**Three-level, mixed-level and fractional factorial designs**

The 2*k* and 3*k* experiments are special cases of factorial designs. In a factorial design, one obtains data at every combination of the levels. The importance of factorial designs, especially 2-level factorial designs, was stated by Montgomery (1991): *It is our belief that the two-level factorial and fractional factorial designs should be the cornerstone of industrial experimentation for product and process development and improvement.* He went on to say: *There are, however, some situations in which it is necessary to include a factor (or a few factors) that have more than two levels.*

This section will look at how to add three-level factors starting with two-level designs, obtaining what is called a *mixed-level* design. We will also look at how to add a four-level factor to a two-level design. The section will conclude with a listing of some useful orthogonal three-level and mixed-level designs (a few of the so-called Taguchi "L" orthogonal array designs), and a brief discussion of their benefits and disadvantages.

**Generating a Mixed Three-Level and Two-Level Design**

Montgomery (1991) suggests how to derive a variable at three levels from a 23 design, using a rather ingenious scheme. The objective is to generate a design for one variable, *A*, at 2 levels and another, *X*, at three levels. This will be formed by combining the -1 and 1 patterns for the*B* and *C* factors to form the levels of the three-level factor *X*:

|  |  |  |
| --- | --- | --- |
| **TABLE 3.38: Generating a Mixed Design** | | |
|  | | |
| **Two-Level** | | **Three-Level** |
|  | | |
| **B** | **C** | **X** |
|  | | |
| -1 | -1 | x1 |
| +1 | -1 | x2 |
| -1 | +1 | x2 |
| +1 | +1 | x3 |
|  | | |

Similar to the 3*k* case, we observe that *X* has 2 degrees of freedom, which can be broken out into a linear and a quadratic component. To illustrate how the 23 design leads to the design with one factor at two levels and one factor at three levels, consider the following table, with particular attention focused on the column labels.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **A** | **XL** | **XL** | **AXL** | **AXL** | **XQ** | **AXQ** | **TRT** | **MNT** |
|  | | | | | | | | | |
| **Run** | **A** | **B** | **C** | **AB** | **AC** | **BC** | **ABC** | **A** | **X** |
|  | | | | | | | | | |
| 1 | -1 | -1 | -1 | +1 | +1 | +1 | -1 | Low | Low |
| 2 | +1 | -1 | -1 | -1 | -1 | +1 | +1 | High | Low |
| 3 | -1 | +1 | -1 | -1 | +1 | -1 | +1 | Low | Medium |
| 4 | +1 | +1 | -1 | +1 | -1 | -1 | -1 | High | Medium |
| 5 | -1 | -1 | +1 | +1 | -1 | -1 | +1 | Low | Medium |
| 6 | +1 | -1 | +1 | -1 | +1 | -1 | -1 | High | Medium |
| 7 | -1 | +1 | +1 | -1 | -1 | +1 | -1 | Low | High |
| 8 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | High | High |
|  | | | | | | | | | |

If we believe that the quadratic effect is negligible, we may include a second two-level factor, D, with D = ABC. In fact, we can convert the design to exclusively a main effect (resolution III) situation consisting of four two-level factors and one three-level factor. This is accomplished by equating the second two-level factor to AB, the third to AC and the fourth to ABC. Column BC cannot be used in this manner because it contains the quadratic effect of the three-level factor X.

**More than one three-level factor**

We have seen that in order to create one three-level factor, the starting design can be a 23 factorial. Without proof we state that a 24 can split off 1, 2 or 3 three-level factors; a 25 is able to generate 3 three-level factors and still maintain a full factorial structure. For more on this, see Montgomery (1991).

**Generating a Two- and Four-Level Mixed Design**

We may use the same principles as for the three-level factor example in creating a four-level factor. We will assume that the goal is to construct a design with one four-level and two two-level factors.

Initially we wish to estimate all main effects and interactions. It has been shown (see Montgomery, 1991) that this can be accomplished via a 24 (16 runs) design, with columns A and B used to create the four level factor *X*.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **TABLE 3.39: A Single Four-level Factor and Two Two-level Factors in 16 runs** | | | | | |
| **Run** | **(A** | **B)** | **= X** | **C** | **D** |
|  | | | | | |
| 1 | -1 | -1 | *x*1 | -1 | -1 |
| 2 | +1 | -1 | *x*2 | -1 | -1 |
| 3 | -1 | +1 | *x*3 | -1 | -1 |
| 4 | +1 | +1 | *x*4 | -1 | -1 |
| 5 | -1 | -1 | *x*1 | +1 | -1 |
| 6 | +1 | -1 | *x*2 | +1 | -1 |
| 7 | -1 | +1 | *x*3 | +1 | -1 |
| 8 | +1 | +1 | *x*4 | +1 | -1 |
| 9 | -1 | -1 | *x*1 | -1 | +1 |
| 10 | +1 | -1 | *x*2 | -1 | +1 |
| 11 | -1 | +1 | *x*3 | -1 | +1 |
| 12 | +1 | +1 | *x*4 | -1 | +1 |
| 13 | -1 | -1 | *x*1 | +1 | +1 |
| 14 | +1 | -1 | *x*2 | +1 | +1 |
| 15 | -1 | +1 | *x*3 | +1 | +1 |
| 16 | +1 | +1 | *x*4 | +1 | +1 |

**Some Useful (Taguchi) Orthogonal "L" Array Designs**

**L9 - A 34-2 Fractional Factorial Design 4 Factors at Three Levels (9 runs)**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Run** | **X1** | **X2** | **X3** | **X4** |
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 | 2 |
| 3 | 1 | 3 | 3 | 3 |
| 4 | 2 | 1 | 2 | 3 |
| 5 | 2 | 2 | 3 | 1 |
| 6 | 2 | 3 | 1 | 2 |
| 7 | 3 | 1 | 3 | 2 |
| 8 | 3 | 2 | 1 | 3 |
| 9 | 3 | 3 | 2 | 1 |

**L18 - A 2 x 37-5 Fractional Factorial (Mixed-Level) Design  
1 Factor at Two Levels and Seven Factors at 3 Levels (18 Runs)**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Run | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 1 | 1 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 1 | 2 | 1 | 1 | 2 | 2 | 3 | 3 |
| 5 | 1 | 2 | 2 | 2 | 3 | 3 | 1 | 1 |
| 6 | 1 | 2 | 3 | 3 | 1 | 1 | 2 | 2 |
| 7 | 1 | 3 | 1 | 2 | 1 | 3 | 2 | 3 |
| 8 | 1 | 3 | 2 | 3 | 2 | 1 | 3 | 1 |
| 9 | 1 | 3 | 3 | 1 | 3 | 2 | 1 | 2 |
| 10 | 2 | 1 | 1 | 3 | 3 | 2 | 2 | 1 |
| 11 | 2 | 1 | 2 | 1 | 1 | 3 | 3 | 2 |
| 12 | 2 | 1 | 3 | 2 | 2 | 1 | 1 | 3 |
| 13 | 2 | 2 | 1 | 2 | 3 | 1 | 3 | 2 |
| 14 | 2 | 2 | 2 | 3 | 1 | 2 | 1 | 3 |
| 15 | 2 | 2 | 3 | 1 | 2 | 3 | 2 | 1 |
| 16 | 2 | 3 | 1 | 3 | 2 | 3 | 1 | 2 |
| 17 | 2 | 3 | 2 | 1 | 3 | 1 | 2 | 3 |
| 18 | 2 | 3 | 3 | 2 | 1 | 2 | 3 | 1 |

**L27 - A 313-10 Fractional Factorial Design  
Thirteen Factors at Three Levels (27 Runs)**

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Run | X1 | X2 | X3 | X4 | X5 | X6 | X7 | X8 | X9 | X10 | X11 | X12 | X13 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 1 | 1 | 1 | 1 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| 5 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 1 | 1 | 1 |
| 6 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 1 | 1 | 1 | 2 | 2 | 2 |
| 7 | 1 | 3 | 3 | 3 | 1 | 1 | 1 | 3 | 3 | 3 | 2 | 2 | 2 |
| 8 | 1 | 3 | 3 | 3 | 2 | 2 | 2 | 1 | 1 | 1 | 3 | 3 | 3 |
| 9 | 1 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 2 | 1 | 1 | 1 |
| 10 | 2 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 11 | 2 | 1 | 2 | 3 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 |
| 12 | 2 | 1 | 2 | 3 | 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 |
| 13 | 2 | 2 | 3 | 1 | 1 | 2 | 3 | 2 | 3 | 1 | 3 | 1 | 2 |
| 14 | 2 | 2 | 3 | 1 | 2 | 3 | 1 | 3 | 1 | 2 | 1 | 2 | 3 |
| 15 | 2 | 2 | 3 | 1 | 3 | 1 | 2 | 1 | 2 | 3 | 2 | 3 | 1 |
| 16 | 2 | 3 | 1 | 2 | 1 | 2 | 3 | 3 | 1 | 2 | 2 | 3 | 1 |
| 17 | 2 | 3 | 1 | 2 | 2 | 3 | 1 | 1 | 2 | 3 | 3 | 1 | 2 |
| 18 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 2 | 3 | 1 | 1 | 2 | 3 |
| 19 | 3 | 1 | 3 | 2 | 1 | 3 | 2 | 1 | 3 | 2 | 1 | 3 | 2 |
| 20 | 3 | 1 | 3 | 2 | 2 | 1 | 3 | 2 | 1 | 3 | 2 | 1 | 3 |
| 21 | 3 | 1 | 3 | 2 | 3 | 2 | 1 | 3 | 2 | 1 | 3 | 2 | 1 |
| 22 | 3 | 2 | 1 | 3 | 1 | 3 | 2 | 2 | 1 | 3 | 3 | 2 | 1 |
| 23 | 3 | 2 | 1 | 3 | 2 | 1 | 3 | 3 | 2 | 1 | 1 | 3 | 2 |
| 24 | 3 | 2 | 1 | 3 | 3 | 2 | 1 | 1 | 3 | 2 | 2 | 1 | 3 |
| 25 | 3 | 3 | 2 | 1 | 1 | 3 | 2 | 3 | 2 | 1 | 2 | 1 | 3 |
| 26 | 3 | 3 | 2 | 1 | 2 | 1 | 3 | 1 | 3 | 2 | 3 | 2 | 1 |
| 27 | 3 | 3 | 2 | 1 | 3 | 2 | 1 | 2 | 1 | 3 | 1 | 3 | 2 |

**L36 - A Fractional Factorial (Mixed-Level) Design Eleven Factors at Two Levels and Twelve Factors at 3 Levels (36 Runs)**

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Run** | **X1** | **X2** | **X3** | **X4** | **X5** | **X6** | **X7** | **X8** | **X9** | **X10** | **X11** | **X12** | **X13** | **X14** | **X15** | **X16** | **X17** | **X18** | **X19** | **X20** | **X21** | **X22** | **X23** |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 |
| 5 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 1 | 1 | 1 | 1 |
| 6 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| 7 | 1 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 2 | 2 | 2 | 1 | 1 | 2 | 3 | 1 | 2 | 3 | 3 | 1 | 2 | 2 | 3 |
| 8 | 1 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 3 | 1 | 2 | 3 | 1 | 1 | 2 | 3 | 3 | 1 |
| 9 | 1 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 1 | 2 | 3 | 1 | 2 | 2 | 3 | 1 | 1 | 2 |
| 10 | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 1 | 2 | 1 | 1 | 3 | 2 | 1 | 3 | 2 | 3 | 2 | 1 | 3 | 2 |
| 11 | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 2 | 1 | 3 | 2 | 1 | 3 | 1 | 3 | 2 | 1 | 3 |
| 12 | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 1 | 2 | 3 | 3 | 2 | 1 | 3 | 2 | 1 | 2 | 1 | 3 | 2 | 1 |
| 13 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 1 | 1 | 2 | 3 | 1 | 3 | 2 | 1 | 3 | 3 | 2 | 1 | 2 |
| 14 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 3 | 1 | 2 | 1 | 3 | 2 | 1 | 1 | 3 | 2 | 3 |
| 15 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 1 | 3 | 1 | 2 | 3 | 2 | 1 | 3 | 2 | 2 | 1 | 3 | 1 |
| 16 | 1 | 2 | 2 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 1 | 1 | 2 | 3 | 2 | 1 | 1 | 3 | 2 | 3 | 3 | 2 | 1 |
| 17 | 1 | 2 | 2 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 1 | 2 | 3 | 1 | 3 | 2 | 2 | 1 | 3 | 1 | 1 | 3 | 2 |
| 18 | 1 | 2 | 2 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 1 | 3 | 1 | 2 | 1 | 3 | 3 | 2 | 1 | 2 | 2 | 1 | 3 |
| 19 | 2 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 3 | 3 | 3 | 1 | 2 | 2 | 1 | 2 | 3 |
| 20 | 2 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 3 | 2 | 1 | 1 | 1 | 2 | 3 | 3 | 2 | 3 | 1 |
| 21 | 2 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 3 | 1 | 3 | 2 | 2 | 2 | 3 | 1 | 1 | 3 | 1 | 2 |
| 22 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 3 | 3 | 1 | 2 | 1 | 1 | 3 | 3 | 2 |
| 23 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 1 | 1 | 2 | 3 | 2 | 2 | 1 | 1 | 3 |
| 24 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 2 | 3 | 1 | 1 | 2 | 2 | 3 | 1 | 3 | 3 | 2 | 2 | 1 |
| 25 | 2 | 1 | 1 | 2 | 2 | 2 | 1 | 2 | 2 | 1 | 1 | 1 | 3 | 2 | 1 | 2 | 3 | 3 | 1 | 3 | 1 | 2 | 2 |
| 26 | 2 | 1 | 1 | 2 | 2 | 2 | 1 | 2 | 2 | 1 | 1 | 2 | 1 | 3 | 2 | 3 | 1 | 1 | 2 | 1 | 2 | 3 | 3 |
| 27 | 2 | 1 | 1 | 2 | 2 | 2 | 1 | 2 | 2 | 1 | 1 | 3 | 2 | 1 | 3 | 1 | 2 | 2 | 3 | 2 | 3 | 1 | 1 |
| 28 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 3 | 2 | 2 | 2 | 1 | 1 | 3 | 2 | 3 | 1 | 3 |
| 29 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 3 | 3 | 3 | 2 | 2 | 1 | 3 | 1 | 2 | 1 |
| 30 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 2 | 2 | 1 | 2 | 3 | 2 | 1 | 1 | 1 | 3 | 3 | 2 | 1 | 2 | 3 | 2 |
| 31 | 2 | 2 | 1 | 2 | 1 | 2 | 1 | 1 | 1 | 2 | 2 | 1 | 3 | 3 | 3 | 2 | 3 | 2 | 2 | 1 | 2 | 1 | 1 |
| 32 | 2 | 2 | 1 | 2 | 1 | 2 | 1 | 1 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 3 | 1 | 3 | 3 | 2 | 3 | 2 | 2 |
| 33 | 2 | 2 | 1 | 2 | 1 | 2 | 1 | 1 | 1 | 2 | 2 | 3 | 2 | 2 | 2 | 1 | 2 | 1 | 1 | 3 | 1 | 3 | 3 |
| 34 | 2 | 2 | 1 | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 1 | 1 | 3 | 1 | 2 | 3 | 2 | 3 | 1 | 2 | 2 | 3 | 1 |
| 35 | 2 | 2 | 1 | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 3 | 1 | 3 | 1 | 2 | 3 | 3 | 1 | 2 |
| 36 | 2 | 2 | 1 | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 1 | 3 | 2 | 3 | 1 | 2 | 1 | 2 | 3 | 1 | 1 | 2 | 3 |

**Advantages and Disadvantages of Three-Level and Mixed-Level "L" Designs**

The good features of these designs are:

* They are orthogonal arrays. Some analysts believe this simplifies the analysis and interpretation of results while other analysts believe it does not.
* They obtain a lot of information about the main effects in a relatively few number of runs.
* You can test whether non-linear terms are needed in the model, at least as far as the three-level factors are concerned.

On the other hand, there are several undesirable features of these designs to consider:

* They provide limited information about interactions.

They require more runs than a comparable 2*k*-*p*design, and a two-level design will often suffice when the factors are continuous and monotonic (many three-level designs are used when two-level designs would have been adequate).