The General 2^{k-p} Design

To construct a 2^{-p} fraction of the 2^k full factorial design, we require p independent generators (such as $D = AB$).

Each generator contributes a word to the defining relation (here, $I = ABD$).

The defining relation consists of these words and their products (generalized interactions); $2^p - 1$ in total (2^p including I).

Each effect has $2^p - 1$ aliases; find them by multiplying the given effect by all words in the defining relation.

Design Criteria

High *resolution* = length of shortest word in full defining relation.

Low *aberration* = number of words with that length.

Appendix X gives maximum resolution, minimum aberration designs for many 2^{k-p} designs with $k \leq 15$ and $n = 2^{k-p} \leq 64$.

Example

2_{IV}^{7-2} design; defining relation contains at least one 4-letter word.

Each 4-letter word introduces 4 aliases of a main effect with a 3-factor interaction, and 6 aliases of 2-factor interactions with each other.

Three choices (among many):

- $I = ABCF = BCDG = ADFG$
- $I = ABCF = ADEG = BCDEFG$
- $I = ABCDF = ABDEG = CEFG$ has minimum aberration.

Blocking a Fractional Factorial

Needed, as always, when the design has more runs than can be carried out under homogeneous conditions.

E.g. for 2 blocks, choose an effect to be confounded with blocks.

All of its aliases are then also confounded–choose carefully!

Appendix X has recommended choices (but some are questionable).

Example

2^{6-2} ($2^4 = 16$ runs) in two blocks (each of 8 runs).

Treat "Blocks" as a seventh 2-level factor, G; find a design for 2^{7-3}.

Appendix X(i) suggests generators $E = ABC$, $F = BCD$, $G = ACD$ with defining relation

$$\begin{aligned} I &= ABCE = BCDF = ADEF \\ &= ACDG = BDEG = ABFG = CEFG \end{aligned}$$

and hence resolution IV.

Rewrite the defining relation as

$$I = ABCE = BCDF = ADEF,$$
$$G = ACD = BDE = ABF = CEF.$$

The first line is the defining relation for a 2^{6-2}_{IV} design.

The second line defines the two blocks, and shows which interactions are confounded with blocks.

This is not the design recommended in Appendix X(f) for 2^{6-2} in two blocks, but it has similar confounding: four 3-factor interactions confounded with blocks.

Another example

2^{5-1}, also in two blocks of 8 runs.

Find a design for 2^{6-2}.

Appendix X(f) suggests generators $E = ABC$, $F = BCD$, with defining relation

$$I = ABCE = BCDF = ADEF.$$

Rewrite as

$$I = ABCE, F = BCD = ADE$$

and use F to define the blocks.

This blocked design is of resolution IV:

- two 3-factor interactions, BCD and ADE, are confounded with blocks;
- the 2-factor alias chains are $AB = CE$ and $AC = BE$.

The recommended design in Appendix X(d) is generated by $E = ABCD$, with defining relation $I = ABCDE$, and is of resolution V.

But with the two recommended blocks:

- $AB = CDE$ is confounded with blocks;
- if interactions of blocks with treatments were present, A would be confounded with the $B \times$ block interaction.

Which design is better?

Resolution III Designs

Main effects are aliased with 2-factor interactions, so these designs are useful for suggesting active factors, but may be ambiguous.

For example, if A, B, and D are identified by the half-normal plot, but $D = AB$, which factors are active?

Designs exist for $K = N - 1$ factors in only N runs, when N is a multiple of 4; *saturated* designs.

E.g. 2_{III}^{3-1}, 2_{III}^{7-4}, 2_{III}^{15-11}, 2_{III}^{31-26}.

Example: 2^{7-4}_{III} has $2^{7-4} = 8$ runs, and can estimate main effects of 7 factors.

Begin with basic design in A, B, C:

	Basic Design		
Run	A	B	C
1	-	-	-
2	+	-	-
3	-	+	-
4	+	+	-
5	-	-	+
6	+	-	+
7	-	+	+
8	+	+	+

Add columns for interactions:

	Basic Design						
Run	A	B	C	AB	AC	BC	ABC
1	-	-	-	+	+	+	-
2	+	-	-	-	-	+	+
3	-	+	-	-	+	-	+
4	+	+	-	+	-	-	-
5	-	-	+	+	-	-	+
6	+	-	+	-	+	-	-
7	-	+	+	-	-	+	-
8	+	+	+	+	+	+	+

Alias D, E, F, and G with the four interactions:

	Basic Design						
Run	A	B	C	$D = AB$	$E = AC$	$F = BC$	$G = ABC$
1	-	-	-	+	+	+	-
2	+	-	-	-	-	+	+
3	-	+	-	-	+	-	+
4	+	+	-	+	-	-	-
5	-	-	+	+	-	-	+
6	+	-	+	-	+	-	-
7	-	+	+	-	-	+	-
8	+	+	+	+	+	+	+

Design is generated by

$$D = AB, \quad E = AC, \quad F = BC, \quad G = ABC$$

which imply that

$$I = ABD = ACE = BCF = ABCG$$

Full defining relation is

$$\begin{aligned} I &= ABD = ACE = AFG = BCF = BEG = CDG = DEF \\ &= ABCG = ABEF = ACDF = ADEG = BCDE = BDFG = CEFG \\ &= ABCDEFG \end{aligned}$$

Every main effect is aliased with three 2-factor interactions, four 3-factor interactions, and one 6-factor interaction.

Example

Response is "eye focus time".

Seven factors, with the above 2^{7-4}_{III} design.

R commands

```
table8p21 <- within(MakeTwoLevel(3), {
                      G <- A * B * C;
                      F <- B * C;
                      E <- A * C;
                      D <- A * B
                    })
table8p21$Time <- c(85.5, 75.1, 93.2, 145.4, 83.7, 77.6, 95.0, 141.8)
summary(lm(Time ~ A + B + C + D + E + F + G, table8p21))
# Time ~ . is short-hand for this formula.
library(gplots)
qqnorm(aov(Time ~ ., table8p21), label = TRUE)
```

The half-normal plot identifies A, B, and D as interesting.

$I = ABD$ means that $A = BD$, $B = AD$, and $D = AB$.

So the half-normal plot is consistent with any of:

- $A + B + D$;
- $A + B + A : B$;
- $A + D + A : D$;
- $B + D + B : D$.

More runs are needed to distinguish among these possibilities.

Sequential Experiments: Fold Over

Begin with the *principal* fraction for a resolution III design.

If *one factor* is of special interest, follow up with the alternate fraction in which signs for that factor are reversed.

Combined experiment, a *single-factor fold over*, gives:

- main effect for that factor free of 2-factor and 3-factor aliases;
- all its 2-factor interactions free of 2-factor aliases.

To achieve that for *all* factors, we would need a resolution V design, which would require more runs; the fold over is more efficient.

Or, if *all main effects* are of interest, follow up with the alternate fraction in which signs for *all* factors are reversed.

Combined experiment, a *full fold over*, or reflection, gives all main effects free of 2-factor aliases $\Rightarrow$ a resolution IV design.

Often the two fractions should be treated as blocks, with those effects in the complete defining relation that change sign confounded with blocks.

Example, continued

In the “eye focus time” example, no single factor is of principal interest, so the full fold over idea was used to construct a second fraction.

R commands

```
table8p22 <- -table8p21
table8p22$Time <- c(91.3, 126.7, 82.4, 73.4, 94.1, 143.8, 87.3, 71.9)
fullFoldOver <- rbind(table8p21, table8p22)
summary(lm(Time ~ .^2, fullFoldOver))
qqnorm(aov(Time ~ .^2, fullFoldOver), label = TRUE)
```

The half-normal plot clarifies that the large effects are B, D, and $B : D$, so B and D appear to be the only active factors.

Example, with Blocks

```
table8p21$Block <- 1
table8p22$Block <- 2
fullFoldOverBlocked <- rbind(table8p21, table8p22)
summary(lm(Time ~ Block + (. - Block)^2, fullFoldOverBlocked))
qqnorm(aov(Time ~ Block + (. - Block)^2, fullFoldOverBlocked), label = TRUE)
```

The single degree of freedom for blocks takes out the single degree of freedom for residuals, so the other estimated effects are all unchanged.

Plackett-Burman Designs

Two-level fractional factorial designs for $k = N - 1$ factors in N runs (saturated designs), with N a multiple of 4.

When N is a power of 2, say $N = 2^q$, these are 2_{III}^{k-p} designs with $k = 2^q - 1$ and $p = k - q$ for $q = 2, 3, 4, \dots$.

Plackett-Burman designs for other N have more complicated aliasing structure.

Example

A 12-run Plackett-Burman design was used in a study of the factors that affect injection molding of plastic components.

The design can produce estimates of the main effects of up to 11 factors, but only 8 $(A - H)$ were used in this study.

The response is R1, "cycle time".

The design was extended by adding 4 center point runs, which we ignore.

Run	A	B	C	D	E	F	G	H	J	K	L	R1
1	+	-	+	-	-	-	+	+	+	-	+	15.4
2	+	+	-	+	-	-	-	+	+	+	-	17.3
3	-	+	+	-	+	-	-	-	+	+	+	19.3
4	+	-	+	+	-	+	-	-	-	+	+	17.4
5	+	+	-	+	+	-	+	-	-	-	+	21.3
6	+	+	+	-	+	+	-	+	-	-	-	19.3
7	-	+	+	+	-	+	+	-	+	-	-	17.3
8	-	-	+	+	+	-	+	+	-	+	-	21.4
9	-	-	-	+	+	+	-	+	+	-	+	21.3
10	+	-	-	-	+	+	+	-	+	+	-	19.4
11	-	+	-	-	-	+	+	+	-	+	+	15.3
12	-	-	-	-	-	-	-	-	-	-	-	15.3

The design has resolution III, because main effects are *not* aliased with each other, but *are* aliased with 2-factor interactions:

```
pb12plus <- read.csv("data/Plackett-Burman-12.csv")
pb12 <- pb12plus[1:12,]
alias(lm(R1 ~ (A + B + C + D + E + F + G + H)^2, pb12))
```

Estimate all main effects

```
summary(lm(R1 ~ A + B + C + D + E + F + G + H, pb12))
```

Stepwise regression

Use step-wise regression to explore main effects and 2-factor interactions (`k` controls over-fitting; default is `k = 2`):

```
summary(step(lm(R1 ~ 1, pb12),
             scope = ~ (A + B + C + D + E + F + G + H)^2, k = 4))
```

The step-wise regression indicates that D and E have strong effects, and B is marginally significant.

Projection

In D and E, the design is three replicates of the full 2^2 factorial.

In B, D, and E, the design is a single replicate of the full 2^3 factorial design, plus the one-half fraction with $BDE = -I$.

Fitting `R1 ~ B * D * E` shows that no interactions are significant.

Supersaturated Designs

The additive model in k factors has $p = k + 1$ parameters:

- the intercept;
- k main effects.

A *saturated* design has $N = p$ runs, so that all parameters can be estimated, but with zero degrees of freedom for error.

A *supersaturated* design,with $N < p$ runs, cannot provide estimates of all p parameters.

Modern methods focus on identifying a subset of parameters that appear to be non-zero, and providing estimates of them.