FINITE DIFFERENCES

Lecture 1: **(a) Operators**

(b) Forward Differences

and their calculations.

(c) Backward Differences

and their calculations.

1. Introduction

When a function $f(x)$ is known explicitly, it is easy to calculate the value (or values) of f(x), corresponding to a fixed given value x. However, when the explicit form of the function $f(x)$ is not known, it is possible to obtain an approximate value of the function up to a desired level of accuracy with the help of finite differences. A function $y = f(x)$, x being an independent variable and y, a dependent variable, is considered. Let x takes equidistant values $a, a + h, a + 2h, a + 3h, \ldots$ (which are finite in numbers); h is the equal spacing, then $f(a), f(a + h), f(a + 2h), f(a + 3h), \ldots$are the corresponding values of y. The values of the independent variable x are termed as **arguments** and the corresponding values of the dependent variable y are called **entries**.

2. Operators (Δ, ∇ and E)

Forward Difference: The forward difference, denoted by Δ, is defined as

$$\Delta y = \Delta f(x) = f(x + h) - f(x);$$

h is called the interval of differencing; $\Delta f(x)$ is the first order differences. We get the second order differences (denoted by Δ^2) when Δ is operated twice on f(x), Thus

$$\Delta^2 y = \Delta^2 f(x) = \Delta\,[\Delta f(x)] = \Delta\,[f(x + h) - f(x)]$$
$$= \Delta f(x + h) - \Delta f(x)$$
$$= [f(x + 2h) - f(x + h)] - [f(x + h) - f(x)]$$
$$= f(x + 2h) - 2f(x + h) + f(x).$$

Similarly, $\Delta^3 y, \Delta^4 y$ may be calculated.

Forward Difference Table with 4 arguments

Argument x	Entries y	First Differences Δy	Second Differences $\Delta^2 y$	Third Differences $\Delta^3 y$
$x_0 = a$	$y_0 = f(a)$	$\Delta y_0 = y_1 - y_0$		
$x_1 = a + h$	$y_1 = f(a + h)$		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
$x_2 = a + 2h$	$y_2 = f(a + 2h)$	$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
$x_3 = a + 3h$	$y_3 = f(a + 3h)$	$\Delta y_2 = y_3 - y_2$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	

Note: If $f(x)$ is a polynomial of degree n in x, then $\Delta^n f(x)$ is a constant and $\Delta^{n+1} f(x)$ is zero. Conversely, if the $(n + 1)^{\text{th}}$ difference is zero, then the polynomial is less or equal to degree n.

Example 1: Let $f(x) = x^2 + 8x - 5$, then

$$\begin{aligned} \Delta f(x) &= f(x + h) - f(x) \\ &= [(x+h)^2 + 8(x+h) - 5] - [x^2 + 8x - 5] \\ &= 2xh + h^2 + 8h \end{aligned}$$

Now,

$$\begin{aligned} \Delta^2 f(x) &= \Delta f(x + h) - \Delta f(x) \\ &= [2h(x+h) + h^2 + 8h] - [2xh + h^2 + 8h] \\ &= 2h^2, \text{ which is a constant} \end{aligned}$$

Hence,

$$\Delta^3 f(x) = \Delta^2 f(x + h) - \Delta^2 f(x) = 2h^2 - 2h^2 = 0$$

Backward Difference: We now define a difference operator, known as backward difference operator, given by

$$\nabla f(x+h) = f(x+h) - f(x)$$

Please note that the backward difference of $f(x + h)$ is same as the forward difference of $f(x)$, that is,

$$\nabla f(x+h) = \Delta f(x)$$

Backward Difference Table with 4 arguments

Argument x	Entries x	First Differences Δy	Second Differences$\Delta^2 y$	Third Differences $\Delta^3 y$
$x_0 = a$	$y_0 = f(a)$	$\nabla y_1 = y_1 - y_0$		
$x_1 = a + h$	$y_1 = f(a+h)$	$\nabla y_2 = y_2 - y_1$	$\nabla^2 y_1 = \nabla y_1 - \nabla y_0$	$\nabla^3 y_1$ $= \nabla^2 y_1 - \nabla^2 y_0$
$x_2 = a + 2h$	$y_2 = f(a+2h)$			
$x_3 = a + 3h$	$y_3 = f(a+3h)$	$\nabla y_3 = y_3 - y_2$	$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	

***E*-Operator:** We now define the E operator as

$$Ef(x) = f(x+h),$$

that is, the operator gives an increment of h to the argument [Note: $E^{-1} f(x) = f(x-h)$].

3. Relation between E and Δ

Let $y = f(x)$ be a function of an independent variable x and the dependent variable y. We have

$$\Delta f(x) = f(x + h) - f(x), \text{ where } h \text{ is the interval of differencing}$$

$$= E[f(x)] - f(x)$$

$$\text{or, } \Delta f(x) = (E-1)\, f(x) \Rightarrow \Delta \equiv E - 1 \Rightarrow E \equiv 1 + \Delta$$

Relation between operator E of finite differences and differential operator D of differential calculus

We know

$$\frac{d}{dx} f(x) = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = Df(x), \text{ where } D \equiv \frac{d}{dx}$$

Now, by Taylor series expansion, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \cdots = \left[1 + hD + \frac{h^2}{2!} D^2 + \cdots\right] f(x)$$

$$\text{or, } E[f(x)] = e^{hD} f(x)$$

$$\Rightarrow E \equiv e^{hD} \equiv 1 + \Delta\,, \;(\,since\; E = 1 + \Delta)$$

$$\text{or, } hD \equiv \log(1+\Delta) \equiv \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots$$

$$\text{or, } D \equiv \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots\right]$$

Example 2. Evaluate the following:

(i) $\Delta \log[f(x)]$

Solution: $\Delta \log[f(x)] = \log[f(x+h)] - \log[f(x)] = \log\left[\frac{f(x+h)}{f(x)}\right] = \log\left[\frac{Ef(x)}{f(x)}\right]$

$$= \log\left[\frac{(1+\Delta)f(x)}{f(x)}\right] = \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right]$$

$$= \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

(ii) $\Delta^2(ab^{cx})$

Solution: $\Delta^2(ab^{cx}) = a\Delta^2(b^{cx}) = a\Delta\left[b^{c(x+h)} - b^{cx}\right]$

$= a\Delta b^{c(x+h)} - a\Delta b^{cx}$

$= a\left[b^{c(x+2h)} - b^{c(x+h)}\right] - a\left[b^{c(x+h)} - b^{cx}\right]$

$= a\left[b^{c(x+2h)} - 2b^{c(x+h)} + b^{cx}\right]$

$= ab^{cx}[b^{2hc} - 2b^{hc} + 1]$

$= ab^{cx}[b^{hc} - 1]^2$ (the interval of differencing being h).

(iii) $\Delta^2(2^x)$

Solution: $\Delta^2(2^x) = \Delta[\Delta(2^x)]$

$= \Delta[2^{x+h} - 2^x] = \Delta 2^{x+h} - \Delta 2^x$

$= [2^{x+2h} - 2^{x+h}] - [2^{x+h} - 2^x]$

$= 2^{x+2h} - 2.2^{x+h} + 2^x$

(iv) $\nabla^2(2^x)$

Solution: $\nabla^2(2^x) = \nabla[\nabla(2^x)]$

$= \nabla[2^x - 2^{x-h}] = \nabla 2^x - \nabla 2^{x-h}$

$= [2^x - 2^{x-h}] - [2^{x-h} - 2^{x-2h}]$

$= 2^x - 2.2^{x-h} + 2^{x-2h}$

(v) $E^2(e^x)$

Solution: $E^2(e^x) = E[E(e^x)] = E[e^{x+h}] = e^{x+2h}$ (the interval of differencing being h).

(vi) $\left(\frac{\Delta^2}{E}\right) f(x)$, where f(x) = x^3.

Solution: $\left(\frac{\Delta^2}{E}\right) f(x) = \left(\frac{\Delta^2}{E}\right) x^3$

$= \Delta^2 E^{-1} x^3$

$= \Delta^2 (x-h)^3$

$= \Delta[(x)^3 - (x-h)^3] = \Delta(x)^3 - \Delta(x-h)^3$

$= [(x+h)^3 - (x)^3] - [(x)^3 - (x-h)^3]$

$$= 6xh^2$$

(vii) $\frac{\Delta^2 f(x)}{Ef(x)}$, where f(x) = x^3.

Solution:

$$\frac{\Delta^2 \mathrm{f(x)}}{\mathrm{Ef(x)}} = \frac{\Delta^2 \mathrm{x}^3}{\mathrm{Ex}^3} = \frac{\Delta[(\mathrm{x+h})^3 - (\mathrm{x})^3]}{(\mathrm{x+h})^3}$$

$$= \frac{[(\mathrm{x+2h})^3 - (\mathrm{x+h})^3] - [(\mathrm{x+h})^3 - (\mathrm{x})^3]}{(\mathrm{x+h})^3}$$

$$= \frac{6\mathrm{h}^3 + 6\mathrm{xh}^2}{(\mathrm{x+h})^3}$$

(viii) $\Delta tan^{-1} x$

Solution: $\Delta \tan^{-1}\mathrm{x} = \tan^{-1}(\mathrm{x+h}) - \tan^{-1}\mathrm{x}$

$$= \tan^{-1}\left(\frac{\mathrm{x+h-x}}{1+(\mathrm{x+h})\mathrm{x}}\right) = \tan^{-1}\left(\frac{\mathrm{h}}{1+\mathrm{x}^2+\mathrm{hx}}\right)$$

(ix) $\Delta^3[(1-ax)(1-bx)(1-cx)]$

Solution: $\Delta^3[(1-\mathrm{ax})(1-\mathrm{bx})(1-\mathrm{cx})]$

$$= \Delta^3[(1-(\mathrm{a+b+c})\mathrm{x} + (\mathrm{ab+bc+ca})\mathrm{x}^2 - \mathrm{abcx}^3]$$

$$= \Delta^3[-\mathrm{abcx}^3], (\mathrm{since}, \Delta^3\mathrm{x} = 0 = \Delta^3\mathrm{x}^2)$$

$$= -\mathrm{abc}\Delta^3(\mathrm{x}^3) = -(\mathrm{abc})(3!) = -6\mathrm{abc}$$

(x) $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$

Solution:

$$\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$$

$$= \Delta^{10}[1 - ax - bx^2 + (ab-c)x^3 + (ac-d)x^4 + (bc+ad)x^5 + (bd-abc)x^6 + (c-ab)x^7 - acdx^8 - bcx^9 + abcdx^{10}]$$

$$= \Delta^{10}[abcdx^{10}]$$

(since all differences of order less than 10, vanishes)

$$= (abcd)(10!).$$

Example 3. Prove the following relations:

(i) $E\nabla \equiv \nabla E \equiv \Delta$

Solution:

Let $y = f(x)$ be a function of an independent variable *x* and the dependent variable *y*. Now,

$E\nabla f(x) = E[f(x) - f(x-h)]$, h being the interval of differencing

$$= Ef(x) - Ef(x-h) = f(x+h) - f(x) = \Delta f(x) \Longrightarrow E\nabla \equiv \Delta$$

Again,

$$\nabla E f(x) = \nabla f(x+h)$$

$$= f(x+h) - f(x)$$

$$= \Delta f(x) \Longrightarrow \nabla E \equiv \Delta$$

Hence, $E\nabla \equiv \nabla E \equiv \Delta$

(ii) $(1+\Delta)(1-\nabla) \equiv 1$

Solution: Let *y* = *f*(*x*) be a function of an independent variable *x* and the dependent variable *y*.

L.H.S. = $(1+\Delta)(1-\nabla)f(x) = (1+\Delta)[f(x) - \nabla f(x)]$

$$= (1+\Delta)[f(x) - \{f(x) - f(x-h)\}]$$

$$= (1+\Delta)f(x-h)$$

$$= f(x-h) + \Delta f(x-h)$$

$$= f(x-h) + \{f(x) - f(x-h)\}$$

$$= f(x)$$

$$\Longrightarrow (1-\Delta)(1-\nabla) \equiv 1$$

(iii) $(\Delta - \nabla) \equiv \Delta.\nabla$

Solution: Let *y* = *f*(*x*) be a function of an independent variable *x* and the dependent variable *y*.

R.H.S. = $\Delta.\nabla f(x) = \Delta[f(x) - f(x-h)]$

$$= \Delta f(x) - \Delta f(x-h)$$

$$= \Delta f(x) - [f(x) - f(x-h)]$$

$$= \Delta f(x) - \nabla f(x)$$

$$\Longrightarrow \Delta.\nabla \equiv (\Delta - \nabla)$$

Hence, $(\Delta - \nabla) \equiv \Delta.\nabla$

Example 4. $f(x)$ is polynomial in x with the following functional values: $f(2) = f(3) = 27$, $f(4) = 78$, $f(5) = 169$. Find the function $f(x)$.

Solution: Since four entries (i.e., four functional values) are given, $f(x)$ can be represented by a polynomial of degree 3. Let $f(x) = a+bx+cx^2+dx^3$, where a, b, c, d are constants to be determined. Now,

$$f(2) = a + 2b + 4c + 8d = 27$$

$$f(3) = a + 3b + 9c + 27d = 27$$

$$f(4) = a + 4b + 16c + 64d = 78$$

$$f(5) = a + 5b + 25c + 125d = 169$$

Solving these equations, we get, a = 221, b = $\frac{-1051}{6}$, $c = 42$ and d = $\frac{-11}{6}$

Therefore, the required function is $f(x) = 224 - \frac{1051}{6}x + 42x^2 - \frac{11}{6}x^3$.

Example 5. Compute the missing terms in the following table:

x	2	3	4	5	6	7	8
$f(x)$	0.135	-	0.111	0.100	-	0.082	0.074

Solution: Since five entries are given, $f(x)$ can be represented by a polynomial of degree four.

Hence $\Delta^4 f(x)$ = constant and

$$\Delta^5 f(x) = 0$$

$$\Rightarrow (E-1)^5 f(x) = 0$$

$$\Rightarrow (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)f(x) = 0$$

$$\Rightarrow E^5 f(x) - 5E^4 f(x) + 10E^3 f(x) - 10E^2 f(x) + 5Ef(x) - f(x) = 0$$

$$\Rightarrow f(x+5) - 5f(x+4) + 10f(x+3) - 10f(x+2) + 5f(x+1) - f(x) = 0$$

(as the interval of differencing is unity)

Now, putting $x = 2$ and $x = 3$, we get

$$f(7) - 5f(6) + 10f(5) - 10f(4) + 5f(3) - f(2) = 0$$

$$f(8) - 5f(7) + 10f(6) - 10f(5) + 5f(4) - f(3) = 0$$

Substituting the values of $f(8)$, $f(7)$, $f(5)$, $f(4)$ and $f(2)$ we get,

$$f(3) - f(6) = 0.0326 \text{ and } 10f(6) - f(3) = 0.781;$$

Solving we get $f(3) = 0.123$ and $f(6) = 0.0904$.

Example 6. For a function u_x, $u_0 + u_8 = 1.9243, u_1 + u_7 = 1.95590, u_2 + u_6 = 1.9843,$

$u_3 + u_5 = 1.9956$; find u_4

Solution: Since eight entries are given, u_x can be represented by a polynomial of degree 7, that is, $\Delta_x^7 u = \text{constant}$

$\Longrightarrow \Delta_x^8 u = 0$

$\Longrightarrow (E-1)^8 u_x = 0$ (since, $E \equiv 1 + \Delta$)

$\Longrightarrow (E^8 - 8_{C_1} E^7 + 8_{C_2} E^6 - 8_{C_3} E^5 + 8_{C_4} E^4 - 8_{C_5} E^3 + 8_{C_6} E^2 - 8_{C_7} E + 8_{C_8}) u_0 = 0$

$\Longrightarrow (E^8 - 8_{C_1} u_7 + 8_{C_2} u_6 - 8_{C_3} u_5 + 8_{C_4} u_4 - 8_{C_5} u_3 + 8_{C_6} u_2 - 8_{C_7} u_1 + 8_{C_8} u_0) = 0$

(assuming the interval of differencing to be unity)

$\Longrightarrow (u_8 + u_0) - 8(u_7 + u_1) + 28(u_6 + u_2) - 56(u_5 + u_3) + 70u_4 = 0$

$\Longrightarrow 1.9243 - 8 \times 1.9590 + 28 \times 1.9843 - 56 \times 1.9956 + 70u_4 = 0$

$\Longrightarrow u_4 = 0.999916$

Exercises

1. Evaluate the following:

   (i) $\left(\frac{\Delta^2}{E}\right)e^x$ (ii) $\frac{Ee^x}{\Delta^2 e^x}$ (iii) $\Delta\left(\frac{2^x}{(x+1)!}\right)$ (iv) $\frac{\Delta^2 x^3}{Ex}$

   (v) $\Delta\sin(2x)$ (vi) $\Delta\log(cx)$ (vii) $\Delta\cot(2^x)$ (viii) $\frac{\Delta}{E}\sin\,(2x)$

   (ix) $\Delta^3[(1-ax)(1-bx)(1-cx)]$ (x) $\frac{\Delta^2}{E}\sin(x+h)+\frac{\Delta^2\sin\,(x+h)}{Esin(x+h)}$

2. Find f (1.1) from the following table:

x	1	2	3	4	5
$f(x)$	7	12	29	64	123

3. Given, $u_0=1, u_1=11, u_2=21, u_3=29$, find $\Delta^4 u_0$.
4. Prove that $e^{-hD}\equiv 1-\nabla$

   [Hint: Already proved $\nabla E\equiv\Delta$, therefore, $E\equiv 1+\Delta\equiv 1+\nabla E, \Longrightarrow E-\nabla E$]

5. Find u_0, given u_0 =-3, $u_1=6, u_2=8, u_3=12$.
6. Given that $u_x$is a polynomial of second degree and u_0 =1, $u_1+u_2=10, u_3+u_4+u_5=$ 65. Find the value of u_{10}.