2^k Factorial Models

Design of Experiments - Montgomery
Chapter 6

2^k Factorial Design

- Each factor has two levels (often labeled + and −)
- Very useful design for preliminary analysis
- Can "weed out" unimportant factors
- Also allows initial study of interactions
- For general two-factor factorial model

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

- Have $1 + (a-1) + (b-1) + (a-1)(b-1)$ parameters
- 2^2 factorial has only four parameters

$$\alpha_1 = -\alpha_2 \qquad \beta_1 = -\beta_2$$
$$(\alpha\beta)_{11} = (\alpha\beta)_{22} \quad (\alpha\beta)_{12} = -(\alpha\beta)_{22} \quad (\alpha\beta)_{21} = -(\alpha\beta)_{22}$$

- Can study 2 factors through repeated 2^2 designs
  - Select different levels of each factor

Design for 2^2 Factorial

- Label the levels of factors A and B using + and −
- Factors need not be on numeric scale
- There are 4 experimental combinations labeled

A	B	Symbol
-	-	(1)
+	-	a
-	+	b
+	+	ab

- Can express combination in terms of model params

$$\begin{aligned}
ab = (+,+): &\quad E(y) = \mu + \alpha_2 + \beta_2 + (\alpha\beta)_{22} \\
a = (+,-): &\quad E(y) = \mu + \alpha_2 + \beta_1 + (\alpha\beta)_{21} \\
b = (-,+): &\quad E(y) = \mu + \alpha_1 + \beta_2 + (\alpha\beta)_{12} \\
(1) = (-,-): &\quad E(y) = \mu + \alpha_1 + \beta_1 + (\alpha\beta)_{11}
\end{aligned}$$

$$\begin{aligned}
ab = (+,+): &\quad E(y) = \mu + \alpha_2 + \beta_2 + (\alpha\beta)_{22} \\
a = (+,-): &\quad E(y) = \mu + \alpha_2 - \beta_2 - (\alpha\beta)_{22} \\
b = (-,+): &\quad E(y) = \mu - \alpha_2 + \beta_2 - (\alpha\beta)_{22} \\
(1) = (-,-): &\quad E(y) = \mu - \alpha_2 - \beta_2 + (\alpha\beta)_{22}
\end{aligned}$$

Estimating Model Parameters

- Have four equations and four unknowns

$$\begin{aligned}
\widehat{\mu} = &\quad && = \frac{ab + a + b + (1)}{4n} \\
\widehat{\alpha}_2 = &\quad .5\left(\frac{ab+a}{2n} - \frac{b+(1)}{2n}\right) && = \frac{ab + a - b - (1)}{4n} \\
\widehat{\beta}_2 = &\quad .5\left(\frac{ab+b}{2n} - \frac{a+(1)}{2n}\right) && = \frac{ab - a + b - (1)}{4n} \\
\widehat{(\alpha\beta)}_{22} = &\quad .5\left(\frac{ab-b}{2n} - \frac{a-(1)}{2n}\right) && = \frac{ab - a - b + (1)}{4n}
\end{aligned}$$

where n represents the number of replications of each combination and ab, b, etc represent the sample sums of each combination.

- Main effect of A defined as $2\widehat{\alpha}_2$
- Main effect of B defined as $2\widehat{\beta}_2$
- Interaction effect defined as $2\widehat{(\alpha\beta)}_{22}$
- Each effect function of ±1(combination sums)

Using Effects To Draw Inference

- Look at magnitude and direction of effects
- ANOVA then used to verify conclusions (Chapt 5)
- Effects and contrasts related as shown

	Contrast coefficients			
Effect	(1)	a	b	ab
A	-1	+1	-1	+1
B	-1	-1	+1	+1
AB	1	-1	-1	+1

- AB coefficients simply product of A and B coeffs
- Sum of squares related to model effects

$$\text{SS}_{\text{Contrast}} = \frac{\left(\sum c_i y_{i.}\right)^2}{n \sum c_i^2} \quad i = 1,2,3,4$$

$$\text{SS}_{\text{A}} = \frac{(ab+a-b-(1))^2}{4n} = 4n\widehat{\alpha_2}^2$$

Regression / Response Surface

- Code each factor as ± 1 where $+1$ is high level
- x_1 associated with Factor 1 = A
- x_2 associated with Factor 2 = B
- Regression Model

$$y_{ijk} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \epsilon_{ijk}$$

- Can show

$$\beta_0 = \widehat{\mu} \qquad \beta_1 = \widehat{\alpha}_2 \qquad \beta_2 = \widehat{\beta}_2$$
$$\beta_{12} = \widehat{(\alpha\beta)}_{22}$$

- Parameters based on unit increase (not -1 to +1)
- Can use regression model to form contours
- No interaction → linear relationship
- Can often see direction for improvement in response

Hormone Example

Combination	Total
(1)	467
a	394
b	642
ab	571

$$\widehat{\mu} = \frac{571+394+642+467}{4(6)} = 90.75$$
$$\widehat{\alpha}_2 = \frac{571+394-642-467}{4(6)} = -6$$
$$\widehat{\beta}_2 = \frac{571-394+642-467}{4(6)} = 14.6666$$
$$\widehat{(\alpha\beta)}_{22} = \frac{571-394-642+467}{4(6)} = .08333$$

- $\text{SS}_{\text{A}} = 4(6)\widehat{\alpha}_2^2 = 864$
- $\text{SS}_{\text{B}} = 4(6)\widehat{\beta}_2^2 = 5162.67$
- $\text{SS}_{\text{AB}} = 4(6)\widehat{\alpha\beta}_{22}^2 = .17$
- Response Surface model

$$y = 90.75 + 14.67 x_2 + \epsilon$$

To increase response, model says to increase level. Be careful of interpolation.

2^3 Factorial Designs

- Eight combinations displayed as cube

A	B	C	Symbol
-	-	-	(1)
+	-	-	a
-	+	-	b
+	+	-	ab
-	-	+	c
+	-	+	ac
-	+	+	bc
+	+	+	abc

- A main effect is equal to

$$2\widehat{\alpha}_2$$
$$\frac{(a-(1))+(ab-b)+(ac-c)+(abc-bc)}{4n}$$
$$\frac{(a-(1))(b+(1))(c+(1))}{4n}$$

- The AB interaction is equal to

$$2\widehat{(\alpha\beta)}_{22}$$
$$\frac{((abc-bc)+(ab-b))-((ac-c)+(a-(1)))}{4n}$$
$$\frac{(a-(1))(b-(1))(c+(1))}{4n}$$

2^3 Factorial

Contrast Coefficients

Symbol	A	B	AB	C	AC	BC	ABC
(1)	-	-	+	-	+	+	-
a	+	-	-	-	-	+	+
b	-	+	-	-	+	-	+
ab	+	+	+	-	-	-	-
c	-	-	+	+	-	-	+
ac	+	-	-	+	+	-	-
bc	-	+	-	+	-	+	-
abc	+	+	+	+	+	+	+

Contrast = $\sum c_i y_{i.} = (\pm(1) \pm a \pm b \pm ab \pm c \pm ac \pm bc \pm abc)$
Parameter = Contrast/$8n$
Effect = Contrast/$4n$=2Parameter

- SS_{Effect}=(Contrast)$^2/8n$
- SS_{Effect}=$2n$(Effect)^2
- Var(Contrast) = $8n\sigma^2$
- Var(Effect) = $\sigma^2/2n$

General 2^k Factorial

- Contrast is sum of trt combinations ($\pm$1 coeff)

  Var(Contrast) = $n2^k\sigma^2$

- Effect is Contrast divided by $n2^{k-1}$

  Var(Effect) = $\sigma^2/n2^{k-2}$

- Can use variances to test significance of effect

  Approximate 95% CI for Effect

  Effect $\pm 1.96\sqrt{\sigma^2/n2^{k-2}}$

- Look at effects whose intervals do not contain 0

Yates Algorithm

- For general 2^k design
  - List data in standard order
  - Create k new columns in the following manner
  - The first 2^{k-1} entries are obtained by adding adjacent values
  - The last 2^{k-1} entries are obtained by subtracting adjacent values
  - Estimates of effects obtained by dividing kth column by $n2^{k-1}$
  - Sum of Squares obtained by squaring kth column and dividing by $n2^k$

Combination	y	1	2	3	Estimate
(1)	15				
a	10				
b	5				
ab	25				
c	15				
ac	25				
bc	10				
abc	5				

Single Replicate In 2^k Design

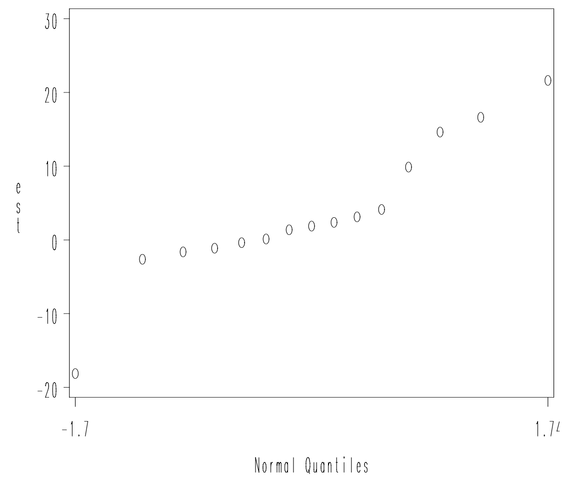
- With only one replicate
  - Cannot estimate all interactions and σ^2
- Often assume 3 or higher interactions do not occur
- Pool estimates together for error
- Warning: may pool significant interaction

- Create Normal Probability Plot of Effects
- If negligible, effect will be N$(0, \sigma^2)$
- Will fall along straight line
- Significant effects will not lie along same line
- Use Yates Algorithm or SAS to create plot

Example 6-2

```
data new;
 input fact1 fact2 fact3 fact4 est;
 cards;
  2 1 1 1  21.625
  1 2 1 1   3.125
  2 2 1 1   0.125
  1 1 2 1   9.875
  2 1 2 1 -18.125
  1 2 2 1   2.375
  2 2 2 1   1.875
  1 1 1 2  14.625
  2 1 1 2  16.625
  1 2 1 2  -0.375
  2 2 1 2   4.125
  1 1 2 2  -1.125
  2 1 2 2  -1.625
  1 2 2 2  -2.625
  2 2 2 2   1.375
 ;

proc univariate noprint;
 qqplot est / normal;
```


```
data filter;
 do D = -1 to 1 by 2;do C = -1 to 1 by 2;
 do B = -1 to 1 by 2;do A = -1 to 1 by 2;
 input y @@;  output;
 end; end; end; end;
cards;
45 71 48 65 68 60 80 65 43 100 45 104 75 86 70 96
;

data inter;                              /* Define Interaction Terms */
 set filter;
 AB=A*B; AC=A*C; AD=A*D; BC=B*C; BD=B*D; CD=C*D; ABC=AB*C; ABD=AB*D;
 ACD=AC*D; BCD=BC*D; ABCD=ABC*D;

proc glm data=inter;                   /* GLM Proc to Obtain Effects */
 class A B C D AB AC AD BC BD CD ABC ABD ACD BCD ABCD;
 model y=A B C D AB AC AD BC BD CD ABC ABD ACD BCD ABCD;
 estimate ’A’ A 1 -1; estimate ’AC’ AC 1 -1;

proc reg outest=effects data=inter;    /* REG Proc to Obtain Effects */
 model y=A B C D AB AC AD BC BD CD ABC ABD ACD BCD ABCD;
data effect2; set effects;
 drop y intercept _RMSE_;
proc transpose data=effect2 out=effect3;
data effect4; set effect3; effect=col1*2;
proc sort data=effect4; by effect;
proc print data=effect4;
proc rank data=effect4 normal=blom;
 var effect; ranks neff;

symbol1 v=circle;
proc gplot;
 plot neff*effect=_NAME_;
```