31. Consider the general first-order *linear* differential equation

$$\frac{dy}{dx} + p(x)y = q(x), \qquad (1.9.25)$$

where $p(x)$ and $q(x)$ are continuous functions on some interval (a, b).

(a) Rewrite Equation (1.9.25) in differential form, and show that an integrating factor for the resulting equation is

$$I(x) = e^{\int p(x)dx}. \qquad (1.9.26)$$

(b) Show that the general solution to Equation (1.9.25) can be written in the form

$$y(x) = I^{-1} \left\{ \int^x I(t)q(t)\, dt + c \right\},$$

where I is given in Equation (1.9.26), and c is an arbitrary constant.

1.10 Numerical Solution to First-Order Differential Equations

So far in this chapter we have investigated first-order differential equations geometrically via slope fields, and analytically by trying to construct exact solutions to certain types of differential equations. Certainly, for most first-order differential equations, it simply is not possible to find analytic solutions, since they will not fall into the few classes for which solution techniques are available. Our final approach to analyzing first-order differential equations is to look at the possibility of constructing a numerical approximation to the unique solution to the initial-value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0. \qquad (1.10.1)$$

We consider three techniques that give varying levels of accuracy. In each case, we generate a sequence of approximations $y_1, y_2, \ldots$ to the value of the exact solution at the points $x_1, x_2, \ldots$, where $x_{n+1} = x_n + h, n = 0, 1, \ldots$, and h is a real number. We emphasize that numerical methods do not generate a formula for the solution to the differential equation. Rather they generate a sequence of approximations to the value of the solution at specified points. Furthermore, if we use a sufficient number of points, then by plotting the points (x_i, y_i) and joining them with straight-line segments, we are able to obtain an overall approximation to the solution curve corresponding to the solution of the given initial-value problem. This is how the approximate solution curves were generated in the preceding sections via the computer algebra system Maple. There are many subtle ideas associated with constructing numerical solutions to initial-value problems that are beyond the scope of this text. Indeed, a full discussion of the application of numerical methods to differential equations is best left for a future course in numerical analysis.

Euler's Method

Suppose we wish to approximate the solution to the initial-value problem (1.10.1) at $x = x_1 = x_0 + h$, where h is small. The idea behind Euler's method is to use the tangent line to the solution curve through (x_0, y_0) to obtain such an approximation. (See Figure 1.10.1.)

The equation of the tangent line through (x_0, y_0) is

$$y(x) = y_0 + m(x - x_0),$$

where m is the slope of the curve at (x_0, y_0). From Equation (1.10.1), $m = f(x_0, y_0)$, so

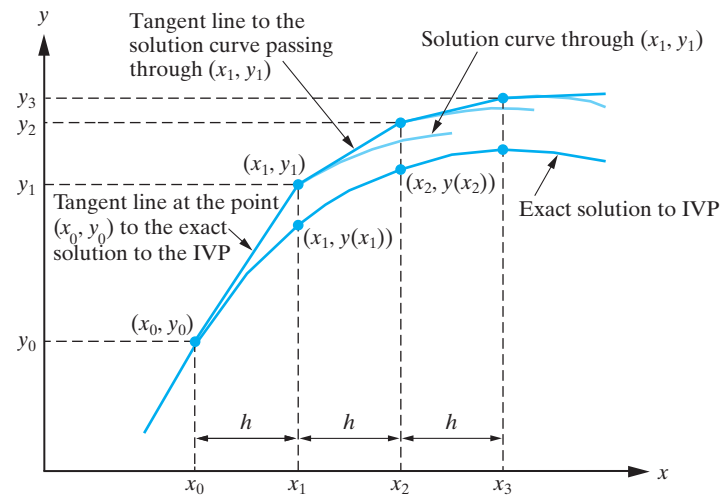$$y(x) = y_0 + f(x_0, y_0)(x - x_0).$$

Figure 1.10.1: Euler's method for approximating the solution to the initial-value problem $dy/dx = f(x, y)$, $y(x_0) = y_0$.

Setting $x = x_1$ in this equation yields the Euler approximation to the exact solution at x_1, namely,

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0),$$

which we write as

$$y_1 = y_0 + hf(x_0, y_0).$$

Now suppose we wish to obtain an approximation to the exact solution to the initial-value problem (1.10.1) at $x_2 = x_1 + h$. We can use the same idea, except we now use the tangent line to the solution curve through (x_1, y_1). From (1.10.1), the slope of this tangent line is $f(x_1, y_1)$, so that the equation of the required tangent line is

$$y(x) = y_1 + f(x_1, y_1)(x - x_1).$$

Setting $x = x_2$ yields the approximation

$$y_2 = y_1 + hf(x_1, y_1),$$

where we have substituted for $x_2 - x_1 = h$, to the solution to the initial-value problem at $x = x_2$. Continuing in this manner, we determine the sequence of approximations

$$y_{n+1} = y_n + hf(x_n, y_n), \qquad n = 0, 1, \dots$$

to the solution to the initial-value problem (1.10.1) at the points $x_{n+1} = x_n + h$. In summary, **Euler's method** for approximating the solution to the initial-value problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

at the points $x_{n+1} = x_0 + nh$ $(n = 0, 1, \dots)$ is

$$y_{n+1} = y_n + hf(x_n, y_n), \qquad n = 0, 1, \dots . \tag{1.10.2}$$

Example 1.10.1   Consider the initial-value problem

$$y' = y - x, \quad y(0) = \tfrac{1}{2}.$$

Use Euler's method with (a) $h = 0.1$ and (b) $h = 0.05$ to obtain an approximation to $y(1)$. Given that the exact solution to the initial-value problem is

$$y(x) = x + 1 - \tfrac{1}{2}e^x,$$

compare the errors in the two approximations to $y(1)$.

Solution: In this problem we have

$$f(x, y) = y - x, \qquad x_0 = 0, \qquad y_0 = \tfrac{1}{2}.$$

(a) Setting $h = 0.1$ in (1.10.2) yields

$$y_{n+1} = y_n + 0.1(y_n - x_n).$$

Hence,

$$y_1 = y_0 + 0.1(y_0 - x_0) = 0.5 + 0.1(0.5 - 0) = 0.55,$$
$$y_2 = y_1 + 0.1(y_1 - x_1) = 0.55 + 0.1(0.55 - 0.1) = 0.595.$$

Continuing in this manner, we generate the approximations listed in Table 1.10.1, where we have rounded the calculations to six decimal places.

n	x_n	y_n	Exact Solution	Absolute Error
1	0.1	0.55	0.547414	0.002585
2	0.2	0.595	0.589299	0.005701
3	0.3	0.6345	0.625070	0.009430
4	0.4	0.66795	0.654088	0.013862
5	0.5	0.694745	0.675639	0.019106
6	0.6	0.714219	0.688941	0.025278
7	0.7	0.725641	0.693124	0.032518
8	0.8	0.728205	0.687229	0.040976
9	0.9	0.721026	0.670198	0.050828
10	1.0	0.703129	0.640859	0.062270

Table 1.10.1: The results of applying Euler's method with $h = 0.1$ to the initial-value problem in Example 1.10.1.

We have also listed the values of the exact solution and the absolute value of the error. In this case, the approximation to $y(1)$ is $y_{10} = 0.703129$, with an absolute error of

$$|y(1) - y_{10}| = 0.062270. \tag{1.10.3}$$

(b) When $h = 0.05$, Euler's method gives

$$y_{n+1} = y_n + 0.05(y_n - x_n), \qquad n = 0, 1, \ldots, 19,$$

which generates the approximations given in Table 1.10.2, where we have listed only every other intermediate approximation. We see that the approximation to $y(1)$ is

$$y_{20} = 0.673351$$

and that the absolute error in this approximation is

$$|y(1) - y_{20}| = 0.032492.$$

n	x_n	y_n	Exact Solution	Absolute Error
2	0.1	0.54875	0.547414	0.001335
4	0.2	0.592247	0.589299	0.002948
6	0.3	0.629952	0.625070	0.004881
8	0.4	0.661272	0.654088	0.007185
10	0.5	0.685553	0.675639	0.009913
12	0.6	0.702072	0.688941	0.013131
14	0.7	0.710034	0.693124	0.016910
16	0.8	0.708563	0.687229	0.021333
18	0.9	0.696690	0.670198	0.026492
20	1.0	0.686525	0.640859	0.032492

Table 1.10.2: The results of applying Euler's method with $h = 0.05$ to the initial-value problem in Example 1.10.1.
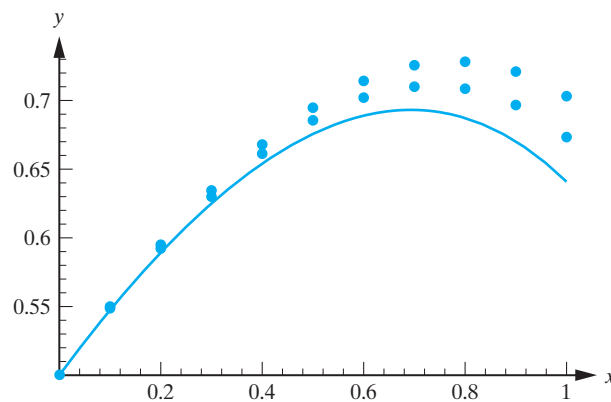

Figure 1.10.2: The exact solution to the initial-value problem considered in Example 1.10.1 and the two approximations obtained using Euler's method.

Comparing this with (1.10.3), we see that the smaller step size has led to a better approximation. In fact, it has almost halved the error at $y(1)$. In Figure 1.10.2 we have plotted the exact solution and the Euler approximations just obtained. □

In the preceding example we saw that halving the step size had the effect of essentially halving the error. However, even then the accuracy was not as good as we probably would have liked. Of course we could just keep decreasing the step size (provided we did not take h to be so small that round-off errors started to play a role) to increase the accuracy, but then the number of steps we would have to take would make the calculations very cumbersome. A better approach is to derive methods that have a higher order of accuracy. We will consider two such methods.

Modified Euler Method (Heun's Method)

The method that we consider here is an example of what is called a **predictor-corrector method**. The idea is to use the formula from Euler's method to obtain a first approximation to the solution $y(x_{n+1})$. We denote this approximation by y^*_{n+1}, so that

$$y^*_{n+1} = y_n + hf(x_n, y_n).$$

We now improve (or "correct") this approximation by once more applying Euler's method. But this time, we use the average of the slopes of the solution curves through (x_n, y_n) and (x_{n+1}, y^*_{n+1}). This gives

$$y_{n+1} = y_n + \tfrac{1}{2}h[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})].$$

As illustrated in Figure 1.10.3 for the case $n = 1$, we can interpret the modified Euler approximations as arising from first stepping to the point
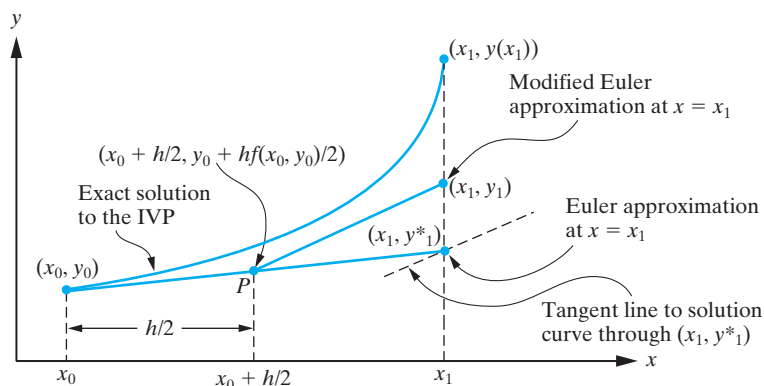

Figure 1.10.3: Derivation of the first step in the modified Euler method.

$$P\left(x_n + \frac{h}{2},\, y_n + \frac{hf(x_n, y_n)}{2}\right)$$

along the tangent line to the solution curve through (x_n, y_n) and then stepping from P to (x_{n+1}, y_{n+1}) along the line through P whose slope is $f(x_n, y^*_n)$.

In summary, the **modified Euler method** for approximating the solution to the initial-value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

at the points $x_{n+1} = x_0 + nh$ $(n = 0, 1, \dots)$ is

$$y_{n+1} = y_n + \tfrac{1}{2}h\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right],$$

where

$$y^*_{n+1} = y_n + hf(x_n, y_n), \qquad n = 0, 1, \dots.$$

Example 1.10.2   Apply the modified Euler method with $h = 0.1$ to determine an approximation to the solution to the initial-value problem

$$y' = y - x, \qquad y(0) = \tfrac{1}{2}$$

at $x = 1$.

Solution: Taking $h = 0.1$ and $f(x, y) = y - x$ in the modified Euler method yields

$$y_{n+1}^* = y_n + 0.1(y_n - x_n),$$
$$y_{n+1} = y_n + 0.05(y_n - x_n + y_{n+1}^* - x_{n+1}).$$

Hence,

$$y_{n+1} = y_n + 0.05 \{y_n - x_n + [y_n + 0.1(y_n - x_n)] - x_{n+1}\}.$$

That is,

$$y_{n+1} = y_n + 0.05(2.1 y_n - 1.1 x_n - x_{n+1}), \qquad n = 0, 1, \ldots, 9.$$

When $n = 0$,

$$y_1 = y_0 + 0.05(2.1 y_0 - 1.1 x_0 - x_1) = 0.5475,$$

and when $n = 1$,

$$y_2 = y_1 + 0.05(2.1 y_1 - 1.1 x_1 - x_2) = 0.5894875.$$

n	x_n	y_n	Exact Solution	Absolute Error
1	0.1	0.5475	0.547414	0.000085
2	0.2	0.589487	0.589299	0.000189
3	0.3	0.625384	0.625070	0.000313
4	0.4	0.654549	0.654088	0.000461
5	0.5	0.676277	0.675639	0.000637
6	0.6	0.689786	0.688941	0.000845
7	0.7	0.694213	0.693124	0.001089
8	0.8	0.688605	0.687229	0.001376
9	0.9	0.671909	0.670198	0.001711
10	1.0	0.642959	0.640859	0.002100

Table 1.10.3: The results of applying the modified Euler method with $h = 0.1$ to the initial-value problem in Example 1.10.2.

Continuing in this manner, we generate the results displayed in Table 1.10.3. From this table, we see that the approximation to $y(1)$ according to the modified Euler method is

$$y_{10} = 0.642960.$$

As seen in the previous example, the value of the exact solution at $x = 1$ is

$$y(1) = 0.640859.$$

Consequently, the absolute error in the approximation at $x = 1$ using the modified Euler approximation with $h = 0.1$ is

$$|y(1) - y_{10}| = 0.002100.$$

Comparing this with the results of the previous example, we see that the modified Euler method has picked up approximately one decimal place of accuracy when using a step size $h = 0.1$. This is indicative of the general result that the error in the modified Euler method behaves as order h^2 as compared to the order h behavior of the Euler method. In Figure 1.10.4 we have sketched the exact solution to the differential equation and the modified Euler approximation with $h = 0.1$.
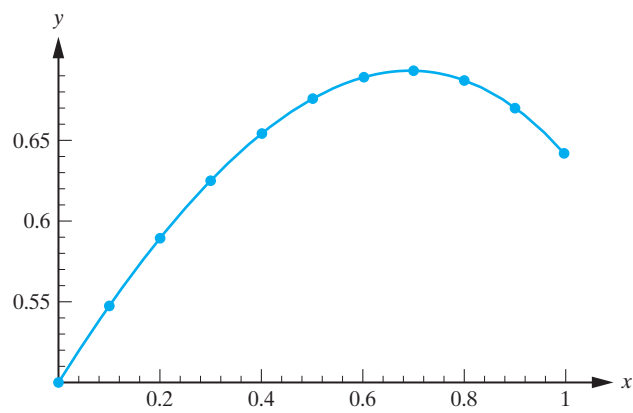
Figure 1.10.4: The exact solution to the initial-value problem in Example 1.10.2 and the approximations obtained using the modified Euler method with $h = 0.1$.

$\square$

Runge-Kutta Method of Order Four

The final method that we consider is somewhat more tedious to use in hand calculations, but is very easily programmed into a calculator or computer. It is a fourth-order method, which, in the case of a differential equation of the form $y' = f(x)$, reduces to Simpson's rule (which the reader has probably studied in a calculus course) for numerically evaluating definite integrals. Without justification, we state the algorithm.

The **fourth-order Runge-Kutta method** for approximating the solution to the initial-value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

at the points $x_{n+1} = x_0 + nh \ (n = 0, 1, \dots)$ is

$$y_{n+1} = y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$k_1 = hf(x_n, y_n), \quad k_2 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1), \quad k_3 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2),$$

$$k_4 = hf(x_{n+1}, y_n + k_3),$$

$n = 0, 1, 2, \dots$.

Remark   In the previous sections, we used Maple to generate slope fields and approximate solution curves for first-order differential equations. The solution curves were in fact generated using a Runge-Kutta approximation.

Example 1.10.3   Apply the fourth-order Runge-Kutta method with $h = 0.1$ to determine an approximation to the solution to the initial-value problem below at $x = 1$:

$$y' = y - x, \quad y(0) = \tfrac{1}{2}$$

Solution: We take $h = 0.1$, and $f(x, y) = y - x$ in the fourth-order Runge-Kutta method, and we need to determine y_{10}. First we determine k_1, k_2, k_3, k_4.

$$k_1 = 0.1 f(x_n, y_n) = 0.1(y_n - x_n),$$
$$k_2 = 0.1 f(x_n + 0.05, y_n + 0.5k_1) = 0.1(y_n + 0.5k_1 - x_n - 0.05),$$
$$k_3 = 0.1 f(x_n + 0.05, y_n + 0.5k_2) = 0.1(y_n + 0.5k_2 - x_n - 0.05),$$
$$k_4 = 0.1 f(x_{n+1}, y_n + k_3) = 0.1(y_n + k_3 - x_{n+1}).$$

When $n = 0$,

$$k_1 = 0.1(0.5) = 0.05,$$
$$k_2 = 0.1[0.5 + (0.5)(0.05) - 0.05] = 0.0475,$$
$$k_3 = 0.1[0.5 + (0.5)(0.0475) - 0.05] = 0.047375,$$
$$k_4 = 0.1(0.5 + 0.047375 - 0.1) = 0.0447375,$$

so that

$$y_1 = y_0 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.5 + \tfrac{1}{6}(0.2844875) = 0.54741458,$$

rounded to eight decimal places. Continuing in this manner, we obtain the results displayed in Table 1.10.4.

n	x_n	y_n	Exact Solution	Absolute Error
1	0.1	0.54741458	0.54741454	0.00000004
2	0.2	0.58929871	0.58929862	0.00000009
3	0.3	0.62507075	0.62507060	0.00000015
4	0.4	0.65408788	0.65408765	0.00000022
5	0.5	0.67563968	0.67563936	0.00000032
6	0.6	0.68894102	0.68894060	0.00000042
7	0.7	0.69312419	0.69312365	0.00000054
8	0.8	0.68723022	0.68722954	0.00000068
9	0.9	0.67019929	0.67019844	0.00000085
10	1.0	0.64086013	0.64085909	0.00000104

Table 1.10.4: The results of applying the fourth-order Runge-Kutta method with $h = 0.1$ to the initial-value problem in Example 1.10.3.

In particular, we see that the fourth-order Runge-Kutta method approximation to $y(1)$ is

$$y_{10} = 0.64086013,$$

so that

$$|y(1) - y_{10}| = 0.00000104.$$

Clearly this is an excellent approximation. If we increase the step size to $h = 0.2$, the corresponding approximation to $y(1)$ becomes

$$y_5 = 0.640874,$$

with absolute error

$$|y(1) - y_5| = 0.000015,$$

which is still very impressive. $\square$

Exercises for 1.10

Key Terms

Euler's method, Predictor-corrector method, Modified Euler method (Heun's method), Fourth-order Runge-Kutta method.

Skills

- Be able to apply Euler's method to approximate the solution to an initial-value problem at a point near the initial value x_0.

- Be able to use the modified Euler method (Heun's method) to approximate the solution to an initial-value problem at a point near the initial value x_0.

- Be able to use the fourth-order Runge-Kutta method to approximate the solution to an initial-value problem at a point near the initial value x_0.

True-False Review

For Questions 1–4, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. Generally speaking, the smaller the step size in Euler's method, the more accurate the approximation to the solution of an initial-value problem at a point near the initial value x_0.

2. Euler's method is based on the equation of a tangent line to a curve at a given point (x_0, y_0).

3. With each additional step that is taken in Euler's method, the error in the approximation obtained from the method can only grow in size.

4. At each step of length h, Heun's method requires two applications of Euler's method with step size $h/2$.

Problems

For Problems 1–5, use Euler's method with the specified step size to determine the solution to the given initial-value problem at the specified point.

1. $y' = 4y - 1$, $y(0) = 1$, $h = 0.05$, $y(0.5)$.

2. $y' = -\dfrac{2xy}{1 + x^2}$, $y(0) = 1$, $h = 0.1$, $y(1)$.

3. $y' = x - y^2$, $y(0) = 2$, $h = 0.05$, $y(0.5)$.

4. $y' = -x^2 y$, $y(0) = 1$, $h = 0.2$, $y(1)$.

5. $y' = 2xy^2$, $y(0) = 0.5$, $h = 0.1$, $y(1)$.

For Problems 6–10, use the modified Euler method with the specified step size to determine the solution to the given initial-value problem at the specified point. In each case, compare your answer to that obtained using Euler's method.

6. The initial-value problem in Problem 1.

7. The initial-value problem in Problem 2.

8. The initial-value problem in Problem 3.

9. The initial-value problem in Problem 4.

10. The initial-value problem in Problem 5.

For Problems 11–15, use the fourth-order Runge-Kutta method with the specified step size to determine the solution to the given initial-value problem at the specified point. In each case, compare your answer to that obtained using Euler's method.

11. The initial-value problem in Problem 1.

12. The initial-value problem in Problem 2.

13. The initial-value problem in Problem 3.

14. The initial-value problem in Problem 4.

15. The initial-value problem in Problem 5.

16. ⋄ Use the fourth-order Runge-Kutta method with $h = 0.5$ to approximate the solution to the initial-value problem

$$y' + \tfrac{1}{10}y = e^{-x/10} \cos x, \quad y(0) = 0$$

at the points $x = 0.5, 1.0, \ldots, 25$. Plot these points and describe the behavior of the corresponding solution.