Exercise 3.3. Prove by induction that we have

$$\sum_{n=1}^{N} n^3 = \frac{N^2(N+1)^2}{4}.$$

Exercise 3.4. Prove that

$$\sum_{n=1}^{N} n^3 = \Big(\sum_{n=1}^{N} n\Big)^2.$$

Exercise 3.5. Prove (3.3).

Exercise 3.6. Prove the following result

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Exercise 3.7. Prove the following result

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

4 First order difference equations

In many cases it is of interest to model the evolution of some system over time. There are two distinct cases. One can think of time as a continuous variable, or one can think of time as a discrete variable. The first case often leads to differential equations. We will not discuss differential equations in these notes.

We consider a time period T and observe (or measure) the system at times $t = nT$, $n \in \mathbf{N}_0$. The result is a sequence $x(0), x(1), x(2), \dots$. In some cases these values are obtained from a function f, which is defined for all $t \geq 0$. In this case $x(n) = f(nT)$. This method of obtaining the values is called periodic sampling. One models the system using a difference equation, or what is sometimes called a recurrence relation.

In this section we will consider the simplest cases first. We start with the following equation

$$x(n+1) = ax(n), \quad n \in \mathbf{N}_0, \tag{4.1}$$

where a is a given constant. The solution is given by

$$x(n) = a^n x(0). \tag{4.2}$$

The value $x(0)$ is called the *initial value*. To prove that (4.2) solves (4.1), we compute as follows.

$$x(n+1) = a^{n+1}x(0) = a(a^n x(0)) = ax(n).$$

Example 4.1. An amount of USD10,000 is deposited in a bank account with an annual interest rate of 4%. Determine the balance of the account after 15 years. This problem leads to the difference equation

$$b(n+1) = 1.04b(n), \quad b(0) = 10,000.$$

The solution is

$$b(n) = (1.04)^n 10,000,$$

in particular $b(15) = 18,009.44$.

We write the equation (4.1) as

$$x(n+1) - ax(n) = 0. \tag{4.3}$$

This equation is called a homogeneous first order difference equation with constant coefficients. The term homogeneous means that the right hand side is zero. A corresponding inhomogeneous equation is given as

$$x(n+1) - ax(n) = c, \tag{4.4}$$

where we take the right hand side to be a constant different from zero.

The equation (4.3) is called linear, since it satisfies the *superposition principle*. Let $y(n)$ and $z(n)$ be two solutions to (4.3), and let $\alpha, \beta \in \mathbf{R}$ be two real numbers. Define $w(n) = \alpha y(n) + \beta z(n)$. Then $w(n)$ also satisfies (4.3), as the following computation shows.

$$\begin{aligned} w(n+1) - aw(n) &= \alpha y(n+1) + \beta z(n+1) - a(\alpha y(n) + \beta z(n)) \\ &= \alpha(y(n+1) - ay(n)) + \beta(z(n+1) - az(n)) = \alpha 0 + \beta 0 = 0. \end{aligned}$$

We now solve (4.4). The idea is to compute a number of terms, guess the structure of the solution, and then prove that we have indeed found the solution. First we compute a number of terms. In the computation of $x(2)$ we give all intermediate steps. These are omitted in the computation of $x(3)$ etc.

$$\begin{aligned} x(1) &= ax(0) + c, \\ x(2) &= ax(1) + c = a(ax(0) + c) + c = a^2x(0) + ac + c, \\ x(3) &= ax(2) + c = a^3x(0) + a^2c + ac + c, \\ x(4) &= ax(3) + c = a^4x(0) + a^3c + a^2c + ac + c, \\ x(5) &= ax(4) + c = a^5x(0) + a^4c + a^3c + a^2c + ac + c, \\ &\ \vdots \\ x(n) &= a^n x(0) + c\sum_{k=0}^{n-1} a^k. \end{aligned}$$

Thus we have guessed that the solution is given by

$$x(n) = a^n x(0) + c\sum_{k=0}^{n-1} a^k. \tag{4.5}$$

To prove that (4.5) is a solution to (4.4), we must prove that (4.5) satisfies this equation. We compute as follows.

$$\begin{aligned} x(n+1) &= a^{n+1}x(0) + c\sum_{k=0}^{n} a^k \\ &= a^{n+1}x(0) + c(1 + a + a_2 + \cdots + a^{n-1} + a^n) \\ &= a(a^n x(0)) + c + a(c(1 + a + a_2 + \cdots + a^{n-1})) \\ &= a\Big(a^n x(0) + c\sum_{k=0}^{n-1} a^k\Big) + c \\ &= ax(n) + c. \end{aligned}$$

Thus we have shown that (4.5) is a solution to (4.4). For $a \neq 1$ the solution (4.5) can be rewritten using the result (3.6):

$$x(n) = a^n x(0) + c\frac{a^n - 1}{a - 1}. \tag{4.6}$$

In the general case both a and c will be functions of n. We have the following result.

Theorem 4.2. *Let $a(n)$ and $c(n)$, $n \in \mathbf{N}_0$, be real sequences. Then the linear first order difference equation*

$$x(n+1) = a(n)x(n) + c(n) \quad \textit{with initial condition } x(0) = y_0 \tag{4.7}$$

has the solution

$$y(n) = \left(\prod_{k=0}^{n-1} a(k)\right) y_0 + \sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n-1} a(j)\right) c(k). \tag{4.8}$$

The solution is unique.

Proof. We define the sequence $y(n)$ by (4.8). We must show that it satisfies the equation (4.7) and the initial condition. Due to the convention (3.1) the initial condition is trivially satisfied. We first write out the expression for $y(n+1)$

$$y(n+1) = \left(\prod_{k=0}^{n} a(k)\right) y_0 + \sum_{k=0}^{n}\left(\prod_{j=k+1}^{n} a(j)\right) c(k).$$

We then rewrite the last term above as follows, using (3.1).

$$\begin{aligned}\sum_{k=0}^{n}\left(\prod_{j=k+1}^{n} a(j)\right) c(k) &= \prod_{j=n+1}^{n} a(j)c(n) + \sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n} a(j)\right) c(k)\\ &= c(n) + \sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n} a(j)\right) c(k) = c(n) + a(n)\sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n-1} a(j)\right) c(k).\end{aligned}$$

Using this result we get

$$y(n+1) = a(n)\left(\prod_{k=0}^{n-1} a(k)\right) y(0) + c(n) + a(n)\sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n-1} a(j)\right) c(k),$$

which implies

$$y(n+1) = a(n)y(n) + c(n).$$

Thus we have shown that $y(n)$ is a solution. Finally we must prove uniqueness. Assume that we have two solutions $y(n)$ and $\tilde{y}(n)$, which satisfy (4.7), i.e. both the equation and the initial condition are satisfied by both solutions. Now consider $\{n \in \mathbf{N}_0 \mid y(n) \neq \tilde{y}(n)\}$. Let n_0 be the smallest integer in this set. We must have $n_0 \geq 1$, since $y(0) = \tilde{y}(0) = y_0$. By the definition of n_0 we have $y(n_0 - 1) = \tilde{y}(n_0 - 1)$, and then

$$y(n_0) = a(n_0 - 1)y(n_0 - 1) + c(n_0 - 1) = a(n_0 - 1)\tilde{y}(n_0 - 1) + c(n_0 - 1) = \tilde{y}(n_0),$$

which is a contradiction. Thus we must have $n_0 = 0$. But $y(0) = \tilde{y}(0)$, since the two equations satisfy the same initial condition. It follows that the solution is unique. □

4.1 Examples

We now give some examples. Details should be worked out by the reader.

Example 4.3. Consider the problem

$$x(n+1) = -x(n), \quad x(0) = 3.$$

Using (4.5) with $c = 0$ we get the solution

$$x(n) = (-1)^n 3.$$

Now consider the inhomogeneous problem

$$x(n+1) = -x(n) + 4, \quad x(0) = 3.$$

Using (4.6) we get the solution

$$x(n) = (-1)^n 3 - 2\big((-1)^n - 1\big) = (-1)^n + 2.$$

Example 4.4. Consider the problem

$$x(n+1) = 2x(n) + n, \quad x(0) = 5.$$

Using the general formula (4.8) we get the solution

$$x(n) = 5 \cdot 2^n + \sum_{k=0}^{n-1} k 2^{n-1-k} = 5 \cdot 2^n + 2^n - n - 1.$$

The last equality requires results that are not covered by this course, so the first expression is sufficient as the solution to the problem.

Example 4.5. Consider the problem

$$x(n+1) = (n-4)x(n), \quad x(0) = 1. \tag{4.9}$$

This problem can be solved in two different manners. One can directly use the general formula (4.8). In this case one gets the solution

$$x(n) = \prod_{k=0}^{n-1} (k-4).$$

But this solution is not very explicit. A more explicit solution can be found by noting that for $n \geq 5$ the product contains the factor 0, hence the product is zero. Thus one has the explicit solution:

$$x(n) = \begin{cases} 1 & n = 0, \\ -4 & n = 1, \\ 12 & n = 2, \\ -24 & n = 3, \\ 24 & n = 4, \\ 0 & n \geq 5. \end{cases} \tag{4.10}$$

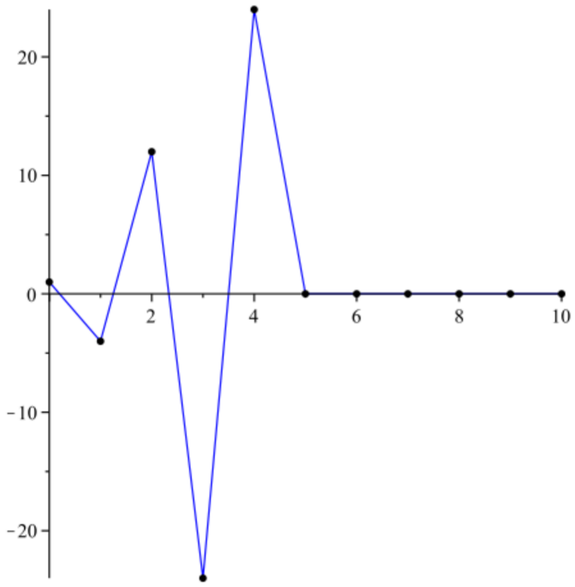

Figure 4.1: Point plot of the solution (4.10). Points connected with blue line segments

We illustrate the solution in Figure 4.1. Here we plot the values of $x(n)$ as filled circles, connected by blue line segments. We include the line segments to visualize the variations in the values.

We note that the solution (4.10) is very sensitive to small changes in the equation. If we add a small constant inhomogeneous term, the solution will rapidly diverge from the solution zero for $n \geq 5$. As an example we consider

$$x(n+1) = (n-4)x(n) + \frac{1}{20}, \quad x(0) = 1. \tag{4.11}$$

A plot of this solution is shown in Figure 4.2.

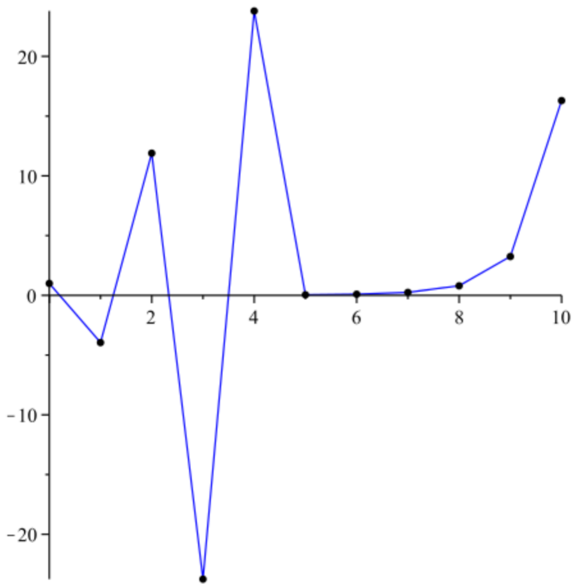

Figure 4.2: Point plot of the solution to (4.11). Points connected with blue line segments

Example 4.6. Let us consider the payment of a loan. Payments are made periodically, e.g. once a month. The interest rate per period is $100r\%$. The payment at the end of each period is denoted $p(n)$. The initial loan is $q(0)$. The outstanding balance after n payments is denoted $q(n)$. Thus $q(n)$ must satisfy the difference equation

$$q(n+1) = (1+r)q(n) - p(n). \tag{4.12}$$

The solution follows from (4.8).

$$q(n) = (1+r)^n q(0) - \sum_{k=0}^{n-1} (1+r)^{n-k-1} p(k). \tag{4.13}$$

Often the loan is paid back in equal installments, i.e. $p(n) = p$ for all n. Then the above sum can be computed. We get the result

$$q(n) = (1+r)^n q(0) - ((1+r)^n - 1)\frac{p}{r}. \tag{4.14}$$

Suppose that we want to pay back the loan in N installments. Then the installment is determined by

$$p = q(0)\frac{r}{1-(1+r)^{-N}} \tag{4.15}$$

Exercises

Exercise 4.1. Fill in the details in Example 4.6. In particular the computations leading to (4.14).

Exercise 4.2. Discuss the applications of the results in Example 4.6.

Exercise 4.3. Adapt the results in the Example 4.6 to the case, where initially no installments are paid.

Exercise 4.4. Discuss the application to loans with a variable interest rate of the results in this section.

Exercise 4.5. Implement the various formulas for interest computation and loan amortization on a programmable calculator or in Maple. In particular, implement the formulas for loans with a variable interest rate and try them out on some real world examples.

Exercise 4.6. Solve each of the following first order difference equations.

1. $x(n+1) = x(n)$, $x(0) = 2$.
2. $x(n+1) = -x(n)$, $x(0) = 4$.
3. $x(n+1) = 2x(n) + 2$, $x(0) = 0$.
4. $x(n+1) = (n+1)x(n)$, $x(0) = 1$.
5. $x(n+1) = \dfrac{1}{n+1}x(n)$, $x(0) = 1$.
6. $x(n+1) = (n-3)x(n)$, $x(0) = 4$.