## 5.4 Predictor-Corrector Methods

The Taylor and Runge-Kutta methods are examples of **one-step methods** for approximating the solution to initial-value problems. These methods use $w_i$ in the approximation $w_{i+1}$ to $y(t_{i+1})$ but do not involve any of the prior approximations $w_0, w_1, \dots, w_{i-1}$. Generally some functional evaluations of $f$ are required at intermediate points, but these are discarded as soon as $w_{i+1}$ is obtained.

Since $|y(t_j) - w_j|$ decreases in accuracy as $j$ increases, better approximation methods can be derived if, when approximating $y(t_{i+1})$, we include in the method some of the approximations prior to $w_i$. Methods developed using this philosophy are called **multistep methods**. In brief, one-step methods consider what occurred at only one previous step; multistep methods consider what happened at more than one previous step.

To derive a multistep method, suppose that the solution to the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad \text{for } a \le t \le b, \quad \text{with } y(a) = \alpha,$$

is integrated over the interval $[t_i, t_{i+1}]$. Then

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} y'(t)\ dt = \int_{t_i}^{t_{i+1}} f(t, y(t))\ dt,$$

and

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\ dt.$$

Since we cannot integrate $f(t, y(t))$ without knowing $y(t)$, which is the solution to the problem, we instead integrate an interpolating polynomial, $P(t)$, determined by some of the previously obtained data points $(t_0, w_0), (t_1, w_1), \dots, (t_i, w_i)$. When we assume, in addition, that $y(t_i) \approx w_i$, we have

$$y(t_{i+1}) \approx w_i + \int_{t_i}^{t_{i+1}} P(t)\ dt.$$

If $w_{m+1}$ is the first approximation generated by the multistep method, then we need to supply starting values $w_0, w_1, \dots, w_m$ for the method. These starting values are generated using a one-step Runge-Kutta method with the same error characteristics as the multistep method.

There are two distinct classes of multistep methods. In an **explicit method**, $w_{i+1}$ does not involve the function evaluation $f(t_{i+1}, w_{i+1})$. A method that does depend in part on $f(t_{i+1}, w_{i+1})$ is **implicit**.

Some of the explicit multistep methods, together with their required starting values and local error terms, are given next.

[Adams-Bashforth Two-Step Explicit Method]

$$\begin{aligned} w_0 &= \alpha,\ w_1 = \alpha_1, \\ w_{i+1} &= w_i + \frac{h}{2}[3f(t_i, w_i) - f(t_{i-1}, w_{i-1})], \end{aligned}$$

where $i = 1, 2, \ldots, N-1$, with local error $\frac{5}{12}y'''(\mu_i)h^3$ for some $\mu_i$ in $(t_{i-1}, t_{i+1})$.

[Adams-Bashforth Three-Step Explicit Method]

$$\begin{aligned} w_0 &= \alpha,\ w_1 = \alpha_1,\ w_2 = \alpha_2, \\ w_{i+1} &= w_i + \frac{h}{12}[23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})] \end{aligned}$$

where $i = 2, 3, \ldots, N-1$, with local error $\frac{3}{8}y^{(4)}(\mu_i)h^4$ for some $\mu_i$ in $(t_{i-2}, t_{i+1})$.

[Adams-Bashforth Four-Step Explicit Method]

$$\begin{aligned} w_0 &= \alpha,\ w_1 = \alpha_1,\ w_2 = \alpha_2,\ w_3 = \alpha_3, \\ w_{i+1} &= w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})] \end{aligned}$$

where $i = 3, 4, \ldots, N-1$, with local error $\frac{251}{720}y^{(5)}(\mu_i)h^5$ for some $\mu_i$ in $(t_{i-3}, t_{i+1})$.

[Adams-Bashforth Five-Step Explicit Method]

$$\begin{aligned} w_0 &= \alpha,\ w_1 = \alpha_1,\ w_2 = \alpha_2,\ w_3 = \alpha_3,\ w_4 = \alpha_4 \\ w_{i+1} &= w_i + \frac{h}{720}[1901f(t_i, w_i) - 2774f(t_{i-1}, w_{i-1}) \\ &\quad + 2616f(t_{i-2}, w_{i-2}) - 1274f(t_{i-3}, w_{i-3}) + 251f(t_{i-4}, w_{i-4})] \end{aligned}$$

where $i = 4, 5, \ldots, N-1$, with local error $\frac{95}{288}y^{(6)}(\mu_i)h^6$ for some $\mu_i$ in $(t_{i-4}, t_{i+1})$.

Implicit methods use $(t_{i+1}, f(t_{i+1}, y(t_{i+1})))$ as an additional interpolation node in the approximation of the integral

$$\int_{t_i}^{t_{i+1}} f(t, y(t))\ dt.$$

Some of the more common implicit methods are listed next. Notice that the local error of an $(m-1)$-step implicit method is $O(h^{m+1})$, the same as that of an $m$-step explicit method. They both use $m$ function evaluations, however, since the implicit methods use $f(t_{i+1}, w_{i+1})$, but the explicit methods do not.

[Adams-Moulton Two-Step Implicit Method]

$$\begin{aligned} w_0 &= \alpha,\ w_1 = \alpha_1 \\ w_{i+1} &= w_i + \frac{h}{12}[5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})] \end{aligned}$$

where $i = 1, 2, \ldots, N-1$, with local error $-\frac{1}{24}y^{(4)}(\mu_i)h^4$ for some $\mu_i$ in $(t_{i-1}, t_{i+1})$.

[Adams-Moulton Three-Step Implicit Method]

$$\begin{aligned} w_0 &= \alpha,\ w_1 = \alpha_1,\ w_2 = \alpha_2, \\ w_{i+1} &= w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})], \end{aligned}$$

where $i = 2, 3, \ldots, N-1$, with local error $-\frac{19}{720}y^{(5)}(\mu_i)h^5$ for some $\mu_i$ in $(t_{i-2}, t_{i+1})$.

[Adams-Moulton Four-Step Implicit Method]

$$\begin{aligned} w_0 &= \alpha,\ w_1 = \alpha_1,\ w_2 = \alpha_2,\ w_3 = \alpha_3, \\ w_{i+1} &= w_i + \frac{h}{720}[251f(t_{i+1}, w_{i+1}) + 646f(t_i, w_i) - 246f(t_{i-1}, w_{i-1}) \\ &\quad + 106f(t_{i-2}, w_{i-2}) - 19f(t_{i-3}, w_{i-3})] \end{aligned}$$

where $i = 3, 4, \ldots, N-1$, with local error $-\frac{3}{160}y^{(6)}(\mu_i)h^6$ for some $\mu_i$ in $(t_{i-3}, t_{i+1})$.

It is interesting to compare an $m$-step Adams-Bashforth explicit method to an $(m-1)$-step Adams-Moulton implicit method. Both require $m$ evaluations of $f$ per step, and both have the terms $y^{(m+1)}(\mu_i)h^{m+1}$ in their local errors. In general, the coefficients of the terms involving $f$ in the approximation and those in the local error are smaller for the implicit methods than for the explicit methods. This leads to smaller truncation and round-off errors for the implicit methods.

EXAMPLE 1 Consider the initial-value problem

$$y' = y - t^2 + 1, \quad \text{for } 0 \le t \le 2, \quad \text{with } y(0) = 0.5,$$

and the approximations given by the explicit Adams-Bashforth Four-Step method and the implicit Adams-Moulton Three-Step method, both using $h = 0.2$. The explicit Adams-Bashforth method has the difference equation

$$w_{i+1} = w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})],$$

for $i = 3, 4, \ldots, 9$. When simplified using $f(t, y) = y - t^2 + 1, h = 0.2$, and $t_i = 0.2i$, it becomes

$$w_{i+1} = \frac{1}{24}[35w_i - 11.8w_{i-1} + 7.4w_{i-2} - 1.8w_{i-3} - 0.192i^2 - 0.192i + 4.736].$$

The implicit Adams-Moulton method has the difference equation

$$w_{i+1} = w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1})] + f(t_{i-2}, w_{i-2})],$$

for $i = 2, 3, \ldots, 9$. This reduces to

$$w_{i+1} = \frac{1}{24}[1.8w_{i+1} + 27.8w_i - w_{i-1} + 0.2w_{i-2} - 0.192i^2 - 0.192i + 4.736].$$

To use this method explicitly, we can solve for $w_{i+1}$, which gives

$$w_{i+1} = \frac{1}{22.2}[27.8w_i - w_{i-1} + 0.2w_{i-2} - 0.192i^2 - 0.192i + 4.736]$$

for $i = 2, 3, \ldots, 9$. The results in Table 5.9 were obtained using the exact values from $y(t) = (t+1)^2 - 0.5e^t$ for $\alpha, \alpha_1$, $\alpha_2$, and $\alpha_3$ in the explicit Adams-Bashforth case and for $\alpha$, $\alpha_1$, and $\alpha_2$ in the implicit Adams-Moulton case. □

Table 5.9

| $t_i$ | $y_i = y(t_i)$ | Adams Bashforth $w_i$ | Error | Adams Moulton $w_i$ | Error |
|---|---|---|---|---|---|
| 0.0 | 0.5000000 | 0.5000000 | 0 | 0.5000000 | 0 |
| 0.2 | 0.8292986 | 0.8292986 | 0.0000000 | 0.8292986 | 0.0000000 |
| 0.4 | 1.2140877 | 1.2140877 | 0.0000000 | 1.2140877 | 0.0000000 |
| 0.6 | 1.6489406 | 1.6489406 | 0.0000000 | 1.6489341 | 0.0000065 |
| 0.8 | 2.1272295 | 2.1273124 | 0.0000828 | 2.1272136 | 0.0000160 |
| 1.0 | 2.6408591 | 2.6410810 | 0.0002219 | 2.6408298 | 0.0000293 |
| 1.2 | 3.1799415 | 3.1803480 | 0.0004065 | 3.1798937 | 0.0000478 |
| 1.4 | 3.7324000 | 3.7330601 | 0.0006601 | 3.7323270 | 0.0000731 |
| 1.6 | 4.2834838 | 4.2844931 | 0.0010093 | 4.2833767 | 0.0001071 |
| 1.8 | 4.8151763 | 4.8166575 | 0.0014812 | 4.8150236 | 0.0001527 |
| 2.0 | 5.3054720 | 5.3075838 | 0.0021119 | 5.3052587 | 0.0002132 |

In Example 1, the implicit Adams-Moulton method gave considerably better results than the explicit Adams-Bashforth method of the same order. Although this is generally the case, the implicit methods have the inherent weakness of first having to convert the method algebraically to an explicit representation for $w_{i+1}$. That this procedure can become difficult, if not impossible, can be seen by considering the elementary initial-value problem

$$y' = e^y, \quad \text{for } 0 \leq t \leq 0.25, \quad \text{with } y(0) = 1.$$

Since $f(t, y) = e^y$, the Adams-Moulton Three-Step method has

$$w_{i+1} = w_i + \frac{h}{24}[9e^{w_{i+1}} + 19e^{w_i} - 5e^{w_{i-1}} + e^{w_{i-2}}]$$

as its difference equation, and this equation cannot be solved explicitly for $w_{i+1}$. We could use Newton's method or the Secant method to approximate $w_{i+1}$, but this complicates the procedure considerably.

In practice, implicit multistep methods are not used alone. Rather, they are used to improve approximations obtained by explicit methods. The combination of an explicit and implicit technique is called a **predictor-corrector method**. The explicit method predicts an approximation, and the implicit method corrects this prediction.

Consider the following fourth-order method for solving an initial-value problem. The first step is to calculate the starting values $w_0, w_1, w_2$, and $w_3$ for the explicit Adams-Bashforth Four-Step method. To do this, we use a fourth-order one-step method, specifically, the Runge-Kutta method of order 4. The next step is to calculate an approximation, $w_4^{(0)}$, to $y(t_4)$ using the explicit Adams-Bashforth Four-Step method as predictor:

$$w_4^{(0)} = w_3 + \frac{h}{24}[55f(t_3, w_3) - 59f(t_2, w_2) + 37f(t_1, w_1) - 9f(t_0, w_0)].$$

This approximation is improved by use of the implicit Adams-Moulton Three-Step method as corrector:

$$w_4^{(1)} = w_3 + \frac{h}{24}\left[9f\left(t_4, w_4^{(0)}\right) + 19f(t_3, w_3) - 5f(t_2, w_2) + f(t_1, w_1)\right].$$

The value $w_4 \equiv w_4^{(1)}$ is now used as the approximation to $y(t_4)$. Then the technique of using the Adams-Bashforth method as a predictor and the Adams-Moulton method as a corrector is repeated to find $w_5^{(0)}$ and $w_5^{(1)}$, the initial and final approximations to $y(t_5)$. This process is continued until we obtain an approximation to $y(t_N) = y(b)$.

The program PRCORM53 is based on the Adams-Bashforth Four-Step method as predictor and one iteration of the Adams-Moulton Three-Step method as corrector, with the starting values obtained from the Runge-Kutta method of order 4.

EXAMPLE 2 Table 5.10 lists the results obtained by using the program PRCORM53 for the initial-value problem

$$y' = y - t^2 + 1, \quad \text{for } 0 \le t \le 2, \quad \text{with } y(0) = 0.5,$$

with $N = 10$. □

Table 5.10

| $t_i$ | $y_i = y(t_i)$ | $w_i$ | Error $\lvert y_i - w_i\rvert$ |
|---|---|---|---|
| 0.0 | 0.5000000 | 0.5000000 | 0 |
| 0.2 | 0.8292986 | 0.8292933 | 0.0000053 |
| 0.4 | 1.2140877 | 1.2140762 | 0.0000114 |
| 0.6 | 1.6489406 | 1.6489220 | 0.0000186 |
| 0.8 | 2.1272295 | 2.1272056 | 0.0000239 |
| 1.0 | 2.6408591 | 2.6408286 | 0.0000305 |
| 1.2 | 3.1799415 | 3.1799026 | 0.0000389 |
| 1.4 | 3.7324000 | 3.7323505 | 0.0000495 |
| 1.6 | 4.2834838 | 4.2834208 | 0.0000630 |
| 1.8 | 4.8151763 | 4.8150964 | 0.0000799 |
| 2.0 | 5.3054720 | 5.3053707 | 0.0001013 |

Other multistep methods can be derived using integration of interpolating polynomials over intervals of the form $[t_j, t_{i+1}]$ for $j \le i - 1$, where some of the data points are omitted. Milne's method is an explicit technique that results when a Newton Backward-Difference interpolating polynomial is integrated over $[t_{i-3}, t_{i+1}]$.

[Milne's Method]

$$w_{i+1} = w_{i-3} + \frac{4h}{3}[2f(t_i, w_i) - f(t_{i-1}, w_{i-1}) + 2f(t_{i-2}, w_{i-2})],$$

where $i = 3, 4, \ldots, N-1$, with local error $\frac{14}{45}h^5 y^{(5)}(\mu_i)$ for some $\mu_i$ in $(t_{i-3}, t_{i+1})$.

This method is used as a predictor for an implicit method called Simpson's method. Its name comes from the fact that it can be derived using Simpson's rule for approximating integrals.

[Simpson's Method]

$$w_{i+1} = w_{i-1} + \frac{h}{3}[f(t_{i+1}, w_{i+1}) + 4f(t_i, w_i) + f(t_{i-1}, w_{i-1})],$$

where $i = 1, 2, \ldots, N-1$, with local error $-\frac{1}{90}h^5 y^{(5)}(\mu_i)$ for some $\mu_i$ in $(t_{i-1}, t_{i+1})$.

Although the local error involved with a predictor-corrector method of the Milne-Simpson type is generally smaller than that of the Adams-Bashforth-Moulton method, the technique has limited use because of round-off error problems, which do not occur with the Adams procedure.

The Adams-Bashforth Four-Step Explicit method is available in Maple using `dsolve` with the `numeric` and `classical` options. Enter the command

```
>eq:= D(y)(t)=y(t)-t^2+1;
```

to define the differential equation, and specify the initial condition with

```
>init:= y(0)=0.5;
```

The Adams-Bashforth method is activated with the command

```
>g:=dsolve({eq,init},numeric,method=classical[adambash],y(t),stepsize=0.2);
```

To approximate $y(t)$ using $g(t)$ at specific values of $t$, for example at $t = 2.0$, enter the command

```
>g(2.0);
```

In a similar manner, the predictor-corrector method using the Adams-Bashforth Four-Step Explicit method with the Adams-Moulton Three-Step Implicit method is called using

```
>g:=dsolve({eq,init},numeric,method=classical[abmoulton],y(t),stepsize=0.2);
```

**E**XERCISE SET 5.4

1. Use all the Adams-Bashforth methods to approximate the solutions to the following initial-value problems. In each case use exact starting values and compare the results to the actual values.

   (a) $y' = te^{3t} - 2y$, for $0 \le t \le 1$, with $y(0) = 0$ and $h = 0.2$; actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$.

   (b) $y' = 1 + (t - y)^2$, for $2 \le t \le 3$, with $y(2) = 1$ and $h = 0.2$; actual solution $y(t) = t + 1/(1 - t)$.

   (c) $y' = 1 + \dfrac{y}{t}$, for $1 \le t \le 2$, with $y(1) = 2$ and $h = 0.2$; actual solution $y(t) = t \ln t + 2t$.

   (d) $y' = \cos 2t + \sin 3t$, for $0 \le t \le 1$ with $y(0) = 1$ and $h = 0.2$; actual solution $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$.

2. Use all the Adams-Moulton methods to approximate the solutions to the Exercises 1(a), 1(c), and 1(d). In each case use exact starting values and explicitly solve for $w_{i+1}$. Compare the results to the actual values.

3. Use each of the Adams-Bashforth methods to approximate the solutions to the following initial-value problems. In each case use starting values obtained from the Runge-Kutta method of order 4. Compare the results to the actual values.

   (a) $y' = \dfrac{y}{t} - \left(\dfrac{y}{t}\right)^2$, for $1 \le t \le 2$, with $y(1) = 1$ and $h = 0.1$; actual solution $y(t) = t/(1 + \ln t)$.

   (b) $y' = 1 + \dfrac{y}{t} + \left(\dfrac{y}{t}\right)^2$, for $1 \le t \le 3$, with $y(1) = 0$ and $h = 0.2$; actual solution $y(t) = t\tan(\ln t)$.

   (c) $y' = -(y+1)(y+3)$, for $0 \le t \le 2$, with $y(0) = -2$ and $h = 0.1$; actual solution $y(t) = -3 + 2/(1 + e^{-2t})$.

   (d) $y' = -5y + 5t^2 + 2t$, for $0 \le t \le 1$, with $y(0) = 1/3$ and $h = 0.1$; actual solution $y(t) = t^2 + \frac{1}{3}e^{-5t}$.

4. Use the predictor-corrector method based on the Adams-Bashforth Four-Step method and the Adams-Moulton Three-Step method to approximate the solutions to the initial-value problems in Exercise 1.

5. Use the predictor-corrector method based on the Adams-Bashforth Four-Step method and the Adams-Moulton Three-Step method to approximate the solutions to the initial-value problem in Exercise 3.

6. The initial-value problem

$$y' = e^y, \quad \text{for } \ 0 \leq t \leq 0.20, \quad \text{with } y(0) = 1$$

has solution

$$y(t) = 1 - \ln(1 - et).$$

Applying the Adams-Moulton Three-Step method to this problem is equivalent to finding the fixed point $w_{i+1}$ of

$$g(w) = w_i + \frac{h}{24}[9e^w + 19e^{w_i} - 5e^{w_{i-1}} + e^{w_{i-2}}].$$

   (a) With $h = 0.01$, obtain $w_{i+1}$ by functional iteration for $i = 2, \ldots, 19$ using exact starting values $w_0, w_1$, and $w_2$. At each step use $w_i$ to initially approximate $w_{i+1}$.

   (b) Will Newton's method speed the convergence over functional iteration?

7. Use the Milne-Simpson Predictor-Corrector method to approximate the solutions to the initial-value problems in Exercise 3.

8. Use the Milne-Simpson Predictor-Corrector method to approximate the solution to

$$y' = -5y, \quad \text{for } 0 \leq t \leq 2, \quad \text{with } y(0) = e,$$

with $h = 0.1$. Repeat the procedure with $h = 0.05$. Are the answers consistent with the local error?