

and making substitutions we obtain

$$u(x,\ t) = (g/2)\ [(at-x)^2/a^2]\ H(t - x/a) - gt^2/2$$

Example 4

Solve the problem using Laplace transform method:

$$u_{xx} = u_{tt},\ 0 < x < 1,\ t > 0 \qquad (1)$$

$$u(0,\ t) = 0,\quad u(1,\ t) = 0 \qquad (2)$$

$$u(x,\ 0) = \sin\pi x,\ u_t(x,\ 0) = -\sin\pi x \qquad (3)$$

Solution

We define

$$U(x,\ s) = L\{u(x,\ t)\} = \int_0^\infty \exp(-st)\,u(x,\ t)\,dt$$

Also

$$L\{u_{tt}\} = s^2U(x,\ s) - su(x,\ 0) - u_t(x,\ 0) \qquad (4)$$

$$= \frac{4}{\pi}(u_1 - u_0) \sum_{n=0}^{\infty} \exp\left[\frac{-2(2n+1)^2 \pi^2 k t}{4\ell^2}\right] \times$$

$$\times \frac{(-1)^n}{(2n+1)^2} \cos\frac{(2n+1)\pi x}{2\ell} + u_1$$

6.10.3 Exercises

1. Solve the problem defined by the equations

$$u_{tt}(x,\ t) = a^2 u_{xx}(x,\ t) - g$$

$$u(x,\ 0) = u_t(x,\ 0) = 0, \quad \lim_{x\to\infty} u(x,\ t) = 0, \quad \lim_{x\to\infty} u_x(x,\ t) = \dot{0}$$

2. Solve the PDE $u_{xx} = u_{tt},\ 0 < x < 1,\ t > 0$, subject to the boundary and initial conditions $u(0,\ t) = 0,\ u(1,\ t) = 0,\ u(x,\ 0) = \sin \pi x$, and $u_t((x,\ 0) = - \sin \pi x$.

[**Ans.** $u(x,\ t) = \sin \pi x \cos \pi t - (1/\pi) \sin \pi t$].

3. A semi-infinite uniform conducting rod is initially at zero temperature. At time $t > 0$ a constant temperature $u_0 > 0$ is applied and maintained at the nearby end of the rod. Formulate and solve the problem, using Laplace transforms.

(**Hint:** $ku_{xx} = u_t,\ \ u(x,0) = 0,\ \ u(0,t)$ and $u(x,t)$ are finite as $x \to \infty$).

(**Ans** : $u(x,t) = L^{-1}\left\{\frac{u_0}{s}\exp(-\sqrt{s/k}\,x)\right\} = u_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)$

4. Find a formal solution of the problem $ku_{xx} = u_t,\ \ t > 0,\ \ 0 < x < \infty$.

$u(0,t) = f(t),\ \ u(x,0) = 0,\ \ u(x,t) \to 0$ as $x \to \infty$

(**Ans:** $U(x,s) = F(s)\exp(-\sqrt{s/k}\,x),\ \ u(x,t) = L^{-1}\left\{F(s)\exp(-\sqrt{s/k}\,x)\right\}$)

****