exact value:

$$\int_{-0.5}^{0} x\ln(x+1)dx = \ln(x+1)\int_{-0.5}^{0} xdx - \int_{-0.5}^{0} [\tfrac{d}{dx}\ln(x+1)\int xdx]dx$$
$$= \ln(x+1)\tfrac{x^2}{2}|_{-0.5}^{0} - \int_{-0.5}^{0} \tfrac{1}{x+1}\tfrac{x^2}{2}dx$$
$$= -0.495713$$

absolute error:

$$= |-0.495713 - 0.0525728|$$
$$= 0.54828$$

6.4.4 Weddles rule

Weddles rule is derived by general form of Newton-Cotes formula 6.3 by placing $n = 6$. Therefore, Weddles rule could be used for $n = 6$ and its multiples. Newton's forward difference interpolation formula which is written as

$$\left.\begin{aligned} f(x) = f(x_0 + rh) \quad &= f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) \\ &+ \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x_0) + \cdots \\ &+ \frac{r(r-1)(r-2)\cdots(r-(n-1))}{n!}\Delta^n f(x_0) \end{aligned}\right\} \tag{6.17}$$

where

$$\Delta f(x_0) = f(x_1) - f(x_0), \quad \text{and} \quad x_n = x_0 + nh$$

For $x = x_0 + rh$, $dx = hdr$, and limits of integration will be changed as for $x = x_0$, $r = 0$ and for $x = x_6$, $r = 6$. as $x_6 = x_0 + 6h$

$$\left.\begin{aligned} \int_0^6 f(x_0 + rh)hdr \quad &= h\Bigg(f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) \\ &+ \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x_0) + \cdots \\ &+ \frac{r(r-1)(r-2)\cdots(r-(n-1))}{n!}\Delta^6 f(x_6)\Bigg)\, dr \end{aligned}\right\} \tag{6.18}$$

$$\left.\begin{aligned}&= h\Bigg(nf(x_0) + \frac{n^2}{2}\Delta f(x_0) + \frac{1}{2!}(\frac{n^3}{3} - \frac{n^2}{2})\Delta^2 f(x_0)\\ &+\frac{1}{3!}(\frac{n^4}{4} - n^3 + n^2)\Delta^3 f(x_0) + \frac{1}{4!}(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2)\Delta^4 f(x_0)\\ &+\frac{1}{5!}(\frac{n^6}{6} - 2n^5 + \frac{35n^4}{4} - \frac{50n^3}{3} + 12n^2)\Delta^5 f(x_0)+\\ &\frac{n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n}{6!}\Delta^6 f(x_0)\Bigg)\end{aligned}\right\} \quad (6.19)$$

$$\int_0^6 f(x)dx = \frac{3h}{10}\left(f(x_0) + 5f(x_1) + f(x_2) + 6f(x_3) + f(x_4) + 5f(x_5) + f(x_6)\right) \quad (6.20)$$

For $n = 12$

$$\begin{aligned}&\int_0^6 f(x)dx == \frac{3h}{10}[f(x_0) + 5(f(x_1) + f(x_5) + f(x_7) + f(x_{11}))\\ &+6(f(x_3) + f(x_9)) + 2f(x_5) + (f(x_2) + f(x_4) + f(x_8) + f(x_{10})) + f(x_{12})]\end{aligned} \quad (6.21)$$

$$\begin{aligned}&\int_0^6 f(x)dx == \frac{3h}{10}[f(x_0) + 5(f(x_1) + f(x_5) + f(x_7) + f(x_{11}) + f(x_{13}) + f(x_{17}))\\ &+6(f(x_3) + f(x_9) + f(x_{15})) + 2(f(x_6) + f(x_{12})) + (f(x_2)\\ &+f(x_4) + f(x_8) + f(x_{10}) + f(x_{14}) + f(x_{16})) + f(x_{18}))]\end{aligned} \quad (6.22)$$

Example 72. *Evaluate the integral* $\int_2^5 \frac{1}{x^2}dx$ *using Weddles Rule*

SOLUTION:

Exact Value:-

$$\int_2^5 x^{-2}dx = \frac{x^{-2+1}}{-2+1}\Big|_2^5 = \frac{3}{10} = 0.3$$

Now for $n = 12$

$$\begin{aligned}&\int_2^5 \frac{1}{x^2}dx = \frac{3h}{10}\left(f(x_0) + 5(f(x_1) + f(x_5) + f(x_7) + (f(x_{11}))\right.\\ &+6(f(x_3) + f(x_9)) + 2(f(x_6)) + (f(x_2) + f(x_4) + f(x_8) + fx_{10})) + f(x_{12}))\end{aligned}$$

$$\int_2^5 \frac{1}{x^2}dx = \frac{3(0.25)}{10}[0.25 + 5(0.1975 + 0.0947 + 0.0711 + 0.0443) + 6(0.1322 + 0.0554)$$

$$+2(0.0816) + (0.16 + 0.1111 + 0.0625 + 0.0494) + 0.04]$$

$$= 0.300001$$

Absolute Error =*Approximated-Exact* $=|0.300001 - 0.3| = 0.0000012$

6.5 Computing Errors

6.5.1 Estimating Errors in Trepezoidal Rule:

For definite integral $\int_a^b f(x)dx$, there exist a function $F(x)$ such that $F'(x) = f(x)$. Therefore,

$$\int_a^b F'(x)dx = F(b) - F(a) \tag{6.23}$$

If the interval $[a,\, b]$ be divided into n-equal sub-intervals such that $x_i = x_o + ih$, for $i = 1, 2, \cdots, n-1$. We will have $n+1$ points so that the integral could take the form as given below

$$\int_a^b f(x)dx = \int_{x_o}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx \tag{6.24}$$

$$I = I_1 + I_2 + \cdots + I_n$$

Let us start in computing error for the interval $[x_0,\, x_1]$. Since

$$\text{Absolute Error} = \text{Exact value} - \text{Approximated value}$$

Therefore,

$$\text{Absolute Error} = \int_{x_o}^{x_1} f(x)dx - \frac{h}{2}\left(f(x_0) + f(x_1)\right)$$

$$= F(x_1) - F(x_0) - \frac{h}{2}\left(f(x_o) + f(x_1)\right)$$
$$\text{For} \quad x_1 = x_0 + h$$
$$= F(x_o + h) - F(x_o) - \frac{h}{2}\left(f(x_o) + f(x_o + h)\right)$$

Apply Taylor's series exapnsion

$$\begin{aligned}
\text{Error} &= \left(F(x_0) + hF'(x_0) + \frac{h^2}{2!}F''(x_0) + \cdots\right) - F(x_0) \\
&\quad - \frac{h}{2}\left(f(x_0) + f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots\right) \\
&= hf(x_0) + \frac{h^2}{2!}f'(x_o) + \frac{h^3}{3!}f''(x_0) + \cdots \\
&\quad - \frac{h}{2}\left(2f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0)\right) \\
&= \left(\frac{1}{3!} - \frac{1}{4})h^3 f''(x_o\right) + \cdots \\
&= -\frac{1}{12}h^3 f''(x_0) + \cdots
\end{aligned} \tag{6.25}$$

$$\text{Local Error} = -\frac{1}{12}h^3 f''(c_0) + \cdots = O(h^3) \tag{6.26}$$

where $c_0 \in \,]x_0,\, x_1[$. In a similar way local error for each sub-interval can be computed. Therefore, local error for interval $[x_1,\, x_2]$ is given by

$$\text{Local Error} = -\frac{1}{12}h^3 f''(c_1) + \cdots = O(h^3) \tag{6.27}$$

where $c_1 \in \,]x_1,\, x_2[$. Similarly, for the interval $[x_{n-1},\, x_n]$

$$\text{Local Error} = -\frac{1}{12}h^3 f''(c_{n-1}) + \cdots = O(h^3) \tag{6.28}$$

where $c_{n-1} \in \,]x_{n-1}, x_n[$ Global error is the sum of the local errors

$$E_G = -\frac{h^3}{12}f''(c_o) - \frac{h^3}{12}f''(c_1) - \frac{h^3}{12}f''(c_2) - \cdots - \frac{h^3}{12}f''(c_{n-1})$$

let

$$|f''(c)| = \max\left(|f''(c_o)|\,, |f''(c_1)|\,, \cdots, |f''(c_{n-1})|\right)$$

Then, Global error will be given

$$|E_G| < \frac{h^3}{12} n \left|f''(c)\right| = \frac{h^3}{12} \frac{|b-a|}{h} \left|f''(c)\right|$$
$$= h^2 \frac{b-a}{12} \left|f''(c)\right| = O(h^2)$$

It is observed that order of global error in Trapezoidal rule is greater than local error.

6.5.2 Error in Simpson's 1/3 Rule:

let

$$I = \int_a^b f(x)dx \tag{6.29}$$

and there exist a function F(x)such that

$$F'(x) = f(x)$$

then

$$\int_a^b F'(x)dx = F(b) - F(a) \tag{6.30}$$

Let the interval [a,b] be sub-divided into n-equal sub-intervals such that

$$x_i = x_o + ih; i = 1, 2, 3, ...n - 1$$

and

$$I = \int_a^b f(x)dx$$

$$I = \int_{x_o}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + ... + \int_{x_{n-2}}^{x_n} f(x)dx \tag{6.31}$$

$$I = I_1 + I_2 + ... + I_{\frac{n}{2}} \tag{6.32}$$

Firstly,determine the error in the interval

$$[x_o, x_2]$$

Error=Exact value-Approximate value

$$= \int_{x_o}^{x_2} f(x)dx - \frac{h}{3}[f(x_o) + 4f(x_1) + f(x_2)]$$

$$Error = F(x_2) - F(x_o) - \frac{h}{3}f(x_o) - \frac{4}{3}f(x_1) - \frac{h}{3}f(x_2)$$

since

$$x_2 = x_o + 2h, x_1 = x_o + h$$

$$= F(x_o + 2h) - F(x_o) - \frac{h}{3}f(x_o) - \frac{4}{3}hf(x_o + h) - \frac{h}{3}f(x_o + 2h)$$

Using Taylor's Series

$$\begin{aligned}
&= F(x_o) + 2hF'(x_o) + \tfrac{(2h)^2}{2!}F''(x_o) + \tfrac{(2h)^3}{3!}F'''(x_o) \\
&+\tfrac{(2h)^4}{4!}F^{(iv)}(x_o) + \tfrac{(2h)^5}{5!}F^{(v)}(x_o) + ... - F(x_o) \\
&-\tfrac{h}{3}f(x_o) - \tfrac{h}{4}[f(x_o) + hf'(x_o) + \tfrac{h^2}{2!}f''(x_o) + \tfrac{h^3}{3!}f'''(x_o) \\
&+\tfrac{h^4}{4!}f^{(iv)}(x_o) + \tfrac{h^5}{5!}f^{(v)}(x_o) + ...] - \tfrac{h}{3}[f(x_o) \\
&+2hf'(x_o) + \tfrac{(2h)^2}{2!}f''(x_o) + \tfrac{(2h)^3}{3!}f'''(x_o) + \tfrac{(2h)^4}{4!}f^{(iv)}(x_o) \\
&+\tfrac{(2h)^5}{5!}f^{(v)}(x_o) + ...]
\end{aligned}$$

$$\begin{aligned}
&Error = 2hF'(x_o) + \tfrac{4h^2}{2}F''(x_o) + \tfrac{4}{3}h^3F'''(x_o) + \tfrac{2}{3}h^4F^{(iv)}(x_o) \\
&+\tfrac{4}{15}h^5F^{(v)}(x_o) + ... - \tfrac{h}{3}f(x_o) - \tfrac{4}{3}hf(x_o) - \tfrac{4}{3}h^2f'(x_o) \\
&-\tfrac{2}{3}h^3f''(x_o) - \tfrac{2}{9}h^4f'''(x_o) - \tfrac{1}{18}h^5f^{(iv)}(x_o) + ... \\
&-\tfrac{h}{3}f(x_o) - \tfrac{2}{3}h^2f'(x_o) - \tfrac{2}{3}h^3f''(x_o) - \tfrac{4}{9}h^4f'''(x_o) \\
&-\tfrac{2}{9}h^5f^{(iv)}(x_o) + ...
\end{aligned}$$

since

$$F'(x) = f(x)$$

$$\begin{aligned}
&Error = 2hf(x_o) + 2h^2f'(x_o) + \tfrac{4}{3}h^3f''(x_o) + \tfrac{2}{3}h^4f'''(x_o) \\
&+\tfrac{4}{15}h^5f^{(iv)}(x_o) + ... - \tfrac{h}{3}f(x_o) - \tfrac{4}{3}hf(x_o) - \tfrac{4}{3}h^2f'(x_o) \\
&-\tfrac{2}{3}h^3f''(x_o) - \tfrac{2}{9}h^4f'''(x_o) - \tfrac{h}{18}h^5f^{(iv)}(x_o) + ... - \tfrac{h}{3}f(x_o) \\
&-\tfrac{2}{3}h^2f'(x_o) - \tfrac{2}{3}h^3f''(x_o) - \tfrac{4}{9}h^4f^{(iv)}(x_o) - \tfrac{2}{9}h^5f^{(v)}(x_o) + ...
\end{aligned}$$

$$Error = \frac{(24 - 5 - 20)}{90}h^5f^{(iv)}(x_o) + ...$$

$$Error = (\frac{-1}{90})h^5 f^{(iv)}(x_o) + ...$$
$$Error = \frac{-1}{90}h^5 f^{(iv)}(c_o) where c_o \in]x_o, x_2[$$
$$Error = O(h^5)$$

is local error. Next,yo determine the error in

$$]x_2, x_4[$$

Error=Exact value-Approximate value

$$= \int_{x_2}^{x_4} f(x)dx - \frac{h}{3}[f(x_2) + 4f(x_3) + f(x_4)]$$

$$= F(x_4) - F(x_2) - \frac{h}{3}f(x_2) - \frac{4}{3}hf(x_3) - \frac{h}{3}f(x_4)$$
$$Since x_4 = x_2 + 2h, x_3 = x_2 + h$$
$$= F(x_2 + 2h) - F(x_2) - \frac{h}{3}f(x_2) - \frac{4}{3}hf(x_2 + h) - \frac{h}{3}f(x_2 + 2h)$$

Using Taylor's Series

$$Error = F(x_2) + 2hF(x_2) + \tfrac{(2h)^2}{2!}F''(x_2) + \tfrac{(2h)^3}{3!}F'''(x_2)$$
$$+\tfrac{(2h)^4}{4!}F^{(iv)}(x_2) + \tfrac{(2h)^5}{5!}F^{(v)}(x_2) + ... - F(x_2) - \tfrac{h}{3}f(x_2)$$
$$-\tfrac{4}{3}h[f(x_2) + hf'(x_2) + \tfrac{h^2}{2!}f''(x_2) + \tfrac{h^3}{3!}f'''(x_2) + \tfrac{h^4}{4!}f^{(iv)}(x_2)$$
$$+...] - \tfrac{h}{3}[f(x_2) + 2hf'(x_2) + \tfrac{(2h)^2}{2!}f''(x_2) + \tfrac{(2h)^3}{3!}f'''(x_2) + ...]$$

$$Error = 2hF'(x_2) + 2h^2F''(x_2) + \tfrac{4}{3}h^3F'''(x_2) + \tfrac{2}{3}h^4F^{(iv)}(x_2)$$
$$+\tfrac{4}{15}h^5F^{(v)}(x_2) + ... - \tfrac{h}{3}f(x_2) - \tfrac{4}{3}hf(x_2) - \tfrac{4}{3}h^2f'(x_2) - \tfrac{2}{3}h^3f''(x_2)$$
$$-\tfrac{4}{9}h^4f'''(x_2) - \tfrac{2}{9}h^5f^{(iv)}(x_2) + ...$$
$$Since F'(x) = f(x)$$
$$Error = 2hf(x_2) + 2h^2f'(x_2) + \tfrac{4}{3}h^3f''(x_2) + \tfrac{2}{3}h^4f'''(x_2)$$
$$+\tfrac{4}{15}h^5f^{(iv)}(x_2) + ... - \tfrac{h}{3}f(x_2) - \tfrac{4}{3}hf(x_2) - \tfrac{4}{3}h^2f'(x_2)$$
$$-\tfrac{2}{3}h^3f''(x_2) - \tfrac{2}{9}h^4f'''(x_2) - \tfrac{1}{18}h^5f^{(iv)}(x_2) + ...$$
$$-\tfrac{h}{3}f(x_2) - \tfrac{2}{3}h^2f'(x_2) - \tfrac{2}{3}h^3f''(x_2) - \tfrac{4}{9}h^4f'''(x_2)$$
$$-\tfrac{2}{9}h^5f^{(iv)}(x_2) + ...$$

$$Error = (2 - \tfrac{1}{3} - \tfrac{4}{3} - \tfrac{1}{3})hf(x_2) + (2 - \tfrac{4}{3} - \tfrac{2}{3})h^2 f'(x_2)$$
$$+(\tfrac{4}{3} - \tfrac{2}{3} - \tfrac{2}{3})h^3 f''(x_2) + (\tfrac{2}{3} - \tfrac{2}{9} - \tfrac{4}{9})h^4 f'''(x_2)$$
$$+(\tfrac{4}{15} - \tfrac{1}{18} - \tfrac{2}{9})h^5 f^{(iv)}(x_2) + ...$$

$$Error = (\frac{4}{15} - \frac{1}{18} - \frac{2}{9})h^5 f^{(iv)}(x_2) + ...$$
$$Error = (\frac{24 - 5 - 20}{90})h^5 f^{(iv)}(x_2) + ...$$
$$Error = -\frac{1}{90}h^5 f^{(iv)}(x_2) + ...$$
$$Error = -\frac{1}{90}h^5 f^{(iv)}(c_1) where c_1 \in]x_2, x_4[$$
$$Error = O(h^5)$$

is local error. Similarly,the error in the interval

$$]x_{n-2}, x_n[$$

is

$$-\frac{1}{90}h^5 f^{(iv)}(c_{\frac{n}{2}-1}) where c_{\frac{n}{2}-1} \in]x_{n-2}, x_n[$$

The Global error is

$$E_G = -\frac{1}{90}h^5 f^{(iv)}(c_o) - \frac{1}{90}h^5 f^{(iv)}(c_2) - ... - \frac{1}{90}h^5 f^{(iv)}(c_{\frac{n}{2}-1})$$

Let

$$\left|f^{(iv)}(c)\right|$$

be maximum of

$$\{\left|f^{(iv)}(c_o)\right|, \left|f^{(iv)}(c_2)\right|, ... \left|f^{(iv)}(c_{\frac{n}{2}-1})\right|\}$$

Then,Global error becomes

$$|E_G| < \frac{1}{90}h^5 \frac{n}{2}\left|f^{(iv)}(c)\right|$$

$$= \frac{h^5}{90}\frac{n}{2}\left|f^{(iv)}(c)\right| \sin cen = \frac{b-a}{h}$$

$$= \frac{h^5}{180}\left|\frac{b-a}{h}\right|\left|f^{(iv)}(c)\right|$$
$$= \frac{h^4}{180}\left|b-a\right|\left|f^{(iv)}(c)\right|$$
$$|E_G| < O(h^4)$$

Hence,order of Global error of Simpson's 1/3 rule is greater than local error.

6.5.3 Error in Simpson's 3/8 Rule:

Let

$$I = \int_a^b f(x)dx \tag{6.33}$$

and there exist a function F(x)such that

$$F'(x) = f(x)$$

then,

$$\int_a^b f(x)dx = F(b) - F(a) \tag{6.34}$$

Let the interval [a,b] be sub-divided into n-equal sub-intervals such that

$$x_i = x_o + ih; i = 1, 2, 3, ...n - 1$$

and

$$I = \int_a^b f(x)dx = \int_{x_o}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + ... + \int_{x_{n-3}}^{x_n} f(x)dx \tag{6.35}$$

$$I = I_1 + I_2 + ...I_{\frac{n}{3}} \tag{6.36}$$

Firstly,determine the error for interval

$$]x_o, x_3[$$

Error=Exact value-Approximate value

$$Error = \int_{x_o}^{x_3} f(x)dx - \frac{3}{8}h[f(x_o) + 3f(x_1) + 3f(x_7) + f(x_3)]$$