Chapter 6

The Laplace Transform and its Applications

6.1 Integral Transforms

The concept of integral transformation is related to that of a linear transformation defined by an integral. To understand this particular linear transformation, we consider the set of functions of x over an interval $I : a \leq x \leq b$ which may be finite or infinite. Next we choose a fixed function $K(x, y)$ of variables (x, y). Then the integral transformation is defined by

$$T\{f(x)\} \equiv F(y) = \int_a^b f(x)\, K(x, y)\, dx$$

The function $K(x, y)$ is called the *kernel* of the transformation T. This concept has been very seminal and conducive to opening up new vistas in Modern Mathematics. In classical analysis certain special types of integral transformations such as Laplace, Fourier, Chebyshev have been extensively studied and used in solving various types of problems. Different transformations correspond to different forms for the kernel $K(x, y)$. The limits of the integral $[a, b]$ are also different in each case.

The function $T\{f(x)\}$ obtained by means of such a transformation is called *integral transform* of the given function $f(t)$.

Laplace transform operator is a linear operator, i.e.

$$L\{c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\} + \cdots + c_n L\{f_n(t)\}$$

This result follows directly from the definition

6.2 Laplace Transform Preliminaries

6.2.1 Piecewise continuous function

A function $f(t)$ is piecewise continuous on the interval $[a,\ b]$ if the inte be broken up into subintervals $[a,\ t_1),\ (t_1,\ t_2),\ \cdots (t_{n-1},\ b]$ such tl is continuous in each subinterval $(t_j,\ t_{j+1})$ and has jump discontinuit endpoints of the subintervals. A jump discontinuity is a finite disconti

In other words the function is continuous inside each subinterval but ha (left and right) limits at the endpoints of the sub-intervals.

For example, the function

$$f(t) = \begin{cases} 2t^2, & 0 \le t \le 1 \\ 3+t, & 1 < t \le 2 \\ t+2, & 2 < t \le 3 \end{cases}$$

is piecewise continuous on the interval $[0,\ 3]$.

whereas the function

$$f(t) = \begin{cases} 1/t, & -1 \le t \le 1 \\ t+2, & 1 < t \le 2 \end{cases}$$

is not piecewise continuous on the interval $[0,\ 2]$.

Exponential order

A function $f(t)$ is said to be of exponential order if there exists a real po number M and a real number c such that

$$|f(t)| \le M\ exp(ct) \text{ for sufficiently large } t.$$

As an example we show that any polynomial is of exponential order. We with Taylor series expansion for $\exp(x)$:

$$\exp(at) = \sum_{n=0}^{\infty} \frac{t^n a^n}{n!} \ge \frac{t^n a^n}{n!} \Rightarrow t^n \le \frac{n!}{a^n} \exp(at)$$

Taking $M = n!/a^n$ $c = a$, we find that every term of a polynomial exponential order.

Similarly we can see that $f(t) = \exp(t^2)$ is not of exponential order.

A function which is piecewise continuous and of exponential order is said to be a function of *class A*.

6.2.2 Definition and notation

Let $f(t)$ be a continuous or sectionally continuous function of t defined over the interval $[0, \infty)$, then the *Laplace transform* of $f(t)$ is a function $F(s)$ of another variable s defined by

$$F(s) = \int_0^\infty \exp(-st)\, f(t)dt = \lim_{T\to\infty} \int_0^T \exp(-st)\, f(t)\, dt$$

It is to be noted that the existence of Laplace transform of a function depends on the existence of the defining integral. It is clear that every function may not possess its Laplace transform.

Notation: The Laplace transform of a function $f(t)$ is denoted by any one of the notations $F(s)$, $\bar{f}(s)$, $L\{f(t)\}$, $L[f(t)]$, $L\{f(t);\, s\}$. The functions $f(t)$ and $F(s)$ are referred to as it Laplace transform pair.

Theorem

The Laplace transformation operator L is a linear operator.

Proof

Let $f(t) = c_1 f_1(t) + c_2 f_2(t)$, $0 \le t < \infty$. Then

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty \exp(-st)\, \{c_1 f_1(t) + c_2 f_2(t)\} dt \\ &= c_1 \int_0^\infty \exp(-st)\, f_1(t) + c_2 \int_0^\infty \exp(-st)\, f_2(t)\, dt \\ &= c_1 L\{f_1\} + c_2 L\{f_2\} \end{aligned}$$

which shows that the operator L is linear.

6.2.3 Necessary and sufficient conditions for the existence of Laplace transforms

Necessary condition

The necessary condition for the existence of L.T. of a function $f(t)$, $t \ge 0$ is that the improper integral $\int_0^\infty \exp(-st)\, f(t)\, dt$ must exist. In other words the integral $\int_0^T \exp(-st)\, f(t)\, dt$ must exist for all value of T.

Sufficiency condition

Sufficient condition for the existence of Laplace transform of a function is that it should be a function of class A. *i.e.* it should be piece-wise continuous and of exponential order.

Proof

Let $f(t)$ be piecewise continuous in the interval $[0, T]$ and be of exponential order c. Then it will be integrable over $[0, T]$ and moreover

$|f(t)| \leq M \exp(ct)$ for $t > t_0$

Therefore

$|\exp(-st) f(t)| \leq M \exp[-(s-c)t]$ for $t > t_0$.

Hence

$$\begin{aligned}
|F(s)| &= |L\{f(t)\}| \leq \left| \int_0^\infty M \exp(-st) \exp(ct)\, dt \right| \\
&\leq M \left| \int_0^\infty \exp[-(s-c)t]\, dt \right| \\
&\leq \frac{M}{|s-c|}, \text{ provided } s > c
\end{aligned}$$

Thus it is clear that piecewise continuity and exponential order are sufficient conditions for a function to have a Laplace transform. But these are not necessary conditions. This can be seen from the fact that the function $t^{-\frac{1}{2}}$ has Laplace transform; yet it is not piecewise continuous in any interval $[0, T]$ where $T > 0$.

Corollary

From the above it follows that $\lim_{s\to\infty} F(s) = 0$.

This result is quite general and can be proved for the Laplace transform of any function, whether satisfying the conditions of the above theorem or not. Hence it follows from this result (corollary) that if $F(s)$ is any function of s such that its limit as $s \to \infty$ does not exist or is not zero, then it cannot be the Laplace transform of any function $f(t)$. Hence functions such as $F(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n$ or $\ln s$, e^s, $\sin s$, $\cos s$ cannot be Laplace transforms of any functions. On the other hand a rational function is Laplace transform of some function if the degree of the numerator is less than that of the denominator.

6.3 Laplace Transforms of Some Functions and Basic Results

6.3.1 Laplace Transform of a constant

If $f(t) = k$, then

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty \exp(-st)\, k dt = k \lim_{T\to\infty} \int_0^T \exp(-st)\, dt \\ &= k \lim_{T\to\infty} \left.\frac{\exp(-st)}{-s}\right|_0^T \\ &= \frac{k}{s} \lim_{T\to\infty} (1 - \exp(-sT)) = \frac{k}{s} \end{aligned}$$

where it has been assumed that s or $\operatorname{Re} s > 0$; otherwise the limit will not exist. Hence we can write

$L\{k\} = k/s, \quad s > 0.$

6.3.2 Laplace transform of a positive integral power of t

First we will calculate Laplace transforms of t and t^2 and then use these results to obtain $L\{t^n\}$.

(i) Let $f(t) = t$, then using Kronecker's rule for integration by parts

$$\begin{aligned} L\{t\} &= \int_0^\infty \exp(-st) t\, dt = \left[\frac{t \exp(-st)}{-s} - 1 \frac{\exp(-st)}{-s^2}\right]\Bigg|_0^\infty \\ &= 0 - \left(0 - \frac{1}{s^2}\right) = \frac{1}{s^2} \end{aligned}$$

where we have used the result $\lim \left(t^n e^{-st}\right) = 0$, as $t \to \infty$ and for $\operatorname{Re} s > 0$.

(ii) Let $f(t) = t^2$, then again using Kronecker's rule

$$\begin{aligned} L\{t^2\} &= \int_0^\infty \exp(-st)\, t^2 dt \\ &= \left[\frac{t^2 \exp(-\,st)}{-s} - 1\,(2t)\,\frac{\exp(-st)}{-s^2} + \frac{\exp(-st)}{(-s)^3}\right]\Bigg|_0^\infty \\ &= 0 + 0 + \frac{2}{s^3} = \frac{2}{s^3} \end{aligned}$$

(iii) Let $f(t) = t^n$, then from definition

$$
\begin{aligned}
L\{t^n\} &= \int_0^\infty \exp(-st)t^n\,dt \\
&= t^n \frac{\exp(-st)}{-s}\Big|_0^\infty + \frac{n}{s}\int_0^\infty \exp(-st)t^{n-1}dt \\
&= 0 + \frac{n}{s}L\{t^{n-1}\} = \frac{n}{s}L\{t^{n-1}\}
\end{aligned}
$$

Again using the same result

$$
\begin{aligned}
L\{t^n\} &= \frac{n}{s}\frac{n-1}{s}L\{t^{n-2}\} \\
&= \frac{n(n-1)(n-2)\cdots 2\,1}{s^n}L\{1\} \\
&= \frac{n!}{s^n}\frac{1}{s} = \frac{n!}{s^{n+1}}
\end{aligned}
$$

6.3.3 Laplace transforms of exponential and trigonometric functions

(i) Let $f(t) = \exp(kt)$, then by definition

$$
\begin{aligned}
L\{\exp(kt)\} &= \int_0^\infty \exp(-st)\exp(kt)\,dt \\
&= \int_0^\infty \exp[-(s-k)\,t]\,dt \\
&= \int_0^\infty \exp(-s't)\,dt \quad \text{where } s' = s-k \\
&= \frac{1}{s'} = \frac{1}{s-k}
\end{aligned}
$$

where it is assumed that $\mathrm{Re}\, s > k$.

(ii) Let $f(t) = \sin kt$, where k is a constant. In calculating Laplace transforms for $\sin kt$ and $\cos kt$, we will use the following formulas

$$\int \exp(at)\cos bt\,dt = \frac{\exp(at)}{a^2+b^2}[a\cos bt + b\sin bt]$$

and

$$\int \exp(at)\sin bt\,dt = \frac{\exp(at)}{a^2+b^2}[a\sin bt - b\cos bt]$$

By definition

$$L\{\sin k\,t\} = \int_0^\infty \exp(-st)\,\sin kt\,dt$$

Using the above formula with $a = -s$, $b = k$, we have

$$L\{\sin kt\} = \left[\frac{\exp(-st)}{s^2+k^2}\left(-s\sin kt - k\cos kt\right)\right]\Bigg|_0^\infty = \frac{k}{s^2+k^2}$$

(iii) Let $f(t) = \cos kt$, then performing calculation similar to the above, with $a = -s$, $b = k$, we have

$$\begin{aligned} L\{\cos kt\} &= \int_0^\infty \exp(-st)\cos kt\,dt \\ &= \left[\frac{\exp(-st)}{s^2+k^2}\left(-s\cos kt + k\sin kt\right)\right]\Bigg|_0^\infty \\ &= \frac{s}{s^2+k^2} \end{aligned}$$

6.3.4 Laplace transforms of hyperbolic functions

These can be calculated using the definitions

$$\sinh kt = \frac{\exp(kt) - \exp(-kt)}{2} \quad \text{and} \quad \cosh kt = \frac{\exp(kt) + \exp(-kt)}{2}$$

We obtain

$$L\{\sinh kt\} = \frac{k}{s^2-k^2} \quad \text{and} \quad L\{\cosh kt\} = \frac{s}{s^2-k^2}$$

6.3.5 The first shifting theorem and the rule of scales

In this subsection we discuss two important results which are useful additions to our toolkit of elementary rules for computing Laplace transforms.

The *first shifting theorem*, (also called the *first translation theorem*) enables us to calculate Laplace transforms of products of functions of the form $e^{kt} f(t)$ in terms of Laplace transform of $f(t)$. It will also be used in calculating inverse Laplace transforms. It can be stated as

$$L\{\exp(kt)\, f(t)\} = F(s-k) \equiv L\{f(t)\}|_{s\to s-k}$$

Proof

$$\begin{aligned} L\{\exp(kt)\, f(t)\} &= \int_0^\infty \exp(-st)\ \exp(kt)\, f(t)\, dt \\ &= \int_0^\infty \exp[-(s-k)t]\ f(t)\ dt \\ &= \int_0^\infty \exp(-s't)\, f(t)\ dt \quad \text{where } s' = s-k \\ &= F(s') = F(s-k) \equiv L\{f(t)\}|_{s \to s-k} \end{aligned}$$

Rule of scales

It enables us to calculate Laplace transform of a function of the form $f(at)$ where $a > 0$ is a constant. It states

$L\{f(at)\} = (1/a)\, L\{f(t)\}, \quad a > 0$

Proof

$$\begin{aligned} L\{f(at)\} &= \int_0^\infty \exp(-st)\, f(at)dt, \quad a > 0 \\ &= \int_0^\infty \exp(-st/a)\, f(t')\ dt'/a, \quad at = t' \\ &= \frac{1}{a}\int_0^\infty \exp(-s't')\, f(t')\, dt', \quad \text{where } s' = s/a \\ &= \frac{1}{a}\int_0^\infty \exp(-s't')\, f(t')\, dt' \\ &= \frac{1}{a} L\{f(t)\}, \quad a > 0 \end{aligned}$$

6.4 Laplace Transforms of Derivatives and Integrals

6.4.1 Laplace transforms of derivatives of a function

The following theorem provides states the sufficient conditions for the existence of Laplace transforms of derivatives of a function.

Theorem

(i) If $f(t)$ is continuous and $f'(t)$ is piecewise continuous on the interval $[0, \infty)$, and both are of exponential order, *i.e.* both of order $\exp(\alpha x)$, then

$L\{f'(t)\} = s\, L\{f(t)\} - f(0) = sF(s) - f(0)$

(ii) If $f(t)$ and $f'(t)$ are continuous and $f''(t)$ is piecewise continuous on the interval $[0, \infty)$, and all are of exponential order, i.e. both of order $e^{\alpha x}$, then

$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

(iii) If $f(t)$, $f'(t)$, $f''(t), \cdots f^{n-1}(t)$ are continuous and $f^n(t)$ is piecewise continuous on the interval $[0, \infty)$, and all are of exponential order, i.e. both of order $e^{\alpha x}$, then

$$L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Proof

Using definition and integrating by parts

$$\begin{aligned} L\{f'(t)\} &= \int_0^\infty \exp(-st)\, f'(t)\, dt \\ &= \exp(-st)\, f(t)]_0^\infty - (-s)\int_0^\infty e^{-st} f(t)\, dt \\ &= -f(0) + sL\{f(t)\} = sF(s) - f(0) \end{aligned}$$

where $F(s)$ is the Laplace transform of $f(t)$.

(ii) Again using definition and the result (i) above, we have

$L\{f''(t)\} = L\{g'(t)\} = sG(s) - g(0)$

where $g(t) = f'(t)$ and we have used result (i) above and $G(s) = L\{g(t)\}$. Now

$G(s) = L\{g(t)\} = L\{f'(t)\} = s\,F(s) - f(0)$

Therefore on substitution, we obtain

$$\begin{aligned} L\{f''(t)\} &= s[sF(s) - f(0)] - f'(0) \\ &= s^2F(s) - sf(0)] - f'(0) \end{aligned}$$

(iii) By repeated application of the results (i) or (ii) we can derive this result.

6.4.2 Laplace transform of the integral of a function

Let $g(t) = \int_0^t f(\tau)\, d\tau$, then

$$L\{g(t)\} = \frac{1}{s} L\{f(t)\}$$

To prove this theorem, we note that by fundamental theorem of calculus, viz. $g'(t) = f(t)$. Therefore

$$L\{g'(t)\} = sL\{g(t)\} - g(0)$$

But

$$g(0) = \int_0^0 f(\tau)\, d\tau = 0 \text{ and } g'(t) = f(t)$$

Hence

$$L\{g'(t)\} = sL\{g(t)\} \implies L\{g(t)\} = \frac{1}{s} L\{f(t)\}$$

or finally

$$L\left\{\int_0^t f(\tau)\, d\tau)\right\} = \frac{1}{s} L\{f(t)\} = \frac{F(s)}{s}$$

6.5 The Inverse Laplace Transform

If $F(s)$ is the Laplace transform of $f(t)$, then $f(t)$ is called the *inverse Laplace transform* of $F(s)$. The inverse Laplace transform of $F(s)$ is denoted by $L^{-1}\{F(s)\} = f(t)$.

In calculating inverse Laplace transforms of functions, we make use of our knowledge of Laplace transforms of simple functions, and develop special methods. The following table summarizes the results on Laplace transforms.

Table of Laplace Transforms

$f(t)$	$F(s)$
1	$1/s, \quad s > 0$
t^n (n = positive integer)	$n!/s^{n+1}, \quad s > 0$
e^{kt}	$1/(s-k), \quad s > k$
$\sin kt$	$k/(s^2+k^2), \quad s > 0$
$\cos kt$	$s/(s^2+k^2), \quad s > 0$
$\sinh kt$	$k/(s^2-k^2), \quad s > \lvert k\rvert$
$\cosh kt$	$s/(s^2-k^2), \quad s > \lvert k\rvert$
$e^{kt} f(t)$	$F(s-k), \quad s > k$
$t^n e^{kt}$	$n!/(s-k)^{n+1}, \quad s > k$
$f^n(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots f^{n-1}(0)$

6.5.1 Computation of Inverse Laplace Transforms

The first shifting theorem can be written in the form

$$L^{-1}\{F(s-k)\} = e^{kt} f(t) \ldots$$

By using the above theorem we can calculate inverse Laplace transforms of some of the functions. This is illustrated with the following example.

The method of partial fractions may also be used in calculating inverse Laplace transform This is illustrated in example 4 below.

6.5.2 Applications to initial value problems

The Laplace transforms have wide applications in initial and boundary value problems associated with ordinary and partial DEs. Their simplest applications and direct applications are in the solution of initial-value problems consisting of an ODE subject to initial conditions. Such applications are illustrated in example 5. The results on Laplace transforms and inverse Laplace transforms discussed in the preceding sections will be used in these examples.

6.5.3 Illustrative examples

Example 1

(a) Calculate the Laplace transform of t^α, where α is any real number.

(b) Derive the Laplace transforms of $t^{1/2}$ and $t^{-1/2}$.

Solution

(a)

$$L\{t^\alpha\} = \int_0^\infty \exp(-st)\, t^\alpha\, dt$$

Making the substitution $st = u$ on the right side, we have

$$L\{t^\alpha\} = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^\alpha \frac{du}{s} = \frac{1}{s^{\alpha+1}} \int_0^\infty \exp(-u)\, u^\alpha\, du = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

where we used the definition of the gamma function $\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du$.

(b) This follows from (a) by noting that

$\Gamma(1/2) = \sqrt{\pi}$, $L\{t^{1/2}\} = (1/2s)\sqrt{\pi/s}$, $L\{t^{-1/2}\} = \sqrt{\pi/s}$.

Example 2

Using the result $L\{t^\alpha\} = (\alpha/s)\, L\{t^{\alpha-1}\}$, $(s > 0,\ \alpha > -1)$, and the result $L\{t^{-1/2}\} = \sqrt{\pi/s}$, find $L\{t^{k/2}\}$, where k is an odd positive integer.

Solution

Since k is an odd positive integer, we can put $k = 2m+1$, where m is a positive integer.

$$L\{t^{k/2}\} = L\{t^{m+1/2}\} = \frac{m+1/2}{s}\,L\{t^{m-1/2}\}$$

Repeated application of the same result gives

$$L\{t^{k/2}\} = \frac{m+1/2}{s}\,\frac{m-1/2}{s}\,\frac{m-3/2}{s}\cdots\frac{3/2}{s}\,\frac{1/2}{s}\,L\{t^{-1/2}\}$$

Using the result $L\{t^{-1/2}\} = \sqrt{\pi/s}$, we obtain

$$L\{t^{k/2}\} = \frac{(2m+1)\,(2m-1)\,(2m-3)\,\cdots\,3\cdot 1}{(2s)^{m+1}}\sqrt{\frac{\pi}{s}}$$

$$= \frac{k(k-2)(k-4)\,\cdots\,3\cdot 1}{2^{k/2+1/2}}\sqrt{\frac{\pi}{s^{k+2}}}$$

Example 3

Find the inverse Laplace transforms of the following functions.

(a) $1/(s^2+2s)$, (b) $s/(s^2+2s)$.

Solution

$$\text{(a)}\quad \frac{1}{s^2+2s} = \frac{1}{s^2+2s+1-1} = \frac{1}{(s+1)^2-1}$$

Hence $F(s+1) = 1/[(s+1)^2-1]$. Therefore with $k=-1$

$$L^{-1}\{F(s+1)\} = \exp(kt)L^{-1}\{F(s)\} = \exp(-t)\,L^{-1}\left\{\frac{1}{s^2-1}\right\} = \exp(-t)\,\sinh t$$

where we have used the shifting theorem.

(b)

$$\frac{s}{s^2+2s} = \frac{s}{(s+1)^2-1} = \frac{s+1-1}{(s+1)^2-1}$$

Hence

$$F(s+1) = \frac{s+1}{(s+1)^2-1} - \frac{1}{(s+1)^2-1}$$

and therefore

$$L^{-1}\{F(s+1)\} = \exp(-t)\,L^{-1}\left\{\frac{s}{s^2-1}\right\} - \exp(-t)\,L^{-1}\left\{\frac{1}{s^2-1}\right\}$$

$$= \exp(-t)\,\cosh t - \exp(-t)\,\sinh t$$

where we have used the shifting theorem.

Example 4

Calculate $L^{-1}\left\{(s+4)/(s^2+3s+2)\right\}$.

Solution

We have

$$\frac{s+4}{s^2+3s+2} = \frac{s+4}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

which gives

$$s+4 = A(s+2) + B(s+1)$$

Putting $s+1=0$ and $s+2=0$ successively, we obtain $A=3$ and $B=-2$.

Therefore

$$\begin{aligned} L^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} &= L^{-1}\left\{\frac{3}{s+1}\right\} + L^{-1}\left\{\frac{-2}{s+2}\right\} \\ &= 3\exp(-t)L^{-1}\left\{\frac{1}{s}\right\} - 2\exp(-2t)L^{-1}\left\{\frac{1}{s}\right\} \\ &= 3\exp(-t) - 2\exp(-2t) \end{aligned}$$

where we have used the shifting theorem and the result $L^{-1}\{1/s\} = 1$.

Example 5

Solve the I.V.Ps:

(a) $u' - 2u = 0, \quad u(0) = 1$

(b) $u'' + 4u' + 3u = 0, \quad u(0) = 1, \quad u'(0) = 0$

Solution

(a) We take the Laplace transform of both sides of the differential equation $u' - 2u = 0$, and obtain $L\{u'\} - 2L\{u\} = 0$.

Using the formula for Laplace transforms of derivatives, we have

$sU(s) - u(0) - 2U(s) = 0$

Now using the initial condition $u(0) = 1$, we obtain

$U(s)(s-2) = 1$ which gives $U(s) = 1/(s-2)$

wherefrom

$u(t) = L^{-1}\{1/(s-2)\} = \exp(2t)$

Hence $u(t) = e^{2t}$ is the solution of the given I.V.P.

(b) Taking Laplace transform of both sides of the DE, we have

$L\{u''\} + 4L\{u'\} + 3L\{u\} = 0$

Next using the formulas

$L\{u''(t)\} = s^2U(s) - su(0) - u'(0), \quad L\{u'(t)\} = sU(s) - u(0)$

we have

$s^2U(s) - s\,u(0) - u'(0) + 4\{sU(s) - u(0)\} + 3U(s) = 0$

or on using the given initial conditions

$$s^2U(s) - s - 0 + 4\{sU(s) - 1\} + 3U(s) = 0$$

or

$$U(s)\left(s^2 + 4s + 3\right) - s - 4 = 0$$

or

$$U(s) = \frac{s+4}{s^2 + 4s + 3}$$

Hence on taking inverse Laplace transform, we obtain

$$u(t) = L^{-1}\left\{\frac{s+4}{s^2+4s+3}\right\} = L^{-1}\left\{\frac{s+4}{(s+1)(s+3)}\right\}$$

Now

$$\frac{s+4}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

On putting $s+1=0$ and $s+3=0$ one after the other, we obtain $A = 3/2$, $B = -1/2$. Therefore we have

$$\begin{aligned}
L^{-1}\left\{\frac{s+4}{(s+1)(s+3)}\right\} &= \frac{3}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+3}\right\} \\
&= \frac{3}{2}e^{-t}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}e^{-3t}L^{-1}\left\{\frac{1}{s}\right\} \\
&= \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}
\end{aligned}$$

6.5.4 Exercises

1. Compute Laplace transforms of the following functions

(a) $\sin^2 \omega t$, (b) $\cos^2 \omega t$ (c) $\sin(\omega t - \phi)$

(d) $\cos(\omega t - \phi)$, (e) $e^{2(t+1)}$

2. Find the Laplace transforms of the following functions

(a) $\sin \omega t\, e^{-2t}$, (b) $\cos 3\omega t\, e^{4t}$, (c) $e^{3t}\, t^4$.

(Hints for solution

(2 a) $L\{e^{-2t} \sin \omega t\} = L\{\sin \omega t\}\,]_{s \to s+2} = \omega/(s^2 + 4s + 4 + \omega^2)$.

(2 c) $L\{e^{3t}\, t^4\} = L\{t^4\}]_{s \to s-3} = 4!/(s-3)^5$.

3. Derive the Laplace transforms of $t^{1/2}$ and $t^{-1/2}$ from definition.

(Hint: Use the substitutions $st = x$, where $x > 0$, and $x = y^2$, and the result $\int_{-\infty}^{+\infty} e^{-t^2}\, dt = \sqrt{\pi}$.).

Find the inverse Laplace transform of the following functions:

4. $s/(s^2 + 2as + b^2)$, 5. $1/(s^2 + 2s + 10)$.

6. $1/(s^2 - 4s + 8)$, 7. $s/(s^2 + 6s + 13)$.

8. $(2s + 3)/(s + 4)^3$, 9. $s^2/(s - 1)^4$.

10. $(2s - 3)/(s^2 - 4s + 8)$.

Use the method of partial fractions to calculate inverse Laplace transforms of the following functions:

11. $1/(s^2 - 4)$, 12. $1/(s^2 - 4)$.

13. $(s + 3)/[s(s^2 + 2)]$, 14. $4/[s(s + 1)]$.

15. Solve the I.V.Ps:

(a) $u' + 2u = 0, \quad u(0) = 1$.

(b) $u'' + 9u = 0, \quad u(0) = 0, \quad u'(0) = 1$.

6.6 The Convolution Theorem and its Applications

6.6.1 The unit step function

We begin our discussion with defining *unit step function* which is also called *Heaviside step function*. We denote it by the symbol $H(t - t_0)$ and define it

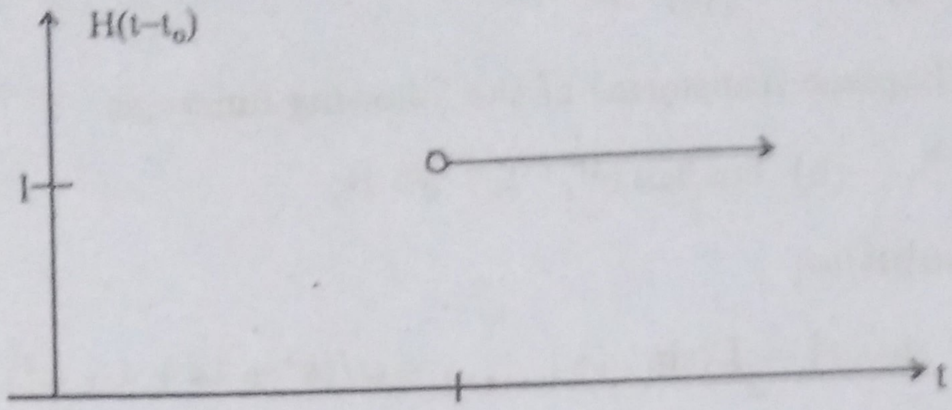

Figure 6.1: Graph of the unit step function $H(t-t_0)$.

by the relations

$$H(t-t_0)=\begin{cases}1, & t\geq t_0\\ 0, & t<t_0\end{cases}$$

It is clear that the step function is discontinuous at $t=t_0$, (see fig. 6.1). Some important properties of the step function are illustrated by the following examples.

To illustrate the a plication of the unit step function sketch the graphs of the following functions:

(a) $H(t)\,t^2$, (b) $H(t-1)\,t^2$, (c) $H(t-2)\,t^3$.

(a) $f(t)=H(t)\,t^2=\begin{cases}t^2, & t>0\\ 0, & t\leq 0\end{cases}$ The graph is shown in fig. 6.2 (a).

(b) $f(t)=H(t-1)\,t^2=\begin{cases}t^2, & t>1\\ 0, & t\leq 1\end{cases}$. The graph is shown in fig. 6.2 (b).

(c) $f(t)=H(t-2)\,t^3=\begin{cases}t^3, & t>2\\ 0, & t\leq 2\end{cases}$. The graph is shown in fig. 6.2 (c).

6.6.2 The convolution theorem (or Faltung theorem)

If $f(t)$ and $g(t)$ are piecewise continuous functions over the interval $[0,\infty)$, then their *convolution* denoted by $f\star g$ is a function of t defined by the integral

$$f\star g=\int_0^t f(\tau)g(t-\tau)\,d\tau$$

From this definition it is easy to prove the following results:

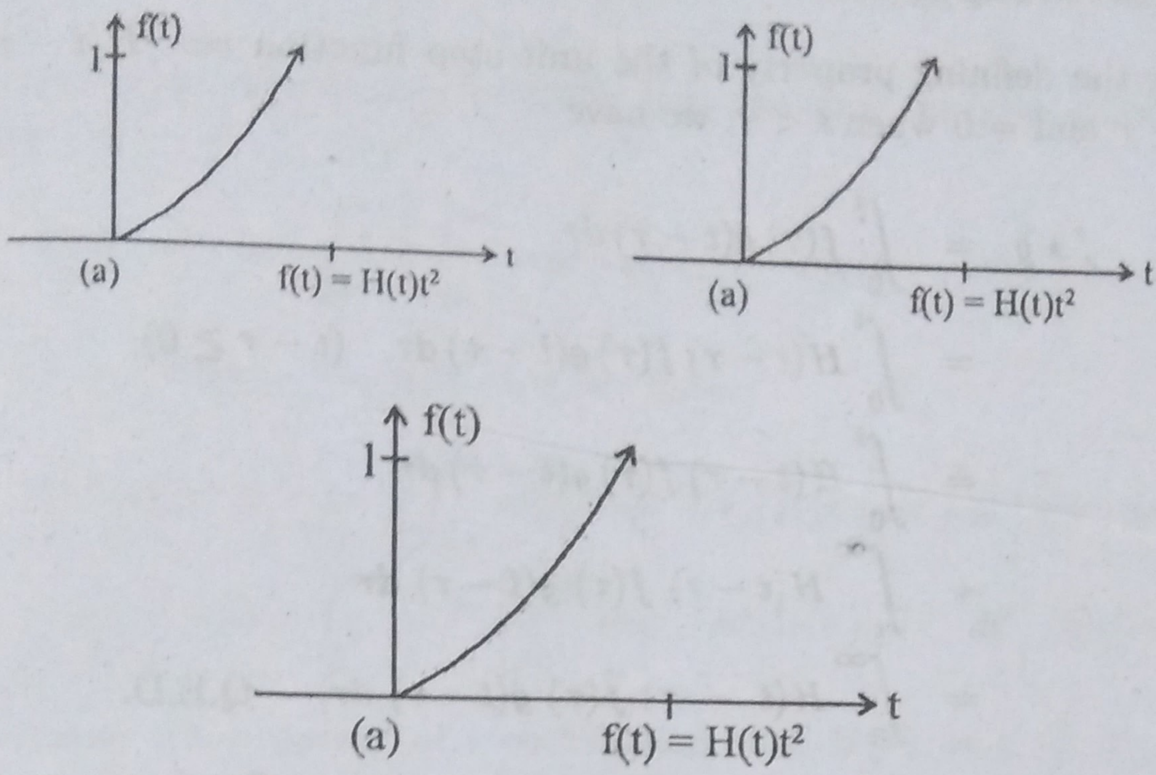

Figure 6.2: Graphs of the functions (a) $H(t)\,t^2$, (b) $H(t-1)\,t^2$, (c) $H(t-2)\,t^3$.

(i) $f \star g = g \star f$

(ii) $f \star (g+h) = f \star g + f \star h$

(iii) $f \star (g \star h) = (f \star g) \star h$

Proof of (i)

$$f \star g = \int_0^t f(\tau) g(t-\tau)\, d\tau$$

We put $t - \tau = t'$, so that $-d\tau = dt'$, and obtain

$$\begin{aligned} f \star g &= \int_t^0 f(t-t')\, g(t')\, (-dt') \\ &= \int_0^t f(t-t')\, g(t')\, dt' \\ &= g \star f, \quad \text{by definition} \end{aligned}$$

Results (ii) and (iii) are left as exercises for the student.

Lemma

We need the following result to prove the convolution theorem.

$$f \star g = \int_0^\infty H(t-\tau)\, f(\tau)\, g(t-\tau)\, d\tau$$

where H is the step function.

Using the defining property of the unit step function *viz.* $H(t-\tau)=1$ when $t \geq \tau$ and $=0$ when $t<\tau$, we have

$$\begin{aligned} f\star g &= \int_0^t f(\tau)\,g(t-\tau)\,d\tau \\ &= \int_0^t H(t-\tau)\,f(\tau)\,g(t-\tau)\,d\tau, \quad (t-\tau \geq 0) \\ &= \int_0^t H(t-\tau)\,f(\tau)\,g(t-\tau)\,d\tau \\ &+ \int_t^\infty H(t-\tau)\;f(\tau)\;g(t-\tau)\;d\tau \\ &= \int_0^\infty H(t\;-\;\tau)\;f(\tau)\;g(t-\tau)\;d\tau, \qquad \text{Q.E.D.} \end{aligned}$$

6.6.3 The convolution theorem

Statement

If $f(t)$ and $g(t)$ are piecewise continuous functions over the interval $[0,\ T)$, where T is any arbitrary positive number, and are of exponential order, then

$$L\{f\star g\} = L\left\{\int_0^t f(\tau)\;g(t-\tau)\right\}\;d\tau = F(s)\,G(s)$$

The same result can also be stated as

$$L^{-1}\{F(s)\,G(s)\} = f\star g$$

where $f(t)=L^{-1}\{F(s)\}$ and $g(t)=L^{-1}\{G(s)\}$.

Proof

We will prove that

$$L\{f\star g\} = F(s)\,G(s)$$

By definition

$$\begin{aligned} L\{f\star g\} &= \int_0^\infty \exp(-st)\;f\star g\;dt \\ &= \int_0^\infty e^{-st}\int_0^t f(\tau)\;g(t-\tau)\;d\tau\;dt \end{aligned}$$

$$= \int_0^\infty \exp(-st) \int_0^\infty H(t-\tau) f(\tau)\, g(t-\tau)\, d\tau\, dt$$

where $H(t-\tau)$ is the unit step function. Now changing the order of integration, we have

$$L\{f \star g\} = \int_0^\infty \left[\int_0^\infty \exp(-st)\, H(t-\tau) g(t-\tau)\, dt \right] f(\tau)\, d\tau$$

Let $t - \tau = t'$, then on substitution, we have

$$\begin{aligned} L\{f \star g\} &= \int_0^\infty \left[\int_{-\tau}^\infty \exp[-s(t'+\tau)]\, H(t')\, g(t')\, dt' \right] f(\tau)\, d\tau \\ &= \int_0^\infty e^{-s\tau} \left[\int_0^\infty \exp(-st') H(t')\, g(t')\, dt' \right] f(\tau)\, d\tau \end{aligned}$$

where using the property of step function that $H(t') = 0$ for $t' < 0$ and $H(t') = 1$ for $t' \geq 0$, we have

$\int_{-\tau}^0 \exp(-s\,t')\; H(t')\; g(t')\; dt' = 0$

Therefore

$$\begin{aligned} L\{f \star g\} &= \int_0^\infty e^{-s\tau} \left[\int_0^\infty e^{-st'} g(t')\, dt' \right] f(\tau)\, d\tau \\ &= \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \int_0^\infty e^{-st'} g(t')\, dt' \\ &= F(s)\, G(s) \end{aligned}$$

6.6.4 Illustrative examples

Example 1

Calculate the following convolutions.

(a) $t \star \exp(t)$, (b) $t \star \sin 2t$, (c) $\sin 2t \star \exp(-t)$

Solution

(a)

$$\begin{aligned} t \star \exp(t) &= \int_0^t \tau \exp(t-\tau)\, d\tau = \exp(t) \int_0^t \tau \exp(-\tau)\, d\tau \\ &= \exp(t)\, [\tau\{-\exp(-tau)\} - \{\exp(-tau)\} \cdot 1]|_0^t \\ &= \exp(t)\, [1 - \exp(-t)\,(1+t)] \end{aligned}$$

(b)

$$\begin{aligned} t \star \sin 2t &= \int_0^t \tau \sin 2(t-\tau)\, d\tau \\ &= \tau \left[\frac{\cos 2(t-\tau)}{2} - \frac{\sin 2(t-\tau)}{-2} \right] \Big|_0^t \\ &= \frac{1}{2} t \cos t + \frac{1}{2} \sin t - \frac{1}{2} \sin 2t \\ &= \frac{1}{2}(t \cos t - \sin 2t + \sin t) \end{aligned}$$

(c)

Use the formula for the integral of $\exp(ax) \cos bx$ given in appendix A.

Example 27

Using the convolution theorem, calculate the inverse Laplace transform of the function

$$\text{(a)} \quad \frac{3}{s^2(s^2+9)}, \qquad \text{(b)} \quad \frac{s}{(s^2+9)^2}$$

Solution

(a) Writing $H(s) = F(s)\,G(s)$. With $F(s) = 1/s^2$ and $G(s) = 3/(s^2+9^2)$, we have $f(t) = t$ and $g(t) = \sin 3t$.

Therefore by convolution theorem

$$\begin{aligned} h(t) &= f(t) \star g(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau \\ &= \int_0^t (t-\tau) \sin 3\tau \, d\tau \\ &= \left(-\frac{t \cos 3\tau}{3} + \frac{\tau \cos 3\tau}{3} - \frac{\sin 3\tau}{9} \right) \Big|_0^t \\ &= -\frac{\sin 3t}{9} + \frac{t}{3} = \frac{1}{9}(3t - \sin 3t) \end{aligned}$$

(b) Here $H(s) = s/(s^2+9)^2$. With $F(s) = s/(s^2+9)$, and $G(s) = 1/(s^2+9)$, we have $f(t) = \cos 3t$, $g(t) = (1/3) \sin 3t$. Therefore by convolution theorem

$$\begin{aligned} h(t) &= L^{-1}\{H(s)\} = f(t) \star g(t) = \frac{1}{3} \int_0^t \cos 3(t-\tau) \sin 3\tau \, d\tau \\ &= \frac{1}{3} \int_0^t (\cos 3t \cos 3\tau + \sin 3t \sin 3\tau) \sin 3\tau \, d\tau \end{aligned}$$

$$= \frac{1}{3}\int_0^t \left(\cos 3t \cos 3\tau \sin 3\tau + \sin 3t \sin^2 3\tau\right) d\tau$$

$$= \frac{1}{6}\cos 3t \int_0^t \sin 6\tau \, d\tau + \frac{1}{3}\sin 3t \int_0^t \frac{1-\cos 6\tau}{2} d\tau$$

$$= \frac{1}{36}\cos 3t\,(1-\cos 6t) + \frac{1}{6}\sin 3t \left(\tau - \frac{\sin 6\tau}{6}\right)\Bigg|_0^t$$

$$= \frac{1}{36}[-\cos 3t + \cos 3t] + \frac{1}{6}t\sin 3t$$

$$= \frac{1}{6}t\sin 3t$$

Example 3

Use convolution theorem to calculate the Laplace transform of

$f(t) = \int_0^t (t-\beta)^3 e^{\beta} \sin\beta \, d\beta$

Solution

By definition of convolution, $f(t) = t^3 \star (e^t \sin t)$. Therefore

$$L\{f(t)\} = L\{t^3\}\, L\{e^t \sin t\} = \frac{3!}{s^4} \left(L\{\sin t\}\right)\Big|_{s\to s-1}$$

$$= \frac{6}{s^4}\frac{1}{(s-1)^2+1} = \frac{6}{s^4(s^2-2s+2)}$$

6.6.5 Exercises

Complete

1. Find the Laplace transform of the given integrals

(a) $\int_0^t (t-\beta)\sin 3\beta \, d\beta$, (b) $\int_0^t \exp[-(t-\beta)]\sin\beta \, d\beta$.

2. Find the inverse Laplace transform using the convolution theorem or otherwise.

(a) $4/[s^2(s-2)]$, (b) $1/(s^2+1)^2$, (c) $1/(s^2-1)^2$.

3. Show that

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\, d\tau$$

⇒ Solving LHS

(Hint: use convolution theorem with $G(s) = 1/s$).

4. Show that

$$L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \left\{\int_0^{\tau} f(\lambda)\, d\lambda\right\} d\tau$$

$= L^{-1}\left\{\frac{1}{s}\frac{F(s)}{s}\right\} = 1 \star \int_0^t f(\tau)\,d\tau$ by Q3.

(**Hint** Apply the convolution theorem to $1/s$ and $F(s)/s$.)

5. In problem 4, show that

L.H.S. $= t\int_0^t f(\lambda)\,d\lambda - \int_0^t t' f(t')\,dt'$

6. Solve the integral equation

$y(t) = a\,t + \int_0^t y(\tau)\,\sin(t-\tau)\,d\tau$

(**Ans.**: $y(t) = at + (a/3!)\,t^3$).

7. Solve the integral equation by using convolution theorem:

$y(t) = f(t) + \int_0^t g(t-\tau)\,y(\tau)\,d\tau$

(**Ans.** $y(t) = L^{-1}\{F(s)/[1-G(s)]\}$).

8. Solve the D.Es. by Laplace transform method.

(a) $y''(t) + k^2 y(t) = f(t)$

(b) $y''(t) - 2ky'(t) + k^2 y(t) = f(t)$

(c) $y''(t) + \lambda y'(t) + k^2 y(t) = f(t)$

In each case discuss the physical significance.

[**Ans.**: (b) $e^{-kt}y(t) = c_1 + c_2 t + \int_0^t (t-\tau)\,e^{-k\tau}\,d\tau$].

9. Solve the problem

$y'' + \omega^2 y = f(t),\ \ y(0) = y_0,\ \ y'(0) = y_1$

and discuss the case when

$$f(t) = \begin{cases} f_0, & t_0 < t < t_1 \\ 0, & \text{for other } t \text{ values} \end{cases}$$

(**Ans.** $y(t) = y_0\cos\omega t + (y_1/\omega)\sin\omega t + (1/\omega)\,f(t) * \sin\omega t$).

10. Solve the inhomogeneous problems with zero initial conditions i.e. $u(0) = 0$ and $u'(0) = 0$.

(a) $u'' + au = 1$

(b) $u'' + u = t$

(c) $u'' + 2u' = 1 - \exp(-t)$

(d) $u'' - u = 1$