$$= \int_0^\infty \exp(-st) \int_0^\infty H(t-\tau) f(\tau)\, g(t-\tau)\, d\tau\, dt$$

where $H(t-\tau)$ is the unit step function. Now changing the order of integration, we have

$$L\{f \star g\} = \int_0^\infty \left[\int_0^\infty \exp(-st)\, H(t-\tau) g(t-\tau)\, dt \right] f(\tau)\, d\tau$$

Let $t - \tau = t'$, then on substitution, we have

$$\begin{aligned} L\{f \star g\} &= \int_0^\infty \left[\int_{-\tau}^\infty \exp[-s(t'+\tau)]\, H(t')\, g(t')\, dt' \right] f(\tau)\, d\tau \\ &= \int_0^\infty e^{-s\tau} \left[\int_0^\infty \exp(-st') H(t')\, g(t')\, dt' \right] f(\tau)\, d\tau \end{aligned}$$

where using the property of step function that $H(t') = 0$ for $t' < 0$ and $H(t') = 1$ for $t' \geq 0$, we have

$\int_{-\tau}^0 \exp(-s\,t')\, H(t')\, g(t')\, dt' = 0$

Therefore

$$\begin{aligned} L\{f \star g\} &= \int_0^\infty e^{-s\tau} \left[\int_0^\infty e^{-st'} g(t')\, dt' \right] f(\tau)\, d\tau \\ &= \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \int_0^\infty e^{-st'} g(t')\, dt' \\ &= F(s)\, G(s) \end{aligned}$$

6.6.4 Illustrative examples

Example 1

Calculate the following convolutions.

(a) $t \star \exp(t)$, (b) $t \star \sin 2t$, (c) $\sin 2t \star \exp(-t)$

Solution

(a)

$$\begin{aligned} t \star \exp(t) &= \int_0^t \tau \exp(t-\tau)\, d\tau = \exp(t) \int_0^t \tau \exp(-\tau)\, d\tau \\ &= \exp(t)\, [\tau\{-\exp(-tau)\} - \{\exp(-tau)\} \cdot 1]|_0^t \\ &= \exp(t)\, [1 - \exp(-t)\,(1+t)] \end{aligned}$$

(b)

$$\begin{aligned} t \star \sin 2t &= \int_0^t \tau \sin 2(t-\tau)\, d\tau \\ &= \tau \left[\frac{\cos 2(t-\tau)}{2} - \frac{\sin 2(t-\tau)}{-2}\right]\Bigg|_0^t \\ &= \frac{1}{2} t \cos t + \frac{1}{2} \sin t - \frac{1}{2} \sin 2t \\ &= \frac{1}{2}(t \cos t - \sin 2t + \sin t) \end{aligned}$$

(c)

Use the formula for the integral of $\exp(ax) \cos bx$ given in appendix A.

Example 27

Using the convolution theorem, calculate the inverse Laplace transform of the function

$$\text{(a)} \quad \frac{3}{s^2(s^2+9)}, \qquad \text{(b)} \quad \frac{s}{(s^2+9)^2}$$

Solution

(a) Writing $H(s) = F(s)\,G(s)$. With $F(s) = 1/s^2$ and $G(s) = 3/(s^2+9^2)$, we have $f(t) = t$ and $g(t) = \sin 3t$.

Therefore by convolution theorem

$$\begin{aligned} h(t) &= f(t) \star g(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau \\ &= \int_0^t (t-\tau) \sin 3\tau \, d\tau \\ &= \left(-\frac{t \cos 3\tau}{3} + \frac{\tau \cos 3\tau}{3} - \frac{\sin 3\tau}{9}\right)\Bigg|_0^t \\ &= -\frac{\sin 3t}{9} + \frac{t}{3} = \frac{1}{9}(3t - \sin 3t) \end{aligned}$$

(b) Here $H(s) = s/(s^2+9)^2$. With $F(s) = s/(s^2+9)$, and $G(s) = 1/(s^2+9)$, we have $f(t) = \cos 3t$, $g(t) = (1/3) \sin 3t$. Therefore by convolution theorem

$$\begin{aligned} h(t) &= L^{-1}\{H(s)\} = f(t) \star g(t) = \frac{1}{3} \int_0^t \cos 3(t-\tau) \sin 3\tau \, d\tau \\ &= \frac{1}{3} \int_0^t (\cos 3t \cos 3\tau + \sin 3t \sin 3\tau) \sin 3\tau \, d\tau \end{aligned}$$

$$= \frac{1}{3}\int_0^t \left(\cos 3t \cos 3\tau \sin 3\tau + \sin 3t \sin^2 3\tau\right) d\tau$$

$$= \frac{1}{6}\cos 3t \int_0^t \sin 6\tau \, d\tau + \frac{1}{3}\sin 3t \int_0^t \frac{1-\cos 6\tau}{2} d\tau$$

$$= \frac{1}{36}\cos 3t\,(1-\cos 6t) + \frac{1}{6}\sin 3t \left(\tau - \frac{\sin 6\tau}{6}\right)\Big|_0^t$$

$$= \frac{1}{36}[-\cos 3t + \cos 3t] + \frac{1}{6} t \sin 3t$$

$$= \frac{1}{6} t \sin 3t$$

Example 3

Use convolution theorem to calculate the Laplace transform of

$f(t) = \int_0^t (t-\beta)^3 e^\beta \sin\beta \, d\beta$

Solution

By definition of convolution, $f(t) = t^3 \star (e^t \sin t)$. Therefore

$$L\{f(t)\} = L\{t^3\}\,L\{e^t \sin t\} = \frac{3!}{s^4}\left(L\{\sin t\}\right)\Big|_{s\to s-1}$$

$$= \frac{6}{s^4}\frac{1}{(s-1)^2+1} = \frac{6}{s^4(s^2-2s+2)}$$

6.6.5 Exercises

1. Find the Laplace transform of the given integrals

(a) $\int_0^t (t-\beta)\sin 3\beta \, d\beta$, (b) $\int_0^t \exp[-(t-\beta)]\sin\beta \, d\beta$.

2. Find the inverse Laplace transform using the convolution theorem or otherwise.

(a) $4/[s^2(s-2)]$, (b) $1/(s^2+1)^2$, (c) $1/(s^2-1)^2$.

3. Show that

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\, d\tau$$

(Hint: use convolution theorem with $G(s) = 1/s$).

4. Show that

$$L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \left\{\int_0^\tau f(\lambda)\, d\lambda\right\} d\tau$$

(**Hint** Apply the convolution theorem to $1/s$ and $F(s)/s$.)

5. In problem 4, show that

L.H.S. $= t\int_0^t f(\lambda)\,d\lambda - \int_0^t t' f(t')\,dt'$

6. Solve the integral equation

$y(t) = a\,t + \int_0^t y(\tau)\ \sin(t-\tau)\,d\tau$

(**Ans.**: $y(t) = at + (a/3!)\,t^3$).

7. Solve the integral equation by using convolution theorem:

$y(t) = f(t) + \int_0^t g(t-\tau)\,y(\tau)\,d\tau$

(**Ans.** $y(t) = L^{-1}\{F(s)/[1-G(s)]\}$).

8. Solve the D.Es. by Laplace transform method.

(a) $y''(t) + k^2 y(t) = f(t)$

(b) $y''(t) - 2ky'(t) + k^2 y(t) = f(t)$

(c) $y''(t) + \lambda y'(t) + k^2 y(t) = f(t)$

In each case discuss the physical significance.

[**Ans.**: (b) $e^{-kt} y(t) = c_1 + c_2 t + \int_0^t (t-\tau)\,e^{-k\tau}\,d\tau$].

9. Solve the problem

$y'' + \omega^2 y = f(t),\ \ y(0) = y_0,\ \ y'(0) = y_1$

and discuss the case when

$$f(t) = \begin{cases} f_0, & t_0 < t < t_1 \\ 0, & \text{for other } t \text{ values} \end{cases}$$

(**Ans.** $y(t) = y_0 \cos\omega t + (y_1/\omega)\sin\omega t + (1/\omega)\,f(t) \star \sin\omega t$).

10. Solve the inhomogeneous problems with zero initial conditions i.e. $u(0) = 0$ and $u'(0) = 0$.

(a) $u'' + au = 1$

(b) $u'' + u = t$

(c) $u'' + 2u' = 1 - \exp(-t)$

(d) $u'' - u = 1$

(e) $u'' + 4u = \sin t$

11. Solve the I.V.P.

$y''(t) + ty'(t) - y(t) = 0, \quad y(0) = 0, \quad y'(0) = 1$

→ formula to study on page 243.

(**Hint:** On applying L.T. and using I.Cs. we obtain

$Y'(s) + (2 - s^2)/s \; Y(s) = -1/s.$

The integrating factor for this DE is $s^2 \exp(-s^2/2)$, and the solution is given by $Y(s) = 1/s^2 + c \exp(s^2/2)$).

6.7 Laplace and Inverse Laplace Transforms of some other Functions

6.7.1 Some useful results

The following results will be used in the sequel.

The Gaussian Integral

$$\int_{-\infty}^{+\infty} \exp(-x^2)\, dx = \sqrt{\pi}$$

Proof

Let

$I = \int_{-\infty}^{+\infty} \exp(-x^2)\, dx = \int_{-\infty}^{+\infty} \exp(-y^2)\, dy$

Therefore

$$\begin{aligned} I^2 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp(x^2)\, \exp(-y^2)\, dx\, dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp[-(x^2 + y^2)\, dx\, dy \end{aligned}$$

Reverting to the polar coordinates (r, θ), we have

$$\begin{aligned} I^2 &= \int_{r=0}^{r=\infty} \int_{\theta=0}^{\theta=2\pi} \exp(-r^2)\, r\, dr\, d\theta \\ &= 2\pi \int_0^{\infty} \exp(-r^2)\, r\, dr \end{aligned}$$

$$= \pi \int_0^{\infty} \exp(-p)\, dp, \quad (p = r^2)$$

Hence

$I = \int_{-\infty}^{+\infty} \exp(-x^2)\, dx = \sqrt{\pi}$ and $\int_0^{\infty} \exp(-x^2)\, dx = \sqrt{\pi}/2$

The Gamma Function

The gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t}\, t^{x-1}\, dt$$

From this definition it follows from integration by parts that

$$\begin{aligned} \Gamma(x+1) &= \int_0^{\infty} e^{-t}\, t^{x}\, dt \\ &= t^{x}\,(-e^{-t})]_0^{\infty} + \int_0^{\infty} e^{-t}\, x\, t^{x-1}\, dt \\ &= 0 + x \int_0^{\infty} e^{-t}\, t^{x-1}\, dt = x\,\Gamma(x) \end{aligned}$$

Hence $\Gamma(x+1) = x\Gamma(x)$.

Also $\Gamma(1) = \int_0^{\infty} e^{-t} t^{1-1}\, dt = \int_0^{\infty} e^{-t}\, dt = 1$.

Therefore with $x = n$, n a positive integer, using the above relation, we obtain

$$\begin{aligned} \Gamma(n+1) &= n\Gamma(n) = n(n-1)(n-2)\cdots 2\cdot 1\cdot \Gamma(1) \\ &= n(n-1)(n-2)\cdots 2\cdot 1\cdot 1 = n! \end{aligned}$$

where we have used the result $\Gamma(1) = 1$.

From this result it is clear that the gamma function can be regarded as the generalization of the factorial function.

6.7.2 Laplace transform of the step function

By definition

$$\begin{aligned} L\{H(t-t_0)\} &= \int_0^{\infty} e^{-st}\, H(t-t_0)\, dt \\ &= \int_0^{t_0} e^{-st}\, H(t-t_0)\, dt + \int_{t_0}^{\infty} e^{-st}\, H(t-t_0)\, dt \end{aligned}$$

$$= 0 + \int_{t_0}^{\infty} e^{-st}\, dt, \quad (H(t-t_0) = 1)$$

$$= \left.\frac{e^{-st}}{-s}\right|_{t_0}^{\infty} = \frac{e^{-st_0}}{s}$$

Hence $L\{H(t-t_0)\} = e^{-st_0}/s$.

In particular when $t_0 = 0$, $L\{H(t)\} = 1/s$.

6.7.3 Laplace transform of the logarithmic function

To calculate $L\{\ln t\}$, proceed as follows.

$$L\{\ln t\} = \int_0^{\infty} e^{-st} \ln t\, dt$$

Now we put $st = u$, in the integral on the right side, and obtain

$$\begin{aligned} L\{\ln t\} &= \int_0^{\infty} e^{-u} (\ln u - \ln s) \frac{du}{s} \\ &= \frac{1}{s}\int_0^{\infty} e^{-u} \ln u\, du - \frac{\ln s}{s}\int_0^{\infty} e^{-u}\, du \\ &= \frac{1}{s}\int_0^{\infty} e^{-u} \ln u\, du - \frac{\ln s}{s} \times 1 \end{aligned}$$

By definition $\Gamma(x+1) = \int_0^{\infty} e^{-u} u^x\, du$. On differentiating

$$\Gamma'(x+1) = \int_0^{\infty} e^{-u} u^x \ln u\, du$$

or on putting $x = 0$, we have

$$\Gamma'(1) = \int_0^{\infty} e^{-u} \ln u\, du$$

Therefore on substitution

$$L\{\ln t\} = (\Gamma'(1) - \ln s)/s$$

where $\Gamma'(1)$ is a constant called *Euler's constant*, whose value is approximately 0.577215665.

6.7.4 Laplace transform of functions of the form $t^n f(t)$

Theorem

If $f(t)$ is a function of exponential order c, then

$$L\{t^n\ f(t)\} = (-1)^n \left(\frac{d}{ds}\right)^n F(s), \quad \text{for } s > a$$

Proof

We know that

$$F(s) = \int_0^\infty \exp(-st)\ f(t)\ dt$$

Differentiating both sides w.r.t. s, we have

$$\begin{aligned} F'(s) &= \int_0^\infty (d/ds) \exp(-st)\ f(t)\ dt \\ &= \int_0^\infty (-t) \exp(-st)\ f(t)\ dt = (-1)\ L\{t\ f(t)\} \end{aligned}$$

which is equivalent to the statement

$$L\{t\ f(t)\} = -\ F'(s) = \left(-\frac{d}{ds}\right)\ F(s)$$

Repeating the same process, we have

$$L\{t^2\ f(t)\} = \left(-\frac{d}{ds}\right)^2\ F(s)$$

and finally

$$L\{t^n\ f(t)\} = (-\frac{d}{ds})^n\ F(s) = (-1)^n \left(\frac{d}{ds}\right)^n\ F(s)$$

which is equivalent to the statement

$$L\{(-t)^n\ f(t)\} = \left(-\frac{d}{ds}\right)^n\ F(s) = \left(\frac{d}{ds}\right)^n\ F(s)$$

Corollary

The following result can be deduced as a corollary from the above.

$$L\{p(t)\ f(t)\} = p(-D)\ F(s)$$

where $p(t)$ is a polynomial in t, $D = d/ds$ and $F(s) = L\{f(t)\}$.

Let

$$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n = \sum_i a_i\, t^i$$

Then

$$\begin{aligned} L\{p(t)\,f(t)\} &= L\{\sum_i (a_i t^i)\,f(t)\} \\ &= \sum_i a_i\,L\{t^i\,f(t)\} = \sum_i a_i \left(-\frac{d}{ds}\right)^i L\{f(t)\} \\ &= \sum_i a_i\,(-D)^i\,F(s) = p(-D)\,F(s) \end{aligned}$$

6.7.5 Laplace transforms of Bessel functions

In many physical problems, we have to calculate Laplace transforms of Bessel functions. The calculation for different Bessel functions of the first kind of order 0 i.e. $J_0(t)$ is illustrated in examples 1 and 2 below.

6.7.6 The second shifting/translation theorem

Just as the first shifting theorem enables us to calculate Laplace transforms of products of functions of the type $e^{kt}\,f(t)$, the second shifting theorem in a similar fashion enables us to compute inverse Laplace transforms of functions of the form $e^{-as}\,F(s)$. It can be stated as

$$L^{-1}\{e^{-as}\,F(s)\} = H(t-a)\,f(t-a), \quad (a > 0)$$

Proof

By definition

$L\{H(t-a)\,f(t-a)\} = \int_0^\infty \exp(-st)\,H(t-a)\,f(t-a)\,dt$

If we let $t - a = t'$, then the limits of integration on the right side will vary from $-a$ to ∞. Therefore

$$\begin{aligned} L\{H(t-a)\,f(t-a)\} &= \int_{-a}^{0} \exp[-s(t'+a)]\,H(t')\,f(t')\,dt' \\ &+ \int_0^\infty \exp[-s(t'+a)]\,H(t')\,f(t')\,dt' \\ &= 0 + \exp(-as)\int_0^\infty \exp(-st')\,H(t')\,f(t')\,dt' \\ &= \exp(-as)\,L\{f\} = \exp(-as)\,F(s) \end{aligned}$$

where we have used the defining properties of the step function.

6.7.7 Illustrative examples

Example 1

Find the Laplace transform of $J_0(t)$ where

$J_0(t) = (1/\pi)\int_0^\pi \cos(t\sin\theta)\,d\theta$

Solution

$$L\{J_0(t)\} = \int_0^\infty \exp(-st)\,\frac{1}{\pi}\left\{\int_0^\pi \cos(t\ \sin\theta)\,d\theta\right\}\,dt$$

Reversing the order of integration, we have

$$L\{J_0(t)\} = \frac{1}{\pi}\int_0^\pi\left\{\int_0^\infty e^{-st}\cos(t\ \sin\theta)\,dt\right\}\,d\theta$$

Now let I denote the integral in braces. Then using the formula

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

we have

$$\begin{aligned} I &= \int_0^\infty \exp(-st)\cos(t\ \sin\theta)\,dt \\ &= \left.\frac{\exp(-st)}{s^2+\sin^2\theta}[-s\cos(t\ \sin\theta)+\sin\theta\cdot\sin(t\sin\theta)]\right|_0^\infty \\ &= 0 - \frac{1}{s^2+\sin^2\theta}\cdot[-s] = \frac{s}{s^2+\sin^2\theta}\end{aligned}$$

On substitution, we obtain

$$\begin{aligned} L\{J_0(t)\} &= \frac{s}{\pi}\int_0^\pi \frac{d\theta}{s^2+\sin^2\theta} \\ &= \frac{s}{\pi}\int_{-\pi/2}^{+\pi/2}\frac{d\theta'}{s^2\ +\ \cos^2\theta'}, \quad (\text{where } \theta = \theta'+\pi/2) \\ &= \frac{2s}{\pi}\int_0^{\pi/2}\frac{d\theta}{s^2+\cos^2\theta} \\ &= \frac{2s}{\pi}\int_0^{\pi/2}\frac{\sec^2\theta\ d\theta}{1+s^2\ \sec^2\theta} \\ &= \frac{2s}{\pi}\int_0^{\pi/2}\frac{\sec^2\theta\ d\theta}{1+s^2\,(1+\tan^2\theta)}\end{aligned}$$

Letting $u = \tan\theta$, $\sec^2\theta\, d\theta = du$, we obtain

$$\begin{aligned}
L\{J_0(t)\} &= \frac{2s}{\pi}\int_0^\infty \frac{du}{1+s^2(1+u^2)} \\
&= \frac{2s}{\pi}\int_0^\infty \frac{du}{(1+s^2)+s^2u^2} \\
&= \frac{2}{\pi s}\int_0^\infty \frac{du}{(1+s^2)/s^2+u^2} \\
&= \frac{2}{\pi s}\int_0^\infty \frac{du}{a^2+u^2}, \quad a^2 = \frac{1+s^2}{s^2} \\
&= \frac{2}{\pi s}\cdot\frac{1}{a}\cdot \tan^{-1}\frac{u}{a}\Big]_0^\infty \\
&= \frac{2}{\pi}\cdot\frac{1}{\sqrt{1+s^2}}\cdot\frac{\pi}{2} \\
&= \frac{1}{\sqrt{1+s^2}}
\end{aligned}$$

Example 2

Given that Bessel functions of the first kind and positive integral order satisfy the recurrence relations

$$J_1 = -J_0', \quad J_{n+1} = J_{n-1} - 2J_n', \quad n \geq 1$$

with $J_0(0) = 1$, $J_n(0) = 0$, $n > 0$, show that

$$L\{J_n(t)\} = \frac{(\sqrt{s^2+1}-s)^n}{\sqrt{s^2+1}}$$

Also find the Laplace transform for $J_0(at)$, $a > 0$.

Solution

From the first recurrence relation, using the formula for Laplace transform of a derivative, we obtain

$$\begin{aligned}
F_1(s) &= L\{J_1(t)\} = -L\{J_0'(t)\} \\
&= -\{sL\{J_0(t)\} - J_0(0)\} \\
&= -sL\{J_0(t)\} + J_0(0), \quad (\text{where } J_0(0) = 1) \\
&= -sF_0(s) + 1 \qquad (1)
\end{aligned}$$

But from example 1

$$F_0(s) = L\{J_0(t)\} = 1/\sqrt{s^2+1}$$

Therefore the given formula for $L\{J_n(t)\}$ is true for $n = 0$. Also

$$\begin{aligned} F_1(s) &= L\{J_1(t)\} = -sL\{J_0(t)\} + 1 \\ &= \frac{-s}{\sqrt{s^2+1}} + 1 = \frac{\sqrt{s^2+1} - s}{\sqrt{s^2+1}} \end{aligned}$$

which shows that the given formula is true for $n = 1$. Next we use the second recurrence relation, *viz.* $J_{n+1} = J_{n-1} - 2J_n'$, which gives $J_2 = J_0 - 2J_1'$. Therefore

$$\begin{aligned} F_2(s) &= L\{J_2(t)\} = L\{J_0(t)\} - 2L\{J_1'(t)\} \\ &= F_0(s) - 2\{sF_1(s) - J_1(0)\} \\ &= F_0 - 2sF_1(s) + J_1(0) \\ &= F_0(s) - 2sF_1(s), \quad \text{because } J_1(0) = 0 \\ &= \frac{1}{\sqrt{s^2+1}} - \frac{2s(\sqrt{s^2+1} - s)}{\sqrt{s^2+1}} \\ &= \frac{1 - 2s\sqrt{s^2+1} + 2s^2}{\sqrt{s^2+1}} \\ &= \frac{(\sqrt{s^2+1} - s)^2}{\sqrt{s^2+1}} \end{aligned}$$

which shows that the formula is true for $n = 2$. Now we use mathematical induction to establish the formula for the general index $n + 1$, assuming that it is true for n and $n - 1$.

From the second recurrence relation

$$\begin{aligned} L\{J_{n+1}(t)\} &= L\{J_{n-1}(t)\} - 2L\{J_n'(t)\} \\ &= L\{J_{n-1}(t)\} - 2[sL\{J_n(t)\} - J_n(0)] \\ &= L\{J_{n-1}(t)\} - 2sL\{J_n(t)\}, \quad J_n(0) = 0,\ n \geq 1 \\ &= \frac{(\sqrt{s^2+1} - s)^{n-1}}{\sqrt{s^2+1}} - \frac{2s(\sqrt{s^2+1} - s)^n}{\sqrt{s^2+1}} \\ &= \frac{(\sqrt{s^2+1} - s)^{n-1} \cdot [1 - 2s\sqrt{s^2+1} + 2s^2)]}{\sqrt{s^2+1}} \\ &= \frac{(\sqrt{s^2+1} - s)^{n-1} \cdot (\sqrt{s^2+1} - s)^2}{\sqrt{s^2+1}} \\ &= \frac{(\sqrt{s^2+1} - s)^{n+1}}{\sqrt{s^2+1}} \end{aligned}$$

Hence the proof. To obtain the same formula when the argument is at, we use the rule of scales, *viz.* $L\{f(at)\} = (1/a)\,F(s/a), a > 0$ and obtain

$$L\{J_n(at)\} = \frac{(\sqrt{s^2+a^2} - s)^n}{a^n\sqrt{s^2+a^2}}$$

Example 3

Evaluate $L\{\exp(at) - (\cos bt)/t\}$ and deduce that

$$L\left\{\frac{\sin^2 t}{t}\right\} = (1/2)\ln\left(\frac{\sqrt{s^2+4}}{s}\right), \quad s > 1$$

Solution

Here we will use the result

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s')\,ds' \qquad (1)$$

provided the limit of $(f(t)/t)$ as $t \to 0$ exists. In this problem

$$\lim_{t\to 0}\frac{f(t)}{t} = \lim_{t\to 0}\frac{\exp(at) - \cos(bt)}{t} = \lim_{t\to 0}\frac{ae^{at} + b\sin(bt)}{1} = a$$

Also

$$F(s) = L\{f(t)\} = L\{e^{at} - \cos(bt)\} = \frac{1}{s-a} - \frac{s}{s^2+b^2}, \quad (s > a)$$

Hence from (1)

$$\begin{aligned}
L\left\{\frac{f(t)}{t}\right\} &= L\left\{\frac{e^{at} - \cos bt}{t}\right\} \\
&= \int_s^\infty \left[\frac{1}{s'-a} - \frac{s'}{s'^2+b^2}\right] ds' \\
&= \left\{\ln(s'-a) - \frac{1}{2}\ln(s'^2+b^2)\right\}\Big|_s^\infty \\
&= \ln\frac{s'-a}{(s'^2+b^2)^{1/2}}\Big]_s^\infty = \ln\frac{(1-0)}{(1+0)} - \ln\frac{s-a}{\sqrt{s^2+b^2}} \\
&= 0 - \ln\frac{s-a}{\sqrt{s^2+b^2}} = \ln\frac{\sqrt{s^2+b^2}}{s-a}, \quad s > a
\end{aligned}$$

To deduce the second part, we put $a = 0,\ b = 2,$ so that $1 - \cos 2t = 2\sin^2 t.$ Hence

$$L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{2}\ln\frac{\sqrt{s^2+4}}{s}, \quad s > 0$$

6.7.8 Exercises

1. Use the result $L\{\exp(-kt)\ f(t)\} = F(s)$, where $f(t)$ is a real function of the real variable t to show that

$$L\{\exp(-\alpha t)\ \cos\ \beta t\ f(t)\} = \operatorname{Re} F(s + \alpha + \iota\beta)$$

and

$$L\{\exp(-\alpha t)\ \sin\ \beta t\ f(t)\} = -\operatorname{Im} F(s + \alpha + \iota\beta)$$

(**Hint:** (Take $k = \alpha + \iota\beta$, and equate real and imaginary parts on both sides)

2. Evaluate $L\{\exp(kt)\, t^n\}$ when n is an integer, and deduce the expressions for $L\{t^n \cos kt\}$ and $L\{t^n \sin kt\}$.

Hint: We have

$$\{L\{\exp(kt)\ t^n\} = \left.\frac{n!}{s^{n+1}}\right|_{s \to s-k} = \frac{n!}{(s-k)^{n+1}},\quad \operatorname{Re}(s-k) > 0$$

Now let $k = \iota\alpha$, where α is real and positive, then

$$\begin{aligned} L\{\exp(\iota\alpha t)\, t^n\} &= L\{(\cos\alpha t + \iota \sin\alpha t)\, t^n\} \\ &= \frac{n!}{(s-\iota\alpha)^{n+1}},\quad \operatorname{Re} s > 0 \end{aligned}$$

Now let $s = r\cos\theta,\ \alpha = r\sin\theta$, so that

$$r = \sqrt{s^2 + \alpha^2},\ \tan\theta = \alpha/s,\ 0 \le \theta < \pi/2$$

Since α and s are positive, $s - \iota\alpha = r(\cos\theta - \iota\sin\theta)$, and therefore

$$\begin{aligned} \frac{1}{(s-\iota\alpha)^{n+1}} &= \frac{1}{r^{-n-1}}\,(\cos\theta + \iota\sin\theta)^{-n-1} \\ &= \frac{1}{r^{n+1}}\,[\cos(n+1)\theta + \iota\sin(n+1)\theta] \end{aligned}$$

Hence

$$L\{\exp(\iota\alpha t)\, t^n\} = \frac{n!}{r^{n+1}}\,[\cos(n+1)\theta + \iota\sin(n+1)\theta]$$

Equating real and imaginary parts.

$$L\{\cos\alpha t\, t^n\} = n!\,(s^2+\alpha^2)^{-(n+1)/2}\cos(n+1)\theta$$

and

$$L\{\sin\alpha t\, t^n\} = n!\,(s^2+\alpha^2)^{-(n+1)/2}\sin(n+1)\theta$$

where $\theta = \tan^{-1}\alpha/s$.)

3. Show that

$$L\left\{\frac{f(t)}{t+a}\right\} = \exp(as)\int_s^\infty \exp(-st')\,F(t')\,dt'$$

(**Hint:** Let $I = L\{f(t)/(t+a)\}$, then with $D = d/ds$

$$\begin{aligned}(D-a)I &= (D-a)\int_0^\infty \exp(-st)\,\frac{f(t)}{t+a}\,dt \\ &= \int_0^\infty (D-a)\exp(-st)\,\frac{f(t)}{t+a}\,dt \\ &= \int_0^\infty (-t-a)\exp(-st)\,\frac{f(t)}{t+a}\,dt \\ &= -\int_0^\infty \exp(-st)\,f(t)\,dt = -F(s)\end{aligned}$$

Also

$$dI/ds - aI = -F(s)$$

which is a nonhomogeneous linear DE with integrating factor $\mu = \exp(-as)$. The solution of this equation is given by

$$\mu I = -\int \mu F(s)\,ds + \text{constant}$$

or

$\exp(-as)\,I = -\int \exp(-as)\,F(s)\,ds + c_1$

To calculate the constant c_1, we take the limit as $s \to \infty$,

$0 = -\int_{s0}^\infty \exp(-as)\;F(s)\;ds + c_1$

which gives

$c_1 = \int_{s0}^\infty \exp(-as)\;F(s)\;ds$

Therefore on substitution

$exp(-as)\,I = \int_s^\infty \exp(-as)\;F(s)\;ds$

or finally

$I = \exp(as)\int_s^\infty \exp(-as')\;F(s')\;ds'$

4. Prove that

$$L\left\{\exp(-at)\int_0^t f(t')\exp(at')\,dt'\right\} = \frac{F(s)}{s+a}$$

(**Hint:** Note that this is general form of a previous result.

$$\text{L.H.S.} = \int_0^\infty \exp[-(s+a)t]\left\{\int_0^t f(t')\;\exp(at')\;dt'\right\}\,dt$$

Integrating by parts w.r.t. t, we have

$$\begin{aligned}
\text{L.H.S.} &= \frac{\exp[-(s+a)t]}{-(s+a)} \cdot \int_0^t f(t') \exp(at')\, dt' \Big|_0^\infty \\
&\quad + \frac{1}{(s+a)} \int_0^\infty \exp[(s+a)t] \cdot e^{at} f(t)\, dt \\
&= 0 + \frac{1}{(s+a)} \int_0^\infty \exp(-st) \cdot f(t)\, dt = \frac{F(s)}{s+a} \;)
\end{aligned}$$

5. Prove that

$$L\{\text{erfc}\,(t^{1/2})\} = \frac{1}{s} - \frac{1}{\sqrt{s+1}}, \text{ where erfc } t = \frac{2}{\sqrt{\pi}} \int_t^\infty \exp(-x^2)\, dx$$

(**Hint**:

$$\begin{aligned}
\text{erfc}\,(t) &= \frac{2}{\sqrt{\pi}} \int_t^\infty \exp(-x^2)\, dx \\
&= \frac{2}{\sqrt{\pi}} \left[\int_0^\infty \exp(-x^2)\, dx - \int_0^t \exp(-x^2)\, dx\right]
\end{aligned}$$

or

$$\text{erfc}\,(t) = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}} \int_0^t \exp(-x^2)\, dx = 1 - \text{erf}\,(t)$$

where

$\text{erf}\,(t) = (2/\sqrt{\pi}) \int_0^t \exp(-x^2)\, dx$

Therefore

$$\begin{aligned}
L\left\{\text{erfc}\,(\sqrt{t})\right\} &= L\{1\} - L\left\{\text{erf}\,(\sqrt{t})\right\} \\
&= \frac{1}{s} - \frac{2}{\sqrt{\pi}} L\left\{\int_0^{\sqrt{t}} \exp(-x^2)\, dx\right\} \\
&= \frac{1}{s} - \frac{2}{\sqrt{\pi}} L\left\{\int_0^t \exp(-v) \frac{dv}{2\sqrt{v}}\right\}, \quad (x^2 = v) \\
&= \frac{1}{s} - \frac{1}{\sqrt{\pi}} L\left\{\int_0^t \frac{\exp(-t')}{\sqrt{t'}}\, dt'\right\} \\
&= \frac{1}{s} - \frac{1}{s\sqrt{\pi}} L\left\{\frac{1}{\sqrt{t}}\right\}\Big|_{s \to s+1} \\
&= \frac{1}{s} - \frac{1}{s\sqrt{s+1}}
\end{aligned}$$

Therefore

$$L\{\text{erfc}\,\sqrt{t}\} = \frac{1}{s}\left(1 - \frac{1}{\sqrt{s+1}}\right)$$

w that

(a) $L\left\{\frac{\exp(-t)}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s+1}}$

(b) $L\left\{\int_0^t \exp(-v)\frac{1}{\sqrt{v}}\,dv\right\} = \frac{1}{s}\sqrt{\frac{\pi}{s+1}}$

(c) $L\left\{\exp(t)\,\text{erfc}\,\sqrt{t}\right\} = \frac{1}{s+\sqrt{s}}$

To Page 269

Further Discussion of Inverse Laplace Transform

1 Uniqueness of the inverse Laplace transform

h's theorem guarantees the uniqueness of inverse Laplace transform up to rtain additive function called the *null function*. A function $N(x)$ is called function if $\int_0^\infty N(x)\,dx = 0$.

The theorem asserts that if $f(t)$ and $g(t)$ are inverse Laplace transforms F(s), then $f(t) - g(t) = N(t)$ where $N(t)$ is a null function. If $f(t)$ is tinuous, then it will be unique inverse Laplace transform of $F(s)$.

3.2 Heaviside expansion theorem

If $M(s)$ and $N(s)$ are polynomials of degree m and n respectively with $< n$, and $N(s)$ has n distinct zeros α_i, $i = 1, 2, 3, \cdots, n$, none of which zero of $M(s)$, then

$$L^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=1}^{n} \frac{M(\alpha_i)}{N'(\alpha_i)} \exp(\alpha_i t)$$

) If $N(s)$ has a repeated root α_1 of multiplicity r while other roots at , α_3, $\cdots$ α_n are not repeated, the corresponding formula is given by

$$L^{-1}\left\{\frac{M(s)}{N(s)}\right\} = \sum_{i=2}^{n} \frac{M(\alpha_i)}{N'(\alpha_i)} \exp(\alpha_i t) + \sum_{j=1}^{r} \frac{1}{(j-1)!}\left\{\frac{d^{j-1}}{ds^{j-1}}(s-\alpha_1)^j F(s)\right\} \exp(\alpha_1 t)\bigg|_{s=\alpha_1}$$

$\frac{1}{s} L\left(\frac{1}{\sqrt{t}}\right)_{s\to s+1}$ and then $L\left(\frac{1}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}}$

$$
\begin{aligned}
c_2 &= \lim_{s \to s_2} (s - s_2) F(s) = \lim \frac{}{s^2(s - s_1)} \\
&= \frac{1}{s_2^2(s_2 - s_1)}
\end{aligned}
$$

Next we express c_1, c_2 in terms of α and β.

$$s_2 = -\alpha + \iota\beta = \sqrt{\alpha^2 + \beta^2}\, e^{\iota\theta}$$

where $\tan\theta = -\beta/\alpha$.

$$s_1 = -\alpha - \iota\beta = \sqrt{\alpha^2 + \beta^2}\, \exp(-\iota\theta)$$

Therefore

$$s_1^2 = (\alpha^2 + \beta^2)\exp(2\iota\theta) = b^2 \exp(2\iota\theta)$$

and

$$s_2^2 = (\alpha^2 + \beta^2)\exp(-2\iota\theta) = b^2 \exp(-2\iota\theta)$$

Also $s_1 - s_2 = 2\iota\beta$. Hence we can write

$$c_1 = \frac{1}{b^2(2\iota\beta)} e^{-2\iota\theta}, \quad c_2 = \frac{1}{b^2(-2\iota\beta)} e^{2\iota\theta}$$

Finally

$$f(t) = \frac{t}{b^2} - \frac{2\alpha}{b^4} + c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

where the last two terms (T_{34}) can be further simplified as follows.

$$
\begin{aligned}
T_{34} &= \frac{1}{2\iota\beta b^2} e^{-2\iota\theta}\, e^{(\alpha+\iota\beta)t} + \frac{1}{-2\iota\beta b^2} e^{2\iota\theta}\, e^{(-\alpha-\iota\beta)t} \\
&= \frac{1}{2\iota\beta b^2} e^{-\alpha t} \left[e^{\iota(2\theta+\beta)} - e^{-\iota(2\theta+\beta t)} \right] \\
&= \frac{e^{-\alpha t}}{2\iota\beta b^2} 2\iota \sin(\beta - 2\theta) \\
&= \frac{e^{-\alpha t}}{\beta b^2} \sin(\beta + 2\theta)
\end{aligned}
$$

Example 3

Find the general solution of the differential equation

$y''(t) + k^2 y(t) = f(t)$

Solution

Taking the Laplace transform of both sides

$$L\{y''(t)\} + k^2 L\{y(t)\} = L\{f(t)\}$$

Now we take Laplace transform of derivatives, assuming that all functions are of class A) and obtain

$$s^2 Y(s) - sy(0) - y'(0) + k^2 Y(s) = F(s)$$

or

$$(s^2 + k^2)\, Y(s) = F(s) + s\, y(0) + y'(0)$$

or

$$Y(s) = [c_1 + c_2 s + F(s)]/(s^2 + k^2)$$

where $c_1 = y'(0)$ and $c_2 = y(0)$ are constants. Therefore

$$\begin{aligned} y(t) &= L^{-1}\{Y(s)\} \\ &= L^{-1}\left\{\frac{c_1}{s^2+k^2}\right\} + L^{-1}\left\{\frac{c_2 s}{s^2+k^2}\right\} \\ &+ L^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\} \\ &= \frac{c_1}{k}\sin kt + c_2 \cos kt + L^{-1}\left\{\frac{F(s)}{s^2+k^2}\right\} \\ &= \frac{c_1}{k}\sin kt + c_2 \cos kt + \frac{1}{k}\sin kt \star f(t) \end{aligned}$$

where we have used the convolution theorem. Finally we have

$$y(t) = \frac{c_1}{k}\sin kt + c_2\,(\cos kt) + \frac{1}{k}\int_0^t \sin(t-\tau)\, f(\tau)\, d\tau$$

Example 4

Solve the I.V.P.

$$y''(t) + ty'(t) - y(t) = 0, \quad y(0) = 0, \quad y'(0) = 1$$

Solution

Taking the Laplace transform of both the sides

$$L\{y''\} + L\{ty'(t)\} - L\{y\,(t)\} = 0$$

or

$$s^2Y(s) - sy(0) - y'(0) + \left(-\frac{d}{ds}\right) L\{y'(t)\} - Y(s) = 0$$

or on substituting for the initial values, we have

$$s^2Y(s) - 1 - \frac{d}{ds}\{sY(s) - y(0)\} - Y(s) = 0$$

or

$$s^2Y(s) - 1 - \{sY'(s) + Y(s)\} - Y(s) = 0$$

or

$$(s^2 - 1)Y(s) - sY'(s) - Y(s) = 1$$

or

$$-sY'(s) + (s^2 - 2)Y(s) = 1$$

or

$$Y'(s) + \frac{2 - s^2}{s}Y(s) = \frac{-1}{s} \tag{1}$$

which is of the form $Y' + p(x)Y = q(x)$, (i.e. it is first order linear nonhomogeneous DE). Its integrating factor $\mu(x)$ is given by

$$\mu(x) = \exp\int\left(\frac{2}{s} - s\right) ds = \exp(2\ln s - s^2/2) = s^2\exp(-s^2/2)$$

Therefore general solution of (1) is given by

$$\begin{aligned} Y(s)s^2e^{-s^2/2} &= \int\left(\frac{-1}{s}\right) s^2\exp(-s^2/2)\,ds + \text{constant} \\ &= \int\{-s\exp(-s^2/2)\}ds + \text{constant} \\ &= \exp(-s^2/2) + c \end{aligned}$$

where c is a constant. Therefore $Y(s) = 1/s^2 + c\exp(s^2/2)$

Now taking limit as $s \to \infty$, we have $c = 0$. Therefore $Y(s) = 1/s^2$.

Taking inverse Laplace transform, we obtain $y(t) = t$, as the required solution.

Example 5

Use Laplace transform to solve the problem

$y'' - ay = f(t),\quad y(0) = y_0,\quad y'(0) = y_1$