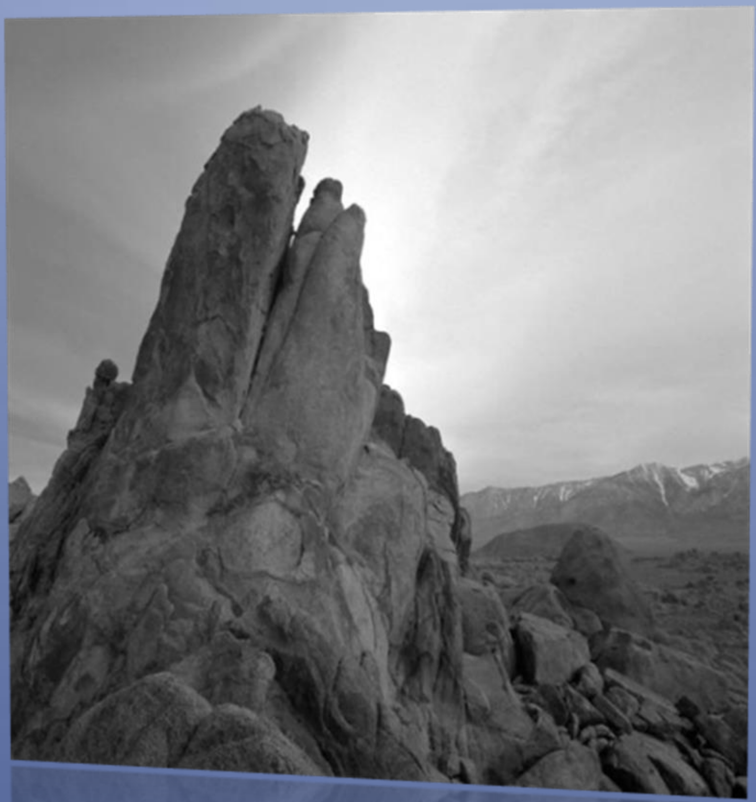

DESIGN AND ANALYSIS OF EXPERIMENT I

Lecture Notes

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Experimental Design

Experimental design is the design of any information gathering exercises where variation is present whether under full control of the experimenter or not. However, in statistics these terms are usually used for controlled experiments. In the design of experiments, the experimenter is often interested in the effect of some process or intervention on some objects, which may be people, parts of people, groups of people, plants, animals, materials etc. Design of experiment is thus a discipline that has a very broad application across all the natural and social sciences.

Design means a statistical plan to get observations relevant to the experiment. The plan usually means

1- Selection of Treatment
2- Specifications of Layout
3- Allocation of Treatment
4- Collection of data analysis
Thus design is logical structure of experiment for the collection of data in an experimental way. A good experimental design is one i.e.
1- Experimental conditions are free from any systematic error.
2- Minimum experimental error for free cost.
3- The inference have wide range of validity.
There are two types of experimental design.
1- Systematic Design
2- Randomized Design
The basic randomized designs are

- Completely Randomized Design (CRD)
- Randomized Complete Block Design (RCBD)
- Latin Square Design (LSD)

Experiment

The word experiment is used in quite different sense to mean an investigation where the system under study is under the control of the investigator. This means that the individuals or material investigated, the nature of the treatments or manipulations under study and the measurement procedures used are all settled, in their important features at least by the investigator e.g. clinical study, fertilizer treatments.

Observational Study

The allocation of individuals to treatment groups, are outside the investigator's control e.g. Penal of interviewers, payment, cash withdrawals

Experimental Unit

A unit or experimental unit in a statistical analysis refers to one member of a set of entities being studied. It is the material source for the mathematical abstraction of a random variable. Common examples of a unit would be a single person, animal, plant or manufactured item that belongs to larger collection of such entities being studied. More formally they correspond to the smallest subdivision of the experimental material such that any two different experimental units might receive different treatments.

Treatments

The treatments are clearly defined procedures one of which is to be applied to each experimental unit. In some cases the treatments are an unstructured set of two or more qualitatively different procedures. In others including many investigations in the physical sciences, the treatments are defined by the levels of one or more quantitative variables such as the amounts per square meter of the constituents nitrogen, potash and potassium etc.

Response Measurement

The response measurement specifies the criterion in terms of which the comparison of treatments is to be effected. In many applications there will be several such measures.

Experimental Error

Experimental material is subject to variation. Experimental error is the measure of variation which exists among observations on experimental units treated alike. There are two source of variation

1. There is inherent variability persists in the experimental material.
2. Variation due to lack of uniformity in the physical conduct of experiment.

Efforts are made to improve the power of the test, decrease the size of the confidence interval and others good results.

Principles of Experimental Design

The three principles of experimental design are:

- **randomization,** to ensure that this estimate is statistically valid;
- **replication,** to provide an estimate of experimental error;
- **local control,** to reduce experimental error by making the experiment more efficient

Randomization

Randomization is the random process of assigning treatments to the experimental units. The random process implies that every possible allotment of treatments has the same probability. An experimental unit is the smallest division of the experimental material and a treatment means an experimental condition whose effect is to be measured and compared. The purpose of randomization is to remove bias and other sources of extraneous variation, which are not controllable. Another advantage of randomization is that it forms the basis of any valid statistical test. Hence the treatments must be assigned at random to the experimental units. Randomization is usually done by drawing numbered cards from a well shuffled pack of cards or by drawing numbered balls from a well shaken container or by using tables of random numbers.

Replication

By replication we means that repetition of the basic experiments. For example, If we need to compare grain yield of two varieties of wheat then each variety is applied to more than one experimental units. The number of times these are applied on experimental units is called their number of replication. A replication is used:

i. to secure more accurate estimate of experimental error, a term which represents the differences that would be observed if the same treatments were applied several times to the same experimental units.

ii. to decrease the experimental error and thereby to increase precision, which is the measure of variability of experimental error

iii. to obtain more precise estimate of the mean effect of a treatment, since $\sigma_{\bar{y}} = \frac{\sigma^2}{n}$ where n denotes the number of replications.

Local Control

It has been observed that all extraneous source of variation are not removed by randomization and replication, i.e. unable to control extraneous source of variation. Thus we need to a refinement in the experimental technique. In other words we need to choose a design in such a way that all extraneous source of variation are brought under control. For this purpose we make use of local control, a term referring to the amount of (i) balancing, (ii) blocking and (iii) grouping of experimental units.

Balancing:

Balancing means that the treatment should be assigned to the experimental units in such a way that the result is a balanced arrangement of treatment.

Blocking:

Blocking means that the like experimental units should be collected together to far relatively homogeneous groups. A block is also a replicate.

The main objective/ purpose of local control is to increase the efficiency of experimental design by decreasing the experimental error.

Completely Randomized Design(CRD)

A completely randomized design, which is the simplest type of the basic designs, may be defined as a design in which the treatments are assigned to experimental units completely random, that is the randomization is done without restrictions. The design is completely flexible i.e. any number of treatments and any number of units per treatment may be used. Moreover, the number of units per treatment need not be equal. A CRD is considered to be more useful where

i. the experimental units are homogenous
ii. the experiments are small such as laboratory experiments
iii. some experimental units are likely to be destroyed or fail to respond

Experimental Layout

The layout of an experiment is the actual placement of the treatments on the experimental units, which may certain to time, space or type of material. Suppose we have k treatments and the experimental material is divided into n experimental units. We shall then assign the k treatments at random to the n experimental units in such a way that treatment τ_i is applied r_j times, with $\sum r_i = n$. When each treatment is applied the same number of times, then $r_1 = r_2 = \cdots = r_t = r$ and $\sum r_i = rt = n$. Usually each treatment is applied equal number of times.

An example of the experimental layout for a CRD using four treatments A, B, C, D, each repeated 3 times is given below:

C	A	B	D
C	B	C	A
A	D	D	B

The result or response of a treatment which may be a real yield, the gain in weight, the ability etc. is generally called yield and is represented by the letter Y.

Statistical Model and Analysis

Each observation may be written in the form

$$Y_{ij} = \mu + \tau_i + e_{ij}, \quad \begin{cases} j = 1,2,\dots,r \\ i = 1,2,\dots,t \end{cases}$$

where μ represents the true mean effect, τ_i represents the effects of the treatment i and e_{ij} denotes the random error, normally and independently distributed with mean 0 and variance σ^2.

Formulation of hypotheses:
$H_0: \tau_i = 0$
$H_0: \tau_i \neq 0$ for all i

Level of significance:

$$\alpha = 0.05$$

Test Statistic:

$$F = \frac{s_t^2}{s_e^2}$$

S.O.V	d.f	SS	MS	F
Treatments	$t-1$	SST	$s_t^2 = \frac{SST}{t-1}$	$F = \frac{s_t^2}{s_e^2}$
Error	$n-t$	SSE	$s_e^2 = \frac{SSE}{n-t}$	
Total	$n-1$			

$$C.F = \frac{Y_{..}^2}{n}, \quad TSS = \sum\sum Y_{ij}^2 - C.F, \quad SST = \frac{1}{r}\sum Y_{i.}^2 - C.F$$

$$SSE = TSS - SST$$

C.R:

$$F \geq F_{(\alpha, t-1, n-t)}$$

Conclusion:
If calculated value of F falls in the critical region then we reject H_0.

Advantages of CRD

- Very flexible design (i.e. number of treatments and replicates is only limited by the available number of experimental units).
- Statistical analysis is simple compared to other designs.
- Loss of information due to missing data is small compared to other designs due to the larger number of degrees of freedom for the error source of variation.

Disadvantages of CRD

- If experimental units are not homogeneous and you fail to minimize this variation using blocking, there may be a loss of precision.
- Usually the least efficient design unless experimental units are homogeneous.
- Not suited for a large number of treatments.

Estimation of Parameters

We estimate the parameters of the statistical model of CRD using OLS technique.

$$S = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(Y_{ij} - \hat{\mu} - \hat{\tau}_i\right)^2$$

Differentiate w.r.t $\hat{\mu}$

$$\frac{\partial S}{\partial \hat{\mu}} = -2\sum_{i=1}^{t}\sum_{j=1}^{r}\left(Y_{ij} - \hat{\mu} - \hat{\tau}_i\right)(-1)$$

Put $\frac{\partial S}{\partial \hat{\mu}} = 0$, we get

$$2\sum_{i=1}^{t}\sum_{j=1}^{r}\left(Y_{ij}-\hat{\mu}-\hat{\tau}_i\right)=0$$

$$\sum_{i=1}^{t}\sum_{j=1}^{r}\left(Y_{ij}-\hat{\mu}-\hat{\tau}_i\right)=0$$

$$\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}-tr\hat{\mu}-r\sum_{i=1}^{t}\hat{\tau}_i=0$$

For unique solutions put $\sum_{i=1}^{t}\hat{\tau}_i = 0$

$$\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}-tr\hat{\mu}-r(0)=0$$

$$\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}-tr\hat{\mu}=0$$

$$tr\hat{\mu}=\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}$$

$$\hat{\mu}=\frac{\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}}{tr}$$

$$\hat{\mu}=\frac{Y_{..}}{tr}=\bar{\bar{Y}}$$

$$S=\sum_{j=1}^{r}\left(Y_{1j}-\hat{\mu}-\hat{\tau}_1\right)^2+\sum_{j=1}^{r}\left(Y_{2j}-\hat{\mu}-\hat{\tau}_2\right)^2+\cdots+\sum_{j=1}^{r}\left(Y_{ij}-\hat{\mu}-\hat{\tau}_i\right)^2+\cdots+\sum_{j=1}^{r}\left(Y_{tj}-\hat{\mu}-\hat{\tau}_t\right)^2$$

Differentiate w.r.t $\tau_1, \tau_2, \ldots, \tau_i, \ldots \tau_t$

$$\frac{\partial S}{\partial \hat{\tau}_1}=-2\sum_{j=1}^{r}\left(Y_{1j}-\hat{\mu}-\hat{\tau}_1\right)(-1)$$

Put $\frac{\partial S}{\partial \hat{\tau}_1}=0$, we get

$$2\sum_{j=1}^{r}\left(Y_{1j}-\hat{\mu}-\hat{\tau}_1\right)=0$$

$$\sum_{j=1}^{r}\left(Y_{1j}-\hat{\mu}-\hat{\tau}_1\right)=0$$

$$\sum_{j=1}^{r}Y_{1j}-r\hat{\mu}-r\hat{\tau}_1=0$$

$$\sum_{j=1}^{r} Y_{1j} = r\hat{\mu} + r\hat{\tau}_1$$

$$Y_{1.} = r\hat{\mu} + r\hat{\tau}_1 \qquad (1)$$

$$\frac{\partial S}{\partial \hat{\tau}_2} = -2\sum_{j=1}^{r}\left(Y_{2j} - \hat{\mu} - \hat{\tau}_2\right)(-1)$$

Put $\frac{\partial S}{\partial \hat{\tau}_2} = 0$, we get

$$2\sum_{j=1}^{r}\left(Y_{2j} - \hat{\mu} - \hat{\tau}_2\right) = 0$$

$$\sum_{j=1}^{r}\left(Y_{2j} - \hat{\mu} - \hat{\tau}_2\right) = 0$$

$$\sum_{j=1}^{r} Y_{2j} - r\hat{\mu} - r\hat{\tau}_2 = 0$$

$$\sum_{j=1}^{r} Y_{2j} = r\hat{\mu} + r\hat{\tau}_2$$

$$Y_{2.} = r\hat{\mu} + r\hat{\tau}_2 \qquad (2)$$

$$\frac{\partial S}{\partial \hat{\tau}_i} = -2\sum_{j=1}^{r}\left(Y_{ij} - \hat{\mu} - \hat{\tau}_i\right)(-1)$$

Put $\frac{\partial S}{\partial \hat{\tau}_i} = 0$, we get

$$2\sum_{j=1}^{r}\left(Y_{ij} - \hat{\mu} - \hat{\tau}_i\right) = 0$$

$$\sum_{j=1}^{r}\left(Y_{ij} - \hat{\mu} - \hat{\tau}_i\right) = 0$$

$$\sum_{j=1}^{r} Y_{ij} - r\hat{\mu} - r\hat{\tau}_i = 0$$

$$\sum_{j=1}^{r} Y_{ij} = r\hat{\mu} + r\hat{\tau}_i$$

$$Y_{i.} = r\hat{\mu} + r\hat{\tau}_i \qquad (3)$$

$$\frac{\partial S}{\partial \hat{\tau}_t} = -2\sum_{j=1}^{r}\left(Y_{tj} - \hat{\mu} - \hat{\tau}_t\right)(-1)$$

Put $\frac{\partial S}{\partial \hat{\tau}_t} = 0$, we get

$$2\sum_{j=1}^{r}\left(Y_{tj} - \hat{\mu} - \hat{\tau}_t\right) = 0$$

$$\sum_{j=1}^{r}(Y_{tj} - \hat{\mu} - \hat{\tau}_t) = 0$$

$$\sum_{j=1}^{r} Y_{tj} - r\hat{\mu} - r\hat{\tau}_t = 0$$

$$\sum_{j=1}^{r} Y_{tj} = r\hat{\mu} + r\hat{\tau}_t$$

$$Y_{t.} = r\hat{\mu} + r\hat{\tau}_t \qquad \text{(t)}$$

Adding equation (1),(2),(i),….,(t)

$$\begin{aligned} Y_{1.} &= r\hat{\mu} + r\hat{\tau}_1 && (1) \\ Y_{2.} &= r\hat{\mu} + r\hat{\tau}_2 && (2) \\ &\vdots \\ Y_{i.} &= r\hat{\mu} + r\hat{\tau}_i && (i) \\ &\vdots \\ Y_{t.} &= r\hat{\mu} + r\hat{\tau}_t && (t) \end{aligned}$$

$$.\sum_{i=1}^{t} Y_{i.} = tr\hat{\mu} + r\sum_{i=1}^{t}\hat{\tau}_i$$

From equation (i)

$$Y_{i.} = r\hat{\mu} + r\hat{\tau}_i$$

$$\hat{\tau}_i = \frac{Y_{i.}}{r} - \frac{r\hat{\mu}}{r}$$

$$\hat{\tau}_i = \bar{Y}_{i.} - \bar{\bar{Y}}$$

Expected Mean square Error for CRD

Fixed Effect Model

Assumptions:

1. $E(e_{ij}) = 0$
2. $E(e_{ij}e_{gh}) = 0$
3. $\sum_{i=1}^{t}\tau_i = 0$

$$Y_{ij} = \mu + \tau_i + e_{ij}$$

$$SSE = TSS - SST$$

$$TSS = \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 - C.F$$

$$C.F = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}\right)^2}{tr} = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r}(\mu + \tau_i + e_{ij})\right)^2}{tr}$$

$$= \frac{\left(tr\mu + r\sum_{i=1}^{t}\tau_i + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{\left(tr\mu + r(0) + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{\left(tr\mu + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{(tr\mu)^2 + \left(\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2 + 2tr\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}}{tr}$$

$$= \frac{t^2r^2\mu^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} {e_{ij}}^2 + 2tr\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh}}{tr}$$

$$= \frac{t^2r^2\mu^2}{tr} + \frac{\sum_{i=1}^{t}\sum_{j=1}^{r} {e_{ij}}^2}{tr} + \frac{2tr\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}}{tr} + \frac{2\sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh}}{tr}$$

$$.= tr\mu^2 + \frac{\sum_{i=1}^{t}\sum_{j=1}^{r} {e_{ij}}^2}{tr} + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + \frac{2\sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh}}{tr}$$

Apply expectation on both sides

$$E(C.F) = E(tr\mu^2) + \frac{\sum_{i=1}^{t}\sum_{j=1}^{r} E({e_{ij}}^2)}{tr} + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} E(e_{ij}) + \frac{2\sum_{i\neq g}\sum_{j\neq h} E(e_{ij}e_{gh})}{tr}$$

$$= tr\mu^2 + \frac{tr\sigma^2}{tr} + 2\mu(0) + \frac{2(0)}{tr}$$

$$= tr\mu^2 + \sigma^2$$

$$\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu + \tau_i + e_{ij}\right)^2$$

$$\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu^2 + \tau_i^2 + e_{ij}^2 + 2\mu\tau_i + 2\mu e_{ij} + 2\tau_i e_{ij}\right)$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 2r\mu\sum_{i=1}^{t}\tau_i + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij}$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 2r\mu(0) + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij}$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij}$$

Apply expectation on both sides

$$= tr\mu^2 + r\sum_{i=1}^{t} E(\tau_i^2) + \sum_{i=1}^{t}\sum_{j=1}^{r} E(e_{ij}^2) + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} E(e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r} \tau_i E(e_{ij})$$

$$= tr\mu^2 + r\sum_{i=1}^{t} \tau_i^2 + tr\sigma^2 + 2\mu(0) + 2(0)$$

$$= tr\mu^2 + r\sum_{i=1}^{t} \tau_i^2 + tr\sigma^2$$

$$E(TSS) = E\left[\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2\right] - E(C.F)$$

$$= tr\mu^2 + r\sum_{i=1}^{t} \tau_i^2 + tr\sigma^2 - tr\mu^2 - \sigma^2$$

$$= r\sum_{i=1}^{t} \tau_i^2 + (tr-1)\sigma^2$$

$$SST = \frac{\sum_{i=1}^{t} Y_{i.}^2}{r} - C.F$$

$$Y_{i.} = \sum_{j=1}^{r} Y_{ij} = \sum_{j=1}^{r}\left(\mu + \tau_i + e_{ij}\right)$$

$$= r\mu + r\tau_i + \sum_{j=1}^{r} e_{ij}$$

$$\frac{\sum_{i=1}^{t} Y_{i.}^2}{r} = \frac{\sum_{i=1}^{t}\left(r\mu + r\tau_i + \sum_{j=1}^{r} e_{ij}\right)^2}{r}$$

$$= \frac{\sum_{i=1}^{t}\left(r^2\mu^2 + r^2\tau_i^2 + \sum_{j=1}^{r} e_{ij}^2 + 2r^2\mu\tau_i + 2r\mu\sum_{j=1}^{r} e_{ij} + 2r\tau_i\sum_{j=1}^{r} e_{ij} + \sum_{j\neq h} e_{ij}e_{gh}\right)}{r}$$

$$= \frac{tr^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 2r^2\mu\sum_{i=1}^{t}\tau_i + 2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2r\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + \sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh}}{r}$$

$$= \frac{tr^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 2r^2\mu(0) + 2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2r\sum_{i=1}^{t}\sum_{j=1}^{r} \tau_i e_{ij} + \sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh}}{r}$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + \frac{\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2}{r} + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + \frac{\sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh}}{r}$$

Apply expectation on both sides

$$E\left[\frac{\sum_{i=1}^{t} Y_{i.}^2}{r}\right] = E\left[tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + \frac{\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2}{r} + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + \frac{\sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh}}{r}\right]$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + \frac{\sum_{i=1}^{t}\sum_{j=1}^{r} E(e_{ij}^2)}{r} + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} E(e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i E(e_{ij}) + \frac{\sum_{i\neq g}\sum_{j\neq h} E(e_{ij}e_{gh})}{r}$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + \frac{tr\sigma^2}{r} + 2\mu(0) + 2(0) + (0)$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + t\sigma^2$$

$$E(SST) = E\left[\frac{\sum_{i=1}^{t} Y_{i.}^2}{r}\right] - E(C.F)$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + t\sigma^2 - tr\mu^2 - \sigma^2$$

$$= r\sum_{i=1}^{t}\tau_i^2 + (t-1)\sigma^2$$

$$E(MSE) = E\left[\frac{SSE}{n-t}\right]$$

$$E(SSE) = E(TSS) - E(SST)$$

$$= r\sum_{i=1}^{t}\tau_i^2 + (tr-1)\sigma^2 - r\sum_{i=1}^{t}\tau_i^2 - (t-1)\sigma^2$$

$$= n\sigma^2 - \sigma^2 - t\sigma^2 + \sigma^2$$

$$= (n-t)\sigma^2$$

$$E(MSE) = \frac{(n-t)\sigma^2}{n-t} = \sigma^2$$

$$E(MST) = E\left[\frac{SST}{t-1}\right] = \frac{r\sum_{i=1}^{t}\tau_i^2 + (t-1)\sigma^2}{t-1}$$

$$= \frac{r\sum_{i=1}^{t}\tau_i^2}{t-1} + \frac{(t-1)\sigma^2}{t-1} = \frac{r\sum_{i=1}^{t}\tau_i^2}{t-1} + \sigma^2$$

Random Effect Model

Assumptions:

1. $\tau_i \sim iidN(0, \sigma_\tau^2)$
2. $E(e_{ij}, e_{gh}) = 0$
3. $E(e_{ij}, \tau_i) = 0$
4. $E(\tau_i, \tau_j) = 0$

$$Y_{ij} = \mu + \tau_i + e_{ij}$$

$$SSE = TSS - SST$$

$$TSS = \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 - C.F$$

$$C.F = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}\right)^2}{tr} = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r}(\mu + \tau_i + e_{ij})\right)^2}{tr}$$

$$= \frac{\left(tr\mu + r\sum_{i=1}^{t}\tau_i + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{(tr\mu)^2 + r^2\left(\sum_{i=1}^{t}\tau_i\right)^2 + \left(\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2 + 2tr^2\mu\sum_{i=1}^{t}\tau_i + 2tr\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij}}{tr}$$

$$= \frac{t^2r^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 2r^2\sum\sum\tau_i\tau_j + \sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh} + 2tr^2\mu\sum_{i=1}^{t}\tau_i + 2tr\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij}}{tr}$$

$$= tr\mu^2 + \frac{r\sum_{i=1}^{t}\tau_i^2}{t} + \frac{\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2}{tr} + \frac{2r\sum\sum\tau_i\tau_j}{t} + \frac{\sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh}}{tr} + 2r\mu\sum_{i=1}^{t}\tau_i + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + \frac{2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij}}{t}$$

Apply expectation on both sides

$$E(C.F) = tr\mu^2 + \frac{r\sum_{i=1}^{t}E(\tau_i^2)}{t} + \frac{\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}^2)}{tr} + \frac{2r\sum\sum E(\tau_i\tau_j)}{t} + \frac{\sum_{i\neq g}\sum_{j\neq h}E(e_{ij}e_{gh})}{tr} + 2r\mu\sum_{i=1}^{t}E(\tau_i)$$
$$+ 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}) + \frac{2\sum_{i=1}^{t}\sum_{j=1}^{r}E(\tau_i e_{ij})}{t}$$

$$= tr\mu^2 + \frac{rt\sigma_\tau^2}{t} + \frac{tr\sigma^2}{tr} + 0 + 0 + 0 + 0 + 0$$

$$= tr\mu^2 + r\sigma_\tau^2 + \sigma^2$$

$$\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu + \tau_i + e_{ij}\right)^2$$

$$\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu^2 + \tau_i^2 + e_{ij}^2 + 2\mu\tau_i + 2\mu e_{ij} + 2\tau_i e_{ij}\right)$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 2r\mu\sum_{i=1}^{t}\tau_i + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij}$$

Apply expectation on both sides

$$= tr\mu^2 + r\sum_{i=1}^{t}E(\tau_i^2) + \sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}^2) + 2r\mu\sum_{i=1}^{t}E(\tau_i) + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij})$$
$$+ 2\sum_{i=1}^{t}\sum_{j=1}^{r}E(\tau_i e_{ij})$$

$$= tr\mu^2 + tr\sigma_\tau^2 + tr\sigma^2 + 2r\mu(0) + 2\mu(0) + 2(0)$$

$$= tr\mu^2 + tr\sigma_\tau^2 + tr\sigma^2$$

$$E(TSS) = E\left[\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2\right] - E(C.F)$$

$$= tr\mu^2 + tr\sigma_\tau^2 + tr\sigma^2 - tr\mu^2 - r\sigma_\tau^2 - \sigma^2$$

$$= r(t-1)\sigma_\tau^2 + (tr-1)\sigma^2$$

$$SST = \frac{\sum_{i=1}^{t} Y_{i.}^2}{r} - C.F$$

$$Y_{i.} = \sum_{j=1}^{r} Y_{ij} = \sum_{j=1}^{r}(\mu + \tau_i + e_{ij})$$

$$= r\mu + r\tau_i + \sum_{j=1}^{r} e_{ij}$$

$$\frac{\sum_{i=1}^{t} Y_{i.}^2}{r} = \frac{\sum_{i=1}^{t}\left(r\mu + r\tau_i + \sum_{j=1}^{r} e_{ij}\right)^2}{r}$$

$$= \frac{\sum_{i=1}^{t}\left(r^2\mu^2 + r^2\tau_i^2 + \sum_{j=1}^{r} e_{ij}^2 + 2r^2\mu\tau_i + 2r\mu\sum_{j=1}^{r} e_{ij} + 2r\tau_i\sum_{j=1}^{r} e_{ij} + \sum_{j\neq h} e_{ij}e_{gh}\right)}{r}$$

$$= \frac{tr^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 2r^2\mu\sum_{i=1}^{t}\tau_i + 2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + \sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh}}{r}$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + \frac{\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2}{r} + 2r\mu\sum_{i=1}^{t}\tau_i + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + \frac{\sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh}}{r}$$

Apply expectation on both sides

$$E\left[\frac{\sum_{i=1}^{t} Y_{i.}^2}{r}\right] = E\left[tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + \frac{\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2}{r} + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + \frac{\sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh}}{r}\right]$$

$$= tr\mu^2 + r\sum_{i=1}^{t} E(\tau_i^2) + \frac{\sum_{i=1}^{t}\sum_{j=1}^{r} E(e_{ij}^2)}{r} + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} E(e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r} E(\tau_i e_{ij}) + \frac{\sum_{i\neq g}\sum_{j\neq h} E(e_{ij}e_{gh})}{r}$$

$$= tr\mu^2 + tr\sigma_\tau^2 + \frac{tr\sigma^2}{r} + 2\mu(0) + 2(0) + (0)$$

$$= tr\mu^2 + tr\sigma_\tau^2 + t\sigma^2$$

$$E(SST) = E\left[\frac{\sum_{i=1}^{t} Y_{i.}^2}{r}\right] - E(C.F)$$

$$= tr\mu^2 + tr\sigma_\tau^2 + t\sigma^2 - tr\mu^2 - r\sigma_\tau^2 - \sigma^2$$

$$= r(t-1)\sigma_\tau^2 + (t-1)\sigma^2$$

$$E(MSE) = E\left[\frac{SSE}{n-t}\right]$$

$$E(SSE) = E(TSS) - E(SST)$$

$$= r(t-1)\sigma_\tau^2 + (tr-1)\sigma^2 - r(t-1)\sigma_\tau^2 - (t-1)\sigma^2$$
$$= tr\sigma^2 - \sigma^2 - t\sigma^2 + \sigma^2$$
$$= (tr-t)\sigma^2 = (n-t)\sigma^2$$
$$E(MSE) = \frac{(n-t)\sigma^2}{n-t} = \sigma^2$$
$$E(MST) = E\left[\frac{SST}{t-1}\right] = \frac{r(t-1)\sigma_\tau^2 + (t-1)\sigma^2}{t-1}$$
$$= r\sigma_\tau^2 + \sigma^2$$

Estimation of Missing Observations

Case I: One missing value

T_1	T_2	…	T_i	…	T_t	
Y_{11}	Y_{21}	…	Y_{i1}	…	Y_{t1}	
Y_{12}	Y_{22}	…	Y_{i2}	…	Y_{t2}	
⋮	⋮		⋮		⋮	
Y_{1j}	Y_{2j}	…	Y_{ij}	…	Y_{tj}	
⋮	⋮	Y_{cd}	⋮		⋮	
Y_{1r}	Y_{2r}	…	Y_{ir}	…	Y_{tr}	
$Y_{1.}$	$Y_{2.}$	$Y'_{c.} + Y_{cd}$	$Y_{i.}$	…	$Y_{t.}$	$Y'_{..} + Y_{cd}$

$$C.F = \frac{\left(\sum_{i=1}^t \sum_{j=1}^r Y_{ij}\right)^2}{tr} = \frac{\left(\sum_{i=1}^t \sum_{j=1}^r Y'_{ij} + \hat{Y}_{cd}\right)^2}{tr}$$
$$SSE = TSS - SST$$
$$= \sum_{i=1}^t \sum_{j=1}^r Y_{ij}^2 + \hat{Y}_{cd}^2 - C.F - \frac{1}{r}\left[\sum_{i=1}^t Y_{i.}^2 + \left(Y'_{c.} + \hat{Y}_{cd}\right)^2\right] + C.F$$
$$= \sum_{i=1}^t \sum_{j=1}^r Y_{ij}^2 + \hat{Y}_{cd}^2 - \frac{\sum_{i=1}^t Y_{i.}^2}{r} - \frac{\left(Y'_{c.} + \hat{Y}_{cd}\right)^2}{r}$$

$$\frac{\partial SSE}{\partial \hat{Y}_{cd}} = 0$$

$$0 + 2\hat{Y}_{cd} - 0 - \frac{2\left(Y'_{c.} + \hat{Y}_{cd}\right)}{r} = 0$$
$$\frac{2r\hat{Y}_{cd} - 2\left(Y'_{c.} + \hat{Y}_{cd}\right)}{r} = 0$$
$$\frac{2\left(r\hat{Y}_{cd} - \left(Y'_{c.} + \hat{Y}_{cd}\right)\right)}{r} = 0$$
$$r\hat{Y}_{cd} - Y'_{c.} - \hat{Y}_{cd} = 0$$
$$(r-1)\hat{Y}_{cd} = Y'_{c.}$$
$$\hat{Y}_{cd} = \frac{Y'_{c.}}{r-1}$$

Case II:Two missing Observations in same treatment

T_1	T_2	…	T_i	…	T_t	
Y_{11}	Y_{21}	…	Y_{i1}	…	Y_{t1}	
Y_{12}	Y_{22}	…	Y_{i2}	…	Y_{t2}	
$\vdots$	$\vdots$		$\vdots$		$\vdots$	
Y_{1j}	Y_{2j}	…	Y_{ij}	…	Y_{tj}	
$\vdots$	$\vdots$	Y_{cd}	$\vdots$		$\vdots$	
		Y_{ef}				
Y_{1r}	Y_{2r}	…	Y_{ir}	…	Y_{tr}	
$Y_{1.}$	$Y_{2.}$	$Y'_{c.}+Y_{cd}+Y_{ef}$	$Y_{i.}$	…	$Y_{t.}$	$Y'_{..}+Y_{cd}+Y_{ef}$

$$C.F=\frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}\right)^2}{tr}=\frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r}Y'_{ij}+\hat{Y}_{cd}+\hat{Y}_{ef}\right)^2}{tr}$$

$$SSE=TSS-SST$$

$$=\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}^2+\hat{Y}_{cd}^2+\hat{Y}_{ef}^2-C.F-\frac{1}{r}\left[\sum_{i=1}^{t}Y_{i.}^2+\left(Y'_{c.}+\hat{Y}_{cd}+\hat{Y}_{ef}\right)^2\right]+C.F$$

$$=\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}^2+\hat{Y}_{cd}^2+\hat{Y}_{ef}^2-\frac{\sum_{i=1}^{t}Y_{i.}^2}{r}-\frac{\left(Y'_{c.}+\hat{Y}_{cd}+\hat{Y}_{ef}\right)^2}{r}$$

$$\frac{\partial SSE}{\partial\hat{Y}_{cd}}=0$$

$$0+2\hat{Y}_{cd}+0-0-\frac{2\left(Y'_{c.}+\hat{Y}_{cd}+\hat{Y}_{ef}\right)}{r}=0$$

$$2\hat{Y}_{cd}-\frac{2\left(Y'_{c.}+\hat{Y}_{cd}+\hat{Y}_{ef}\right)}{r}=0$$

$$\frac{2\left(r\hat{Y}_{cd}-\left(Y'_{c.}+\hat{Y}_{cd}+\hat{Y}_{ef}\right)\right)}{r}=0$$

$$r\hat{Y}_{cd}-Y'_{c.}-\hat{Y}_{cd}-\hat{Y}_{ef}=0$$

$$(r-1)\hat{Y}_{cd}-\hat{Y}_{ef}=Y'_{c.}\qquad(1)$$

$$\frac{\partial SSE}{\partial\hat{Y}_{ef}}=0$$

$$0+0+2\hat{Y}_{ef}-0-\frac{2\left(Y'_{c.}+\hat{Y}_{cd}+\hat{Y}_{ef}\right)}{r}=0$$

$$2\hat{Y}_{ef}-\frac{2\left(Y'_{c.}+\hat{Y}_{cd}+\hat{Y}_{ef}\right)}{r}=0$$

$$\frac{2\left(r\hat{Y}_{ef}-\left(Y'_{c.}+\hat{Y}_{cd}+\hat{Y}_{ef}\right)\right)}{r}=0$$

$$r\hat{Y}_{ef}-Y'_{c.}-\hat{Y}_{cd}-\hat{Y}_{ef}=0$$

$$(r-1)\hat{Y}_{ef}-\hat{Y}_{cd}=Y'_{c.}\qquad(2)$$

Multiply (1) by $r-1$ and add it in equation (2)

$$\begin{array}{rl}
(r-1)^2\hat{Y}_{cd} - (r-1)\hat{Y}_{ef} &= (r-1)Y'_{c.} \\
-\hat{Y}_{cd} + (r-1)\hat{Y}_{ef} &= Y'_{c.} \\
\hline
(r-1)^2\hat{Y}_{cd} - \hat{Y}_{cd} &= (r-1)Y'_{c.} + Y'_{c.}
\end{array}$$

$$((r-1)^2 - 1)\hat{Y}_{cd} = (r-1+1)Y'_{c.}$$

$$(r^2 - 2r + 1 - 1)\hat{Y}_{cd} = rY'_{c.}$$

$$r(r-2)\hat{Y}_{cd} = rY'_{c.}$$

$$\hat{Y}_{cd} = \frac{rY'_{c.}}{r(r-2)}$$

$$\hat{Y}_{cd} = \frac{Y'_{c.}}{r-2}$$

Exercise

Estimate the two missing observations in different treatments.

Randomized Complete Block Design

The RCBD assumes that a population of experimental units can be divided into a number of relatively homogeneous subpopulations or blocks. The treatments are then randomly assigned to experimental units such that each treatment occurs equally often (usually once) in each block (i.e. each block contains all treatments). Blocks usually represent levels of naturally-occurring differences or sources of variation that are unrelated to the treatments, and the characterization of these differences is not of interest to the researcher. In the analysis, the variation among blocks can be partitioned out of the experimental error (MSE), thereby reducing this quantity and increasing the power of the test.

Blocking technique

The purpose of blocking is to reduce the experimental error by eliminating the contribution of known sources of variation among the experimental units. This is done by grouping the experimental units into blocks such that variability within each block is minimized and variability among blocks is maximized. Since only the variation within a block becomes part of the experimental error, blocking is most effective when the experimental area has a predictable pattern of variability.

An ideal source of variation to use as the basis for blocking is one that is large and highly predictable. An example is soil heterogeneity, in a fertilizer or provenance trial where yield data is the primary character of interest. In the case of such experiments, after identifying the specific source of variability to be used as the basis for blocking, the size and the shape of blocks must be selected to maximize variability among blocks. The guidelines for this decision are (i) When the gradient is unidirectional (*i.e.*, there is only one gradient), use long and narrow blocks. Furthermore, orient these blocks so that their length is perpendicular to the direction of the gradient. (ii) When the fertility gradient occurs in two directions with one gradient much stronger than the other, ignore the weaker gradient and follow the preceding guideline for the case of the unidirectional gradient. (iii) When the fertility gradient occurs in two directions with both gradients equally strong and perpendicular to each other, use blocks that are as square as possible or choose other designs like *latin square design* (Gomez and Gomez, 1980).

Whenever blocking is used, the identity of the blocks and the purpose for their use must be consistent throughout the experiment. That is, whenever a source of variation exists that is beyond the control of the researcher, it should be ensured that such variation occurs among blocks rather than within blocks. For example, if certain operations such as application of insecticides or data collection cannot be completed for the whole experiment in one day, the task should be completed for all plots of the same block on the same day. In this way, variation among days (which may be enhanced by weather factors) becomes a part of block variation and is, thus, excluded from the experimental error. If more than one observer is to make measurements in the trial, the same observer should be assigned to make measurements for all plots of the same block. This way, the variation among observers if any, would constitute a part of block variation instead of the experimental error.

Experimental Layout

Suppose there are k treatments and r blocks in a randomized complete block design, then each block contains k homogenous plots, one of each treatment. An experimental layout for such a design using 6 treatments A,B,C,D,E,F in 3 blocks might be as follows:

BLOCK I	D	B	A	C	F	E
BLOCK II	C	B	E	F	D	A
BLOCK III	C	F	B	D	A	E

Statistical Model and Analysis

As each observation in a RCBD is classified by the block to which it belongs and by the treatment it receives, therefore Y_{ij} represents the observation corresponding to block j and treatment i.

The linear statistical model for RCBD is as:

$$Y_{ij} = \mu + \tau_i + \beta_j + e_{ij} \quad \begin{cases} i = 1,2,3, \dots, t \\ j = 1,2,3, \dots., r \end{cases}$$

where

Y_{ij}- any observation for which i is the treatment factor j is the blocking factor
μ - the mean
τ_i- the effect for being in treatment i
B_j is the effect for being in block j

Formulation of Hypotheses:

$$H_0: \tau_i = 0$$
$$H_0': \beta_j = 0$$
$$H_1: \tau_i \neq 0$$
$$H_1': \beta_j \neq 0$$

Level of Significance:

$$\alpha = 0.05$$

Test Statistic:

$$F_1 = \frac{s_t^2}{s_e^2}, \quad F_2 = \frac{s_b^2}{s_e^2}$$

S.O.V	d.f	SS	MS	F
Treatments	$t-1$	SST	$s_t^2 = \frac{SST}{t-1}$	$F_1 = \frac{s_t^2}{s_e^2}$
Blocks	$r-1$	SSB	$s_b^2 = \frac{SSB}{r-1}$	$F_2 = \frac{s_b^2}{s_e^2}$
Error	$(t-1)(r-1)$	SSE	$s_e^2 = \frac{SSE}{(t-1)(r-1)}$	
Total	$tr-1$	TSS		

Where

$$C.F = \frac{Y_{..}^2}{tr}, \quad TSS = \sum\sum Y_{ij}^2 - C.F, \quad SST = \frac{1}{r}\sum Y_{i.}^2 - C.F$$

$$SSB = \frac{1}{t}\sum Y_{.j}^2 - C.F\ , SSE = TSS - SST - SSB$$

C.R:

$F_1 \geq F_{\alpha(t-1,(t-1)(r-1))}$

$F_2 \geq F_{\alpha(r-1,(t-1)(r-1))}$

Conclusion:

If calculated value of F falls in the critical region then we reject H_0.

Advantages of RCBD

1. Generally more precise than the CRD.
2. No restriction on the number of treatments or replicates.
3. Some treatments may be replicated more times than others.
4. Missing plots are easily estimated.
5. Whole treatments or entire replicates may be deleted from the analysis.
6. If experimental error is heterogeneous, valid comparisons can still be made.

Disadvantages of RCBD

1. Error df is smaller than that for the CRD (problem with a small number of treatments).
2. If there is a large variation between experimental units within a block, a large error term may result (this may be due to too many treatments).
3. If there are missing data, a RCBD experiment may be less efficient than a CRD

Estimation of Model Parameters

Let $\hat{\mu}, \hat{\tau}_i, \hat{\beta}_j$ are the OLS estimators of μ, τ_i, β_j respectively.

$$S = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(Y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j\right)^2$$

Differentiate w.r.t $\hat{\mu}$, we get

$$\frac{\partial S}{\partial \hat{\mu}} = 2\sum_{i=1}^{t}\sum_{j=1}^{r}\left(Y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j\right)(-1)$$

Now put $\frac{\partial S}{\partial \hat{\mu}} = 0$, we get

$$-2\sum_{i=1}^{t}\sum_{j=1}^{r}\left(Y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j\right) = 0$$

$$\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij} - tr\hat{\mu} - r\sum_{i=1}^{t}\hat{\tau}_i - t\sum_{j=1}^{r}\hat{\beta}_j = 0$$

$$tr\hat{\mu} = \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij} - r\sum_{i=1}^{t}\hat{\tau}_i - t\sum_{j=1}^{r}\hat{\beta}_j$$

For unique solutions put $\sum_{i=1}^{t}\hat{\tau}_i = 0$, and $\sum_{j=1}^{r}\hat{\beta}_j = 0$

$$tr\hat{\mu} = \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij} - r(0) - t(0)$$

$$\hat{\mu} = \frac{\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}}{tr} = \bar{\bar{Y}}$$

$$S = \sum_{j=1}^{r}\left(Y_{1j} - \hat{\mu} - \hat{\tau}_1 - \hat{\beta}_j\right)^2 + \sum_{j=1}^{r}\left(Y_{2j} - \hat{\mu} - \hat{\tau}_2 - \hat{\beta}_j\right)^2 + \cdots$$
$$+ \sum_{j=1}^{r}\left(Y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j\right)^2 + \cdots + \sum_{j=1}^{r}\left(Y_{tj} - \hat{\mu} - \hat{\tau}_t - \hat{\beta}_j\right)^2$$

Differentiate w.r.t $\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_i, \dots, \hat{\tau}_t$

$$\frac{\partial S}{\partial \hat{\tau}_1} = 2\sum_{j=1}^{r}\left(Y_{1j} - \hat{\mu} - \hat{\tau}_1 - \hat{\beta}_j\right)(-1) = 0$$

$$-2\sum_{j=1}^{r}\left(Y_{1j}-\hat{\mu}-\hat{\tau}_1-\hat{\beta}_j\right)=0$$

$$\sum_{j=1}^{r}Y_{1j}-r\hat{\mu}-r\hat{\tau}_1-\sum_{j=1}^{r}\hat{\beta}_j=0$$

$$\sum_{j=1}^{r}Y_{1j}=r\hat{\mu}+r\hat{\tau}_1+\sum_{j=1}^{r}\hat{\beta}_j$$

$$Y_{1.}=r\hat{\mu}+r\hat{\tau}_1+\textstyle\sum_{j=1}^{r}\hat{\beta}_j \qquad (1)$$

$$\frac{\partial S}{\partial\hat{\tau}_2}=2\sum_{j=1}^{r}\left(Y_{2j}-\hat{\mu}-\hat{\tau}_2-\hat{\beta}_j\right)(-1)=0$$

$$-2\sum_{j=1}^{r}\left(Y_{2j}-\hat{\mu}-\hat{\tau}_2-\hat{\beta}_j\right)=0$$

$$\sum_{j=1}^{r}Y_{2j}-r\hat{\mu}-r\hat{\tau}_2-\sum_{j=1}^{r}\hat{\beta}_j=0$$

$$\sum_{j=1}^{r}Y_{2j}=r\hat{\mu}+r\hat{\tau}_2+\sum_{j=1}^{r}\hat{\beta}_j$$

$$Y_{2.}=r\hat{\mu}+r\hat{\tau}_2+\textstyle\sum_{j=1}^{r}\hat{\beta}_j \qquad (2)$$

$$\frac{\partial S}{\partial\hat{\tau}_i}=2\sum_{j=1}^{r}\left(Y_{ij}-\hat{\mu}-\hat{\tau}_i-\hat{\beta}_j\right)(-1)=0$$

$$-2\sum_{j=1}^{r}\left(Y_{ij}-\hat{\mu}-\hat{\tau}_i-\hat{\beta}_j\right)=0$$

$$\sum_{j=1}^{r}Y_{ij}-r\hat{\mu}-r\hat{\tau}_i-\sum_{j=1}^{r}\hat{\beta}_j=0$$

$$\sum_{j=1}^{r}Y_{ij}=r\hat{\mu}+r\hat{\tau}_i+\sum_{j=1}^{r}\hat{\beta}_j$$

$$Y_{i.}=r\hat{\mu}+r\hat{\tau}_i+\textstyle\sum_{j=1}^{r}\hat{\beta}_j \qquad (i)$$

$$\frac{\partial S}{\partial \hat{\tau}_t} = 2\sum_{j=1}^{r}\left(Y_{tj} - \hat{\mu} - \hat{\tau}_t - \hat{\beta}_j\right)(-1) = 0$$

$$-2\sum_{j=1}^{r}\left(Y_{tj} - \hat{\mu} - \hat{\tau}_t - \hat{\beta}_j\right) = 0$$

$$\sum_{j=1}^{r} Y_{tj} - r\hat{\mu} - r\hat{\tau}_t - \sum_{j=1}^{r}\hat{\beta}_j = 0$$

$$\sum_{j=1}^{r} Y_{tj} = r\hat{\mu} + r\hat{\tau}_t + \sum_{j=1}^{r}\hat{\beta}_j$$

$$Y_{t.} = r\hat{\mu} + r\hat{\tau}_t + \textstyle\sum_{j=1}^{r}\hat{\beta}_j \qquad (t)$$

From equation (i), put $\sum_{j=1}^{r}\hat{\beta}_j = 0$ we get

$$Y_{i.} = r\hat{\mu} + r\hat{\tau}_i + 0$$

$$r\hat{\tau}_i = Y_{i.} - r\hat{\mu}$$

$$\hat{\tau}_i = \frac{Y_{i.}}{r} - \frac{r\hat{\mu}}{r}$$

$$\hat{\tau}_i = \bar{Y}_{i.} - \bar{\bar{Y}}$$

$$S = \sum_{i=1}^{t}\left(Y_{i1} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_1\right)^2 + \sum_{i=1}^{t}\left(Y_{i2} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_2\right)^2 + \cdots$$
$$+ \sum_{i=1}^{t}\left(Y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j\right)^2 + \cdots + \sum_{i=1}^{t}\left(Y_{ir} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_r\right)^2$$

Differentiate w.r.t $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_j, \ldots, \hat{\beta}_r$

$$\frac{\partial S}{\partial \hat{\beta}_1} = 2\sum_{i=1}^{t}\left(Y_{i1} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_1\right)(-1) = 0$$

$$-2\sum_{i=1}^{t}\left(Y_{i1} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_1\right) = 0$$

$$\sum_{i=1}^{t} Y_{i1} - t\hat{\mu} - \sum_{i=1}^{t}\hat{\tau}_i - t\hat{\beta}_1 = 0$$

$$\sum_{i=1}^{t} Y_{i1} = t\hat{\mu} + \sum_{i=1}^{t} \hat{\tau}_i + t\hat{\beta}_1$$

$$Y_{.1} = t\hat{\mu} + \textstyle\sum_{i=1}^{t} \hat{\tau}_i + t\hat{\beta}_1 \qquad (1)$$

$$\frac{\partial S}{\partial \hat{\beta}_2} = 2\sum_{i=1}^{t} \left(Y_{i2} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_2\right)(-1) = 0$$

$$-2\sum_{i=1}^{t} \left(Y_{i2} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_2\right) = 0$$

$$\sum_{i=1}^{t} Y_{i2} - t\hat{\mu} - \sum_{i=1}^{t} \hat{\tau}_i - t\hat{\beta}_2 = 0$$

$$\sum_{i=1}^{t} Y_{i2} = t\hat{\mu} + \sum_{i=1}^{t} \hat{\tau}_i + t\hat{\beta}_2$$

$$Y_{.2} = t\hat{\mu} + \textstyle\sum_{i=1}^{t} \hat{\tau}_i + t\hat{\beta}_2 \qquad (2)$$

$$\frac{\partial S}{\partial \hat{\beta}_j} = 2\sum_{i=1}^{t} \left(Y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j\right)(-1) = 0$$

$$-2\sum_{i=1}^{t} \left(Y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j\right) = 0$$

$$\sum_{i=1}^{t} Y_{ij} - t\hat{\mu} - \sum_{i=1}^{t} \hat{\tau}_i - t\hat{\beta}_j = 0$$

$$\sum_{i=1}^{t} Y_{ij} = t\hat{\mu} + \sum_{i=1}^{t} \hat{\tau}_i + t\hat{\beta}_j$$

$$Y_{.j} = t\hat{\mu} + \textstyle\sum_{i=1}^{t} \hat{\tau}_i + t\hat{\beta}_j \qquad (j)$$

$$\frac{\partial S}{\partial \hat{\beta}_r} = 2\sum_{i=1}^{t} \left(Y_{ir} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_r\right)(-1) = 0$$

$$-2\sum_{i=1}^{t} \left(Y_{ir} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_r\right) = 0$$

$$\sum_{i=1}^{t} Y_{ir} - t\hat{\mu} - \sum_{i=1}^{t} \hat{\tau}_i - t\hat{\beta}_r = 0$$

$$\sum_{i=1}^{t} Y_{ir} = t\hat{\mu} + \sum_{i=1}^{t} \hat{\tau}_i + t\hat{\beta}_r$$

$$Y_{.r} = t\hat{\mu} + \textstyle\sum_{i=1}^{t} \hat{\tau}_i + t\hat{\beta}_r \qquad (r)$$

Put $\sum_{i=1}^{t} \hat{\tau}_i = 0$ equation (j), we get

$$Y_{.j} = t\hat{\mu} + 0 + t\hat{\beta}_j$$

$$t\hat{\beta}_j = Y_{.j} - t\hat{\mu}$$

$$\hat{\beta}_j = \frac{Y_{.j}}{t} - \frac{t\hat{\mu}}{t}$$

$$\hat{\beta}_j = \bar{Y}_{.j} - \bar{\bar{Y}}$$

Expected Mean Square Error

Fixed Effect Model

Assumptions:

1. $E(e_{ij}) = 0$
2. $E(e_{ij}e_{gh}) = 0$
3. $\sum_{i=1}^{t} \tau_i = 0$
4. $\sum_{j=1}^{r} \beta_j = 0$

$$Y_{ij} = \mu + \tau_i + \beta_j + e_{ij} \quad \begin{cases} i = 1,2,3,\dots,t \\ j = 1,2,3,\dots.,r \end{cases}$$

$$SSE = TSS - SST - SSB$$

$$TSS = \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 - C.F$$

$$C.F = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}\right)^2}{tr} = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r}(\mu + \tau_i + \beta_j + e_{ij})\right)^2}{tr}$$

$$= \frac{\left(tr\mu + r\sum_{i=1}^{t}\tau_i + t\sum_{j=1}^{r}\beta_j + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{\left(tr\mu + 0 + 0 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{\left(tr\mu + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{t^2r^2\mu^2 + \left(\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2 + 2tr\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}}{tr}$$

$$= tr\mu^2 + \frac{\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh}}{tr} + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}$$

Apply expectation on both sides

$$E(C.F) = tr\mu^2 + \frac{\sum_{i=1}^{t}\sum_{j=1}^{r} E(e_{ij}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij}e_{gh})}{tr} + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} E(e_{ij})$$

$$= tr\mu^2 + \frac{tr\sigma^2 + 0}{tr} + 0$$

$$= tr\mu^2 + \frac{tr\sigma^2}{tr}$$

$$= tr\mu^2 + \sigma^2$$

$$\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu + \tau_i + \beta_j + e_{ij}\right)^2$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu^2 + {\tau_i}^2 + {\beta_j}^2 + e_{ij}^2 + 2\mu\tau_i + 2\mu\beta_j + 2\mu e_{ij} + 2\tau_i\beta_j + 2\tau_i e_{ij} + 2\beta_j e_{ij}\right)$$

$$= tr\mu^2 + r\sum_{i=1}^{t}{\tau_i}^2 + t\sum_{j=1}^{r}{\beta_j}^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 2r\mu\sum_{i=1}^{t}\tau_i + 2t\mu\sum_{j=1}^{r}\beta_j + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}$$

$$= tr\mu^2 + r\sum_{i=1}^{t}{\tau_i}^2 + t\sum_{j=1}^{r}{\beta_j}^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 0 + 0 + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 0 + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i{}^2 + t\sum_{j=1}^{r}\beta_j{}^2 + \sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2 + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}$$

Apply expectation on both sides

$$E\left[\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}^2\right] = tr\mu^2 + r\sum_{i=1}^{t}\tau_i{}^2 + t\sum_{j=1}^{r}\beta_j{}^2 + \sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}^2) + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i E(e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j E(e_{ij})$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i{}^2 + t\sum_{j=1}^{r}\beta_j{}^2 + tr\sigma^2 + 0 + 0 + 0$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i{}^2 + t\sum_{j=1}^{r}\beta_j{}^2 + tr\sigma^2$$

$$E(TSS) = E\left[\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}^2\right] - E(C.F)$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i{}^2 + t\sum_{j=1}^{r}\beta_j{}^2 + tr\sigma^2 - tr\mu^2 - \sigma^2$$

$$= r\sum_{i=1}^{t}\tau_i{}^2 + t\sum_{j=1}^{r}\beta_j{}^2 + (tr-1)\sigma^2$$

$$SST = \frac{\sum_{i=1}^{t}Y_{i.}^2}{r} - C.F$$

$$Y_{i.} = \sum_{j=1}^{r}Y_{ij} = \sum_{j=1}^{r}(\mu + \tau_i + \beta_j + e_{ij}) = r\mu + r\tau_i + \sum_{j=1}^{r}\beta_j + \sum_{j=1}^{r}e_{ij}$$

$$= r\mu + r\tau_i + 0 + \sum_{j=1}^{r}e_{ij} = r\mu + r\tau_i + \sum_{j=1}^{r}e_{ij}$$

$$\frac{\sum_{i=1}^{t}Y_{i.}^2}{r} = \frac{\sum_{i=1}^{t}\left(r\mu + r\tau_i + \sum_{j=1}^{r}e_{ij}\right)^2}{r}$$

$$= \frac{\sum_{i=1}^{t}\left(r^2\mu^2 + r^2\tau_i^2 + \left(\sum_{j=1}^{r}e_{ij}\right)^2 + 2r^2\mu\tau_i + 2r\mu\sum_{j=1}^{r}e_{ij} + 2r\tau_i\sum_{j=1}^{r}e_{ij}\right)}{r}$$

$$= \frac{\sum_{i=1}^{t}\left(r^2\mu^2 + r^2\tau_i^2 + \sum_{j=1}^{r}e_{ij}^2 + \sum_{j\neq h}e_{ij}e_{ih} + 2r^2\mu\tau_i + 2r\mu\sum_{j=1}^{r}e_{ij} + 2r\tau_i\sum_{j=1}^{r}e_{ij}\right)}{r}$$

$$= \frac{tr^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + \sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij}e_{gh} + 2r^2\mu\sum_{i=1}^{t}\tau_i + 2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij} + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}\tau_i}{r}$$

$$=\frac{tr^2\mu^2+r^2\sum_{i=1}^{t}\tau_i^2+\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2+\sum_{i\neq g}\sum_{j\neq h}e_{ij}e_{gh}+0+2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}+2r\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}\,\tau_i}{r}$$

$$=\frac{tr^2\mu^2+r^2\sum_{i=1}^{t}\tau_i^2+\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2+\sum_{i\neq g}\sum_{j\neq h}e_{ij}e_{gh}+2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}+2r\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}\,\tau_i}{r}$$

Apply expectation on both sides

$$=\frac{tr^2\mu^2+r^2\sum_{i=1}^{t}\tau_i^2+\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}^2)+\sum_{i\neq g}\sum_{j\neq h}E(e_{ij}e_{gh})+2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij})+2r\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij})\,\tau_i}{r}$$

$$=\frac{tr^2\mu^2+r^2\sum_{i=1}^{t}\tau_i^2+tr\sigma^2+0+0+0}{r}$$

$$E\left(\frac{\sum_{i=1}^{t}Y_{i.}^2}{r}\right)=tr\mu^2+r\sum_{i=1}^{t}\tau_i^2+t\sigma^2$$

$$E(SST)=E\left(\frac{\sum_{i=1}^{t}Y_{i.}^2}{r}\right)-E(C.F)$$

$$=tr\mu^2+r\sum_{i=1}^{t}\tau_i^2+t\sigma^2-tr\mu^2-\sigma^2$$

$$=r\sum_{i=1}^{t}\tau_i^2+(t-1)\sigma^2$$

$$SSB=\frac{\sum_{j=1}^{r}Y_{.j}^2}{t}-C.F$$

$$Y_{.j}=\sum_{i=1}^{t}Y_{ij}=\sum_{i=1}^{t}\left(\mu+\tau_i+\beta_j+e_{ij}\right)=t\mu+\sum_{i=1}^{t}\tau_i+t\beta_j+\sum_{i=1}^{t}e_{ij}$$

$$=t\mu+0+t\beta_j+\sum_{i=1}^{t}e_{ij}=t\mu+t\beta_j+\sum_{i=1}^{t}e_{ij}$$

$$\frac{\sum_{j=1}^{r}Y_{.j}^2}{t}=\frac{\sum_{j=1}^{r}\left(t\mu+t\beta_j+\sum_{i=1}^{t}e_{ij}\right)^2}{t}$$

$$=\frac{\sum_{j=1}^{r}\left(t^2\mu^2+t^2\beta_j^2+\left(\sum_{i=1}^{t}e_{ij}\right)^2+2t^2\mu\beta_j+2t\mu\sum_{i=1}^{t}e_{ij}+2t\beta_j\sum_{i=1}^{t}e_{ij}\right)}{t}$$

$$=\frac{t^2r\mu^2+t^2\sum_{j=1}^{r}\beta_j^2+\sum_{j=1}^{r}\sum_{i=1}^{t}e_{ij}^2+\sum_{i\neq g}\sum_{j\neq h}e_{ij}e_{gh}+2t^2\mu\sum_{j=1}^{r}\beta_j+2t\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}+2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_je_{ij}}{t}$$

$$=\frac{t^2r\mu^2+t^2\sum_{j=1}^{r}\beta_j^2+\sum_{j=1}^{r}\sum_{i=1}^{t}e_{ij}^2+\sum_{i\neq g}\sum_{j\neq h}e_{ij}e_{gh}+0+2t\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}+2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_je_{ij}}{t}$$

$$=\frac{t^2r\mu^2+t^2\sum_{j=1}^{r}\beta_j^2+\sum_{j=1}^{r}\sum_{i=1}^{t}e_{ij}^2+\sum_{i\neq g}\sum_{j\neq h}e_{ij}e_{gh}+2t\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}+2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_je_{ij}}{t}$$

Apply expectation on both sides

$$=\frac{t^2r\mu^2+t^2\sum_{j=1}^{r}\beta_j^2+\sum_{j=1}^{r}\sum_{i=1}^{t}E(e_{ij}^2)+\sum_{i\neq g}\sum_{j\neq h}E(e_{ij}e_{gh})+2t\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij})+2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_jE(e_{ij})}{t}$$

$$=\frac{t^2r\mu^2+t^2\sum_{j=1}^{r}\beta_j^2+tr\sigma^2+0+0+0}{t}$$

$$E\left[\frac{\sum_{j=1}^{r}Y_{.j}^2}{t}\right]=tr\mu^2+t\sum_{j=1}^{r}\beta_j^2+r\sigma^2$$

$$E(SSB)=E\left[\frac{\sum_{j=1}^{r}Y_{.j}^2}{t}\right]-E(C.F)$$

$$= tr\mu^2 + t\sum_{j=1}^{r}\beta_j^2 + r\sigma^2 - tr\mu^2 - \sigma^2$$

$$= t\sum_{j=1}^{r}\beta_j^2 + (r-1)\sigma^2$$

$$E(SSE) = E(TSS) - E(SST) - E(SSB)$$

$$= r\sum_{i=1}^{t}{\tau_i}^2 + t\sum_{j=1}^{r}{\beta_j}^2 + (tr-1)\sigma^2 - r\sum_{i=1}^{t}\tau_i^2 - (t-1)\sigma^2 - t\sum_{j=1}^{r}\beta_j^2 - (r-1)\sigma^2$$

$$= (tr-1)\sigma^2 - (t-1)\sigma^2 - (r-1)\sigma^2$$

$$= (tr-1-t+1-r+1)\sigma^2$$

$$= (tr-t-r+1)\sigma^2$$

$$= (t(r-1)-1(r-1))\sigma^2$$

$$= (t-1)(r-1)\sigma^2$$

$$E(MSE) = E\left[\frac{SSE}{(t-1)(r-1)}\right]$$

$$= \frac{E(SSE)}{(t-1)(r-1)} = \frac{(t-1)(r-1)\sigma^2}{(t-1)(r-1)} = \sigma^2$$

$$E(MST) = E\left[\frac{SST}{t-1}\right]$$

$$= \frac{E(SST)}{t-1} = \frac{r\sum_{i=1}^{t}\tau_i^2 + (t-1)\sigma^2}{t-1} = \sigma^2 + \frac{r\sum_{i=1}^{t}\tau_i^2}{t-1}$$

$$E(MSB) = E\left[\frac{SSB}{r-1}\right]$$

$$= \frac{E(SSB)}{r-1} = \frac{t\sum_{j=1}^{r}\beta_j^2 + (r-1)\sigma^2}{r-1} = \sigma^2 + \frac{t\sum_{j=1}^{r}\beta_j^2}{r-1}$$

Random Effect Model

Assumptions:

1. $E(e_{ij}) = 0$
2. $E(e_{ij}e_{gh}) = 0$
3. $e_i \sim iidN(0, \sigma^2)$
4. $\tau_i \sim iidN(0, \sigma_\tau^2)$
5. $\beta_j \sim iidN(0, \sigma_\beta^2)$
6. $E(\tau_i\tau_j) = 0$
7. $E(\beta_i\beta_j) = 0$
8. $E(\tau_i e_{ij}) = 0$
9. $E(\beta_j e_{ij}) = 0$
10. $E(\tau_i\beta_j) = 0$

$$Y_{ij} = \mu + \tau_i + \beta_j + e_{ij} \quad \begin{cases} i = 1,2,3,\dots,t \\ j = 1,2,3,\dots.,r \end{cases}$$

$$SSE = TSS - SST - SSB$$

$$TSS = \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 - C.F$$

$$C.F = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}\right)^2}{tr} = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu + \tau_i + \beta_j + e_{ij}\right)\right)^2}{tr}$$

$$= \frac{\left(tr\mu + r\sum_{i=1}^{t}\tau_i + t\sum_{j=1}^{r}\beta_j + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{\begin{array}{c} t^2r^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + r^2\sum\sum_{i\neq j}\tau_i\tau_j + t^2\sum_{j=1}^{r}\beta_j^2 + t^2\sum\sum_{i\neq j}\beta_i\beta_j + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh} \\ +2tr^2\mu\sum_{i=1}^{t}\tau_i + 2t^2r\mu\sum_{j=1}^{r}\beta_j + 2tr\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2tr\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} \\ 2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij} \end{array}}{tr}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} t^2r^2\mu^2 + r^2\sum_{i=1}^{t}E(\tau_i^2) + r^2\sum\sum_{i\neq j}E(\tau_i\tau_j) + t^2\sum_{j=1}^{r}E(\beta_j^2) + t^2\sum\sum_{i\neq j}E(\beta_i\beta_j) + \sum_{i=1}^{t}\sum_{j=1}^{r} E(e_{ij}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij}e_{gh}) \\ +2tr^2\mu\sum_{i=1}^{t}E(\tau_i) + 2t^2r\mu\sum_{j=1}^{r}E(\beta_j) + 2tr\mu\sum_{i=1}^{t}\sum_{j=1}^{r} E(e_{ij}) + 2tr\sum_{i=1}^{t}\sum_{j=1}^{r}E(\tau_i\beta_j) + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}E(\tau_i e_{ij}) \\ 2t\sum_{i=1}^{t}\sum_{j=1}^{r}E(\beta_j e_{ij}) \end{array}}{tr}$$

$$= \frac{t^2r^2\mu^2 + tr^2\sigma_\tau^2 + 0 + t^2r\sigma_\beta^2 + 0 + tr\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{tr}$$

$$E(C.F) = tr\mu^2 + r\sigma_\tau^2 + t\sigma_\beta^2 + \sigma^2$$

$$\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu + \tau_i + \beta_j + e_{ij}\right)^2$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu^2 + {\tau_i}^2 + {\beta_j}^2 + e_{ij}^2 + 2\mu\tau_i + 2\mu\beta_j + 2\mu e_{ij} + 2\tau_i\beta_j + 2\tau_i e_{ij} + 2\beta_j e_{ij}\right)$$

$$= tr\mu^2 + r\sum_{i=1}^{t}{\tau_i}^2 + t\sum_{j=1}^{r}{\beta_j}^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 2r\mu\sum_{i=1}^{t}\tau_i + 2t\mu\sum_{j=1}^{r}\beta_j + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}$$
$$+ 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}$$

Apply expectation on both sides

$$.= tr\mu^2 + r\sum_{i=1}^{t}E({\tau_i}^2) + t\sum_{j=1}^{r}E({\beta_j}^2) + \sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}^2) + 2r\mu\sum_{i=1}^{t}E(\tau_i) + 2t\mu\sum_{j=1}^{r}E(\beta_j) +$$
$$2\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}E(\tau_i\beta_j) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}E(\tau_i e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}(\beta_j e_{ij})$$

$$= tr\mu^2 + tr\sigma_\tau^2 + tr\sigma_\beta^2 + tr\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0$$

$$E\left(\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}^2\right) = tr\mu^2 + tr\sigma_\tau^2 + tr\sigma_\beta^2 + tr\sigma^2$$

$$E(TSS) = E\left(\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}^2\right) - E(C.F)$$

$$= tr\mu^2 + tr\sigma_\tau^2 + tr\sigma_\beta^2 + tr\sigma^2 - tr\mu^2 - r\sigma_\tau^2 - t\sigma_\beta^2 - \sigma^2$$

$$= r(t-1)\sigma_\tau^2 + t(r-1)\sigma_\beta^2 + (tr-1)\sigma^2$$

$$SST = \frac{\sum_{i=1}^{t}Y_{i.}^2}{r} - C.F$$

$$Y_{i.} = \sum_{j=1}^{r}Y_{ij} = \sum_{j=1}^{r}(\mu + \tau_i + \beta_j + e_{ij}) = r\mu + r\tau_i + \sum_{j=1}^{r}\beta_j + \sum_{j=1}^{r}e_{ij}$$

$$\frac{\sum_{i=1}^{t}Y_{i.}^2}{r} = \frac{\sum_{i=1}^{t}\left(r\mu + r\tau_i + \sum_{j=1}^{r}\beta_j + \sum_{j=1}^{r}e_{ij}\right)^2}{r}$$

$$= \frac{\sum_{i=1}^{t}\begin{pmatrix} r^2\mu^2 + r^2\tau_i^2 + \sum_{j=1}^{r}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + \sum_{j=1}^{r}e_{ij}^2 + \sum_{j\neq h}e_{ij}e_{ih} + 2r^2\mu\tau_i + 2r\mu\sum_{j=1}^{r}\beta_j \\ +2r\mu\sum_{j=1}^{r}e_{ij} + 2r\tau_i\sum_{j=1}^{r}\beta_j + 2r\tau_i\sum_{j=1}^{r}e_{ij} + 2\sum_{j=1}^{r}\beta_j e_{ij}\end{pmatrix}}{r}$$

$$= \frac{\begin{matrix} tr^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + t\sum_{j=1}^{r}\beta_j^2 + t\sum\sum_{i\neq j}\beta_i\beta_j + \sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij}e_{gh} + 2r^2\mu\sum_{i=1}^{t}\tau_i + 2tr\mu\sum_{i=1}^{r}\beta_j \\ +2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij} + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}\end{matrix}}{r}$$

Apply expectation on both sides

$$= \frac{\begin{matrix} tr^2\mu^2 + r^2\sum_{i=1}^{t}E(\tau_i^2) + t\sum_{j=1}^{r}E(\beta_j^2) + t\sum\sum_{i\neq j}E(\beta_i\beta_j) + \sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij}e_{gh}) \\ +2r^2\mu\sum_{i=1}^{t}E(\tau_i) + 2tr\mu\sum_{j=1}^{r}E(\beta_j) + 2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}) + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}E(\tau_i\beta_j) \\ +2r\sum_{i=1}^{t}\sum_{j=1}^{r}E(\tau_i e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}E(\beta_j e_{ij})\end{matrix}}{r}$$

$$= \frac{tr^2\mu^2 + tr^2\sigma_\tau^2 + tr\sigma_\beta^2 + 0 + tr\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{r}$$

$$E\left[\frac{\sum_{i=1}^{t}Y_{i.}^2}{r}\right] = tr\mu^2 + tr\sigma_\tau^2 + t\sigma_\beta^2 + t\sigma^2$$

$$E(SST) = E\left[\frac{\sum_{i=1}^{t}Y_{i.}^2}{r}\right] - E(C.F)$$

$$= tr\mu^2 + tr\sigma_\tau^2 + t\sigma_\beta^2 + t\sigma^2 - tr\mu^2 - r\sigma_\tau^2 - t\sigma_\beta^2 - \sigma^2$$

$$= tr\sigma_\tau^2 + t\sigma^2 - r\sigma_\tau^2 - \sigma^2$$

$$= r(t-1)\sigma_\tau^2 + (t-1)\sigma^2$$

$$SSB = \frac{\sum_{j=1}^{r}Y_{.j}^2}{t} - C.F$$

$$Y_{.j} = \sum_{i=1}^{t}Y_{ij} = \sum_{i=1}^{t}(\mu + \tau_i + \beta_j + e_{ij}) = t\mu + \sum_{i=1}^{t}\tau_i + t\beta_j + \sum_{i=1}^{t}e_{ij}$$

$$\frac{\sum_{j=1}^{r}Y_{.j}^2}{t} = \frac{\sum_{j=1}^{r}\left(t\mu + \sum_{i=1}^{t}\tau_i + t\beta_j + \sum_{i=1}^{t}e_{ij}\right)^2}{t}$$

$$= \frac{\sum_{j=1}^{r}\begin{pmatrix} t^2\mu^2 + \sum_{i=1}^{t}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + t^2\beta_j^2 + \sum_{i=1}^{t}e_{ij}^2 + \sum_{i\neq g}e_{ij}e_{gj} + 2t\mu\sum_{i=1}^{t}\tau_i + 2t^2\mu\beta_j \\ +2t\mu\sum_{i=1}^{t}e_{ij} + 2\sum_{i=1}^{t}\tau_i e_{ij} + 2t\beta_j\sum_{i=1}^{t}\tau_i + 2t\beta_j\sum_{i=1}^{t}e_{ij}\end{pmatrix}}{t}$$

$$= \frac{\begin{matrix} t^2r\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + r\sum\sum_{i\neq j}\tau_i\tau_j + t^2\sum_{j=1}^{r}\beta_j^2 + \sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij}e_{gh} + 2tr\mu\sum_{i=1}^{t}\tau_i + 2t^2\mu\sum_{j=1}^{r}\beta_j \\ +2t\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2t\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}\end{matrix}}{t}$$

Apply expectation on both sides

$$= \frac{\begin{matrix} t^2r\mu^2 + r\sum_{i=1}^{t}E(\tau_i^2) + r\sum\sum_{i\neq j}E(\tau_i\tau_j) + t^2\sum_{j=1}^{r}E(\beta_j^2) + \sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij}e_{gh}) \\ 2tr\mu\sum_{i=1}^{t}E(\tau_i) + 2t^2\mu\sum_{j=1}^{r}E(\beta_j) + 2t\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}E(\tau_i e_{ij}) \\ +2t\sum_{i=1}^{t}\sum_{j=1}^{r}E(\tau_i\beta_j) + 2t\sum_{i=1}^{t}\sum_{j=1}^{r}E(\beta_j e_{ij})\end{matrix}}{t}$$

$$= \frac{t^2r\mu^2 + tr\sigma_\tau^2 + 0 + t^2r\sigma_\beta^2 + tr\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{t}$$

$$E\left[\frac{\sum_{j=1}^{r}Y_{.j}^2}{t}\right] = tr\mu^2 + r\sigma_\tau^2 + tr\sigma_\beta^2 + r\sigma^2$$

$$E(SSB) = E\left[\frac{\sum_{j=1}^{r}Y_{.j}^2}{t}\right] - E(C.F)$$

$$= tr\mu^2 + r\sigma_\tau^2 + tr\sigma_\beta^2 + r\sigma^2 - tr\mu^2 - r\sigma_\tau^2 - t\sigma_\beta^2 - \sigma^2$$

$$= t(r-1)\sigma_\beta^2 + (r-1)\sigma^2$$

$$E(SSE) = E(TSS) - E(SST) - E(SSB)$$

$$= r(t-1)\sigma_\tau^2 + t(r-1)\sigma_\beta^2 + (tr-1)\sigma^2 - r(t-1)\sigma_\tau^2 - (t-1)\sigma^2 - t(r-1)\sigma_\beta^2 - (r-1)\sigma^2$$

$$= (tr - 1 - t + 1 - r + 1)\sigma^2 = (tr - t - r + 1)\sigma^2$$

$$= (t(r-1) - 1(r-1))\sigma^2 = (t-1)(r-1)\sigma^2$$

$$E(MSE) = E\left[\frac{SSE}{(t-1)(r-1)}\right]$$

$$= \frac{E(SSE)}{(t-1)(r-1)} = \frac{(t-1)(r-1)\sigma^2}{(t-1)(r-1)} = \sigma^2$$

$$E(MST) = E\left[\frac{SST}{t-1}\right]$$

$$= \frac{E(SST)}{t-1} = \frac{r(t-1)\sigma_\tau^2 + (t-1)\sigma^2}{t-1} = \sigma^2 + r\sigma_\tau^2$$

$$E(MSB) = E\left[\frac{SSB}{r-1}\right]$$

$$= \frac{E(SSB)}{r-1} = \frac{t(r-1)\sigma_\beta^2 + (r-1)\sigma^2}{r-1} = \sigma^2 + t\sigma_\beta^2$$

Mixed Effect Model

Case I: In this model the effect of τ_i is fixed and effect of β_j is random

Assumptions:

1. $E(e_{ij}) = 0$
2. $E(e_{ij}e_{gh}) = 0$

3. $e_i \sim iidN(0,\sigma^2)$
4. $\beta_j \sim iidN(0,\sigma_\beta^2)$
5. $E(\beta_i\beta_j) = 0$
6. $E(\beta_j e_{ij}) = 0$
7. $\sum_{i=1}^{t}\tau_i = 0$

$$Y_{ij} = \mu + \tau_i + \beta_j + e_{ij} \quad \begin{cases} i = 1,2,3,\dots,t \\ j = 1,2,3,\dots,r \end{cases}$$

$$SSE = TSS - SST - SSB$$

$$TSS = \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 - C.F$$

$$C.F = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}\right)^2}{tr} = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu + \tau_i + \beta_j + e_{ij}\right)\right)^2}{tr}$$

$$= \frac{\left(tr\mu + r\sum_{i=1}^{t}\tau_i + t\sum_{j=1}^{r}\beta_j + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{\left(tr\mu + 0 + t\sum_{j=1}^{r}\beta_j + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{\left(tr\mu + t\sum_{j=1}^{r}\beta_j + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{t^2r^2\mu^2 + t^2\sum_{j=1}^{r}\beta_j^2 + t^2\sum\sum_{i\neq j}\beta_i\beta_j + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh} + 2t^2r\mu\sum_{j=1}^{r}\beta_j + 2tr\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}}{tr}$$

Apply expectation on both sides

$$= \frac{t^2r^2\mu^2 + t^2\sum_{j=1}^{r}E(\beta_j^2) + t^2\sum\sum_{i\neq j}E(\beta_i\beta_j) + \sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij}e_{gh}) + 2t^2r\mu\sum_{j=1}^{r}E(\beta_j) + 2tr\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}) + 2t\sum_{i=1}^{t}\sum_{j=1}^{r}E(\beta_j e_{ij})}{tr}$$

$$= \frac{t^2r^2\mu^2 + t^2r\sigma_\beta^2 + 0 + tr\sigma^2 + 0 + 0 + 0 + 0}{tr}$$

$$E(C.F) = tr\mu^2 + t\sigma_\beta^2 + \sigma^2$$

$$\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu + \tau_i + \beta_j + e_{ij}\right)^2$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu^2 + \tau_i^2 + \beta_j^2 + e_{ij}^2 + 2\mu\tau_i + 2\mu\beta_j + 2\mu e_{ij} + 2\tau_i\beta_j + 2\tau_i e_{ij} + 2\beta_j e_{ij}\right)$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + t\sum_{j=1}^{r}\beta_j^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 2r\mu\sum_{i=1}^{t}\tau_i + 2t\mu\sum_{j=1}^{r}\beta_j + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + t\sum_{j=1}^{r}\beta_j^2 + \sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2 + 0 + 2t\mu\sum_{j=1}^{r}\beta_j + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + t\sum_{j=1}^{r}\beta_j^2 + \sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2 + 2t\mu\sum_{j=1}^{r}\beta_j + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}$$

Apply expectation on both sides

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + t\sum_{j=1}^{r}E(\beta_j^2) + \sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}^2) + 2t\mu\sum_{j=1}^{r}E(\beta_j) + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i E(\beta_j) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i E(e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}E(\beta_j e_{ij})$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + tr\sigma_\beta^2 + tr\sigma^2 + 0 + 0 + 0 + 0 + 0$$

$$E\left(\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}^2\right) = tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + tr\sigma_\beta^2 + tr\sigma^2$$

$$E(TSS) = E\left(\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}^2\right) - E(C.F)$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + tr\sigma_\beta^2 + tr\sigma^2 - tr\mu^2 - t\sigma_\beta^2 - \sigma^2$$

$$= r\sum_{i=1}^{t}\tau_i^2 + tr\sigma_\beta^2 + tr\sigma^2 - t\sigma_\beta^2 - \sigma^2$$

$$= r\sum_{i=1}^{t}\tau_i^2 + t(r-1)\sigma_\beta^2 + (tr-1)\sigma^2$$

$$SST = \frac{\sum_{i=1}^{t}Y_{i.}^2}{r} - C.F$$

$$Y_{i.} = \sum_{j=1}^{r}Y_{ij} = \sum_{j=1}^{r}\left(\mu + \tau_i + \beta_j + e_{ij}\right) = r\mu + r\tau_i + \sum_{j=1}^{r}\beta_j + \sum_{j=1}^{r}e_{ij}$$

$$\frac{\sum_{i=1}^{t}Y_{i.}^2}{r} = \frac{\sum_{i=1}^{t}\left(r\mu + r\tau_i + \sum_{j=1}^{r}\beta_j + \sum_{j=1}^{r}e_{ij}\right)^2}{r}$$

$$= \frac{\sum_{i=1}^{t}\begin{pmatrix} r^2\mu^2 + r^2\tau_i^2 + \sum_{j=1}^{r}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + \sum_{j=1}^{r}e_{ij}^2 + \sum_{j\neq h}e_{ij}e_{ih} + 2r^2\mu\tau_i + 2r\mu\sum_{j=1}^{r}\beta_j \\ +2r\mu\sum_{j=1}^{r}e_{ij} + 2r\tau_i\sum_{j=1}^{r}\beta_j + 2r\tau_i\sum_{j=1}^{r}e_{ij} + 2\sum_{j=1}^{r}\beta_j e_{ij}\end{pmatrix}}{r}$$

$$= \frac{\begin{matrix} tr^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + t\sum_{j=1}^{r}\beta_j^2 + t\sum\sum_{i\neq j}\beta_i\beta_j + \sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij}e_{gh} + 2r^2\mu\sum_{i=1}^{t}\tau_i + 2tr\mu\sum_{j=1}^{r}\beta_j \\ +2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij} + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}\end{matrix}}{r}$$

$$= \frac{\begin{matrix} tr^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + t\sum_{j=1}^{r}\beta_j^2 + t\sum\sum_{i\neq j}\beta_i\beta_j + \sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij}e_{gh} + 0 + 2tr\mu\sum_{j=1}^{r}\beta_j \\ +2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij} + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}\end{matrix}}{r}$$

$$= \frac{\begin{matrix} tr^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + t\sum_{j=1}^{r}\beta_j^2 + t\sum\sum_{i\neq j}\beta_i\beta_j + \sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij}e_{gh} + 2tr\mu\sum_{j=1}^{r}\beta_j \\ +2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij} + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}\end{matrix}}{r}$$

Apply expectation on both sides

$$= \frac{\begin{matrix} tr^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + t\sum_{j=1}^{r}E(\beta_j^2) + t\sum\sum_{i\neq j}E(\beta_i\beta_j) + \sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij}e_{gh}) \\ +2tr\mu\sum_{j=1}^{r}E(\beta_j) + 2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}) + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i E(\beta_j) \\ +2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i E(e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}E(\beta_j e_{ij})\end{matrix}}{r}$$

$$= \frac{tr^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + tr\sigma_\beta^2 + 0 + tr\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{r}$$

$$E\left[\frac{\sum_{i=1}^{t}Y_{i.}^2}{r}\right] = tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + t\sigma_\beta^2 + t\sigma^2$$

$$E(SST) = E\left[\frac{\sum_{i=1}^{t}Y_{i.}^2}{r}\right] - E(C.F)$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + t\sigma_\beta^2 + t\sigma^2 - tr\mu^2 - t\sigma_\beta^2 - \sigma^2$$

$$= r\sum_{i=1}^{t}\tau_i^2 + (t-1)\sigma^2$$

$$SSB = \frac{\sum_{j=1}^{r}Y_{.j}^2}{t} - C.F$$

$$Y_{.j} = \sum_{i=1}^{t}Y_{ij} = \sum_{i=1}^{t}(\mu + \tau_i + \beta_j + e_{ij}) = t\mu + \sum_{i=1}^{t}\tau_i + t\beta_j + \sum_{i=1}^{t}e_{ij}$$

$$= t\mu + 0 + t\beta_j + \sum_{i=1}^{t}e_{ij} = t\mu + t\beta_j + \sum_{i=1}^{t}e_{ij}$$

$$\frac{\sum_{j=1}^{r}Y_{.j}^2}{t} = \frac{\sum_{j=1}^{r}\left(t\mu + t\beta_j + \sum_{i=1}^{t}e_{ij}\right)^2}{t}$$

$$= \frac{\sum_{j=1}^{r}\left(t^2\mu^2 + t^2\beta_j^2 + \sum_{i=1}^{t}e_{ij}^2 + \sum_{i\neq g}e_{ij}e_{gj} + 2t^2\mu\beta_j + 2t\mu\sum_{i=1}^{t}e_{ij} + 2t\beta_j\sum_{i=1}^{t}e_{ij}\right)}{t}$$

$$= \frac{t^2r\mu^2 + t^2\sum_{j=1}^{r}\beta_j^2 + \sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij}e_{gh} + 2t^2\mu\sum_{j=1}^{r}\beta_j + 2t\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij} + 2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}}{t}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} t^2r\mu^2 + t^2\sum_{j=1}^{r}E(\beta_j^2) + \sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij}e_{gh}) + +2t^2\mu\sum_{j=1}^{r}E(\beta_j) \\ +2t\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}) + 2t\sum_{i=1}^{t}\sum_{j=1}^{r}E(\beta_j e_{ij}) \end{array}}{t}$$

$$= \frac{t^2r\mu^2 + t^2r\sigma_\beta^2 + tr\sigma^2 + 0 + 0 + 0 + 0}{t}$$

$$E\left[\frac{\sum_{j=1}^{r}Y_{.j}^2}{t}\right] = tr\mu^2 + tr\sigma_\beta^2 + r\sigma^2$$

$$E(SSB) = E\left[\frac{\sum_{j=1}^{r}Y_{.j}^2}{t}\right] - E(C.F)$$

$$= tr\mu^2 + tr\sigma_\beta^2 + r\sigma^2 - tr\mu^2 - t\sigma_\beta^2 - \sigma^2$$

$$= t(r-1)\sigma_\beta^2 + (r-1)\sigma^2$$

$$E(SSE) = E(TSS) - E(SST) - E(SSB)$$

$$= r\sum_{i=1}^{t}\tau_i^2 + t(r-1)\sigma_\beta^2 + (tr-1)\sigma^2 - r\sum_{i=1}^{t}\tau_i^2 - (t-1)\sigma^2 - t(r-1)\sigma_\beta^2 - (r-1)\sigma^2$$

$$= (tr - 1 - t + 1 - r + 1)\sigma^2 = (tr - t - r + 1)\sigma^2$$

$$= (t(r-1) - 1(r-1))\sigma^2 = (t-1)(r-1)\sigma^2$$

$$E(MSE) = E\left[\frac{SSE}{(t-1)(r-1)}\right]$$

$$= \frac{E(SSE)}{(t-1)(r-1)} = \frac{(t-1)(r-1)\sigma^2}{(t-1)(r-1)} = \sigma^2$$

$$E(MST) = E\left[\frac{SST}{t-1}\right]$$

$$= \frac{E(SST)}{t-1} = \frac{r\sum_{i=1}^{t}\tau_i^2 + (t-1)\sigma^2}{t-1} = \sigma^2 + \frac{r\sum_{i=1}^{t}\tau_i^2}{t-1}$$

$$E(MSB) = E\left[\frac{SSB}{r-1}\right]$$

$$= \frac{E(SSB)}{r-1} = \frac{t(r-1)\sigma_\beta^2 + (r-1)\sigma^2}{r-1} = \sigma^2 + t\sigma_\beta^2$$

Case II: In this model the effect of τ_i is random and effect of β_j is fixed

Assumptions:

1. $E(e_{ij}) = 0$
2. $E(e_{ij}e_{gh}) = 0$
3. $e_i \sim iidN(0, \sigma^2)$
4. $\tau_i \sim iidN(0, \sigma_\tau^2)$
5. $E(\tau_i\tau_j) = 0$
6. $E(\tau_i e_{ij}) = 0$
7. $\sum_{j=1}^{r}\beta_j = 0$

$$Y_{ij} = \mu + \tau_i + \beta_j + e_{ij} \quad \begin{cases} i = 1,2,3,\dots,t \\ j = 1,2,3,\dots.,r \end{cases}$$

$$SSE = TSS - SST - SSB$$

$$TSS = \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 - C.F$$

$$C.F = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}\right)^2}{tr} = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu + \tau_i + \beta_j + e_{ij}\right)\right)^2}{tr}$$

$$= \frac{\left(tr\mu + r\sum_{i=1}^{t}\tau_i + t\sum_{j=1}^{r}\beta_j + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{\left(tr\mu + r\sum_{i=1}^{t}\tau_i + 0 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{\left(tr\mu + r\sum_{i=1}^{t}\tau_i + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\right)^2}{tr}$$

$$= \frac{t^2r^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + r^2\sum\sum_{i\neq j}\tau_i\tau_j + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh} + 2r^2t\mu\sum_{i=1}^{t}\tau_i + 2tr\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_j e_{ij}}{tr}$$

Apply expectation on both sides

$$= \frac{t^2r^2\mu^2 + r^2\sum_{i=1}^{t}E(\tau_i^2) + r^2\sum\sum_{i\neq j}E(\tau_i\tau_j) + \sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij}e_{gh}) + 2r^2t\mu\sum_{i=1}^{t}E(\tau_i) + 2tr\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}) + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}E(\tau_j e_{ij})}{tr}$$

$$= \frac{t^2r^2\mu^2 + tr^2\sigma_\tau^2 + 0 + tr\sigma^2 + 0 + 0 + 0 + 0}{tr}$$

$$E(C.F) = tr\mu^2 + r\sigma_\tau^2 + \sigma^2$$

$$\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu + \tau_i + \beta_j + e_{ij}\right)^2$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r}\left(\mu^2 + \tau_i^2 + \beta_j^2 + e_{ij}^2 + 2\mu\tau_i + 2\mu\beta_j + 2\mu e_{ij} + 2\tau_i\beta_j + 2\tau_i e_{ij} + 2\beta_j e_{ij}\right)$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + t\sum_{j=1}^{r}\beta_j^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 2r\mu\sum_{i=1}^{t}\tau_i + 2t\mu\sum_{j=1}^{r}\beta_j + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + t\sum_{j=1}^{r}\beta_j^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + 2r\mu\sum_{i=1}^{t}\tau_i + 0 + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}$$

$$= tr\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + t\sum_{j=1}^{r}\beta_j^2 + \sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2 + 2r\mu\sum_{i=1}^{t}\tau_i + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i\beta_j + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij}$$

Apply expectation on both sides

$$= tr\mu^2 + r\sum_{i=1}^{t}E(\tau_i^2) + t\sum_{j=1}^{r}\beta_j^2 + \sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}^2) + 2r\mu\sum_{i=1}^{t}E(\tau_i) + 2\mu\sum_{i=1}^{t}\sum_{j=1}^{r}E(e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}E(\tau_i)\beta_j + 2\sum_{i=1}^{t}\sum_{j=1}^{r}E(\tau_i e_{ij}) + 2\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j E(e_{ij})$$

$$= tr\mu^2 + tr\sigma_\tau^2 + t\sum_{j=1}^{r}\beta_j^2 + tr\sigma^2 + 0 + 0 + 0 + 0 + 0$$

$$E\left(\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}^2\right) = tr\mu^2 + tr\sigma_\tau^2 + t\sum_{j=1}^{r}\beta_j^2 + tr\sigma^2$$

$$E(TSS) = E\left(\sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}^2\right) - E(C.F)$$

$$= tr\mu^2 + tr\sigma_\tau^2 + t\sum_{j=1}^{r}\beta_j^2 + tr\sigma^2 - tr\mu^2 - r\sigma_\tau^2 - \sigma^2$$

$$= tr\sigma_\tau^2 + t\sum_{j=1}^{r}\beta_j^2 + tr\sigma^2 - r\sigma_\tau^2 - \sigma^2$$

$$= t\sum_{j=1}^{r}\beta_j^2 + r(t-1)\sigma_\tau^2 + (tr-1)\sigma^2$$

$$SST = \frac{\sum_{i=1}^{t}Y_{i.}^2}{r} - C.F$$

$$Y_{i.} = \sum_{j=1}^{r}Y_{ij} = \sum_{j=1}^{r}\left(\mu + \tau_i + \beta_j + e_{ij}\right) = r\mu + r\tau_i + \sum_{j=1}^{r}\beta_j + \sum_{j=1}^{r}e_{ij}$$

$$= r\mu + r\tau_i + 0 + \sum_{j=1}^{r}e_{ij} = r\mu + r\tau_i + \sum_{j=1}^{r}e_{ij}$$

$$\frac{\sum_{i=1}^{t}Y_{i.}^2}{r} = \frac{\sum_{i=1}^{t}\left(r\mu + r\tau_i + \sum_{j=1}^{r}e_{ij}\right)^2}{r}$$

$$= \frac{\sum_{i=1}^{t}\left(r^2\mu^2 + r^2\tau_i^2 + \sum_{j=1}^{r}e_{ij}^2 + \sum_{j\neq h}e_{ij}e_{ih} + 2r^2\mu\tau_i + 2r\mu\sum_{j=1}^{r}e_{ij} + 2r\tau_i\sum_{j=1}^{r}e_{ij}\right)}{r}$$

$$= \frac{tr^2\mu^2 + r^2\sum_{i=1}^{t}\tau_i^2 + \sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij}e_{gh} + 2r^2\mu\sum_{i=1}^{t}\tau_i + 2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r}e_{ij} + 2r\sum_{i=1}^{t}\sum_{j=1}^{r}\tau_i e_{ij}}{r}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} tr^2\mu^2 + r^2\sum_{i=1}^{t} E(\tau_i^2) + \sum_{i=1}^{t}\sum_{j=1}^{r} E\left(e_{ij}^2\right) + \sum_{i\neq g}\sum_{j\neq h} E\left(e_{ij}e_{gh}\right) + 2r^2\mu\sum_{i=1}^{t} E(\tau_i) \\ +2r\mu\sum_{i=1}^{t}\sum_{j=1}^{r} E(e_{ij}) + 2r\sum_{i=1}^{t}\sum_{j=1}^{r} E(\tau_i e_{ij}) \end{array}}{r}$$

$$= \frac{tr^2\mu^2 + tr^2\sigma_\tau^2 + tr\sigma^2 + 0 + 0 + 0 + 0}{r}$$

$$E\left[\frac{\sum_{i=1}^{t} Y_{i.}^2}{r}\right] = tr\mu^2 + tr\sigma_\tau^2 + t\sigma^2$$

$$E(SST) = E\left[\frac{\sum_{i=1}^{t} Y_{i.}^2}{r}\right] - E(C.F)$$

$$= tr\mu^2 + tr\sigma_\tau^2 + t\sigma^2 - tr\mu^2 - r\sigma_\tau^2 - \sigma^2$$

$$= r(t-1)\sigma_\tau^2 + (t-1)\sigma^2$$

$$SSB = \frac{\sum_{j=1}^{r} Y_{.j}^2}{t} - C.F$$

$$Y_{.j} = \sum_{i=1}^{t} Y_{ij} = \sum_{i=1}^{t}\left(\mu + \tau_i + \beta_j + e_{ij}\right) = t\mu + \sum_{i=1}^{t}\tau_i + t\beta_j + \sum_{i=1}^{t} e_{ij}$$

$$\frac{\sum_{j=1}^{r} Y_{.j}^2}{t} = \frac{\sum_{j=1}^{r}\left(t\mu + \sum_{i=1}^{t}\tau_i + t\beta_j + \sum_{i=1}^{t} e_{ij}\right)^2}{t}$$

$$= \frac{\sum_{j=1}^{r}\left(\begin{array}{c} t^2\mu^2 + \sum_{i=1}^{t}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + t^2\beta_j^2 + \sum_{i=1}^{t} e_{ij}^2 + \sum_{i\neq g} e_{ij}e_{gj} + 2t\mu\sum_{i=1}^{t}\tau_i + 2t^2\mu\beta_j \\ +2t\mu\sum_{i=1}^{t} e_{ij} + 2\sum_{i=1}^{t}\tau_i e_{ij} + 2t\beta_j\sum_{i=1}^{t}\tau_i + 2t\beta_j\sum_{i=1}^{t} e_{ij} \end{array}\right)}{t}$$

$$= \frac{\begin{array}{c} t^2r\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + r\sum\sum_{i\neq j}\tau_i\tau_j + t^2\sum_{j=1}^{r}\beta_j^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh} \\ +2tr\mu\sum_{i=1}^{t}\tau_i + 2t^2\mu\sum_{j=1}^{r}\beta_j + 2t\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\tau_i \\ +2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j\tau_i + 2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij} \end{array}}{t}$$

$$= \frac{\begin{array}{c} t^2r\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + r\sum\sum_{i\neq j}\tau_i\tau_j + t^2\sum_{j=1}^{r}\beta_j^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh} \\ +2tr\mu\sum_{i=1}^{t}\tau_i + 0 + 2t\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\tau_i \\ +2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j\tau_i + 2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij} \end{array}}{t}$$

$$= \frac{\begin{array}{c} t^2r\mu^2 + r\sum_{i=1}^{t}\tau_i^2 + r\sum\sum_{i\neq j}\tau_i\tau_j + t^2\sum_{j=1}^{r}\beta_j^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij}e_{gh} \\ +2tr\mu\sum_{i=1}^{t}\tau_i + 2t\mu\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij} + 2\sum_{i=1}^{t}\sum_{j=1}^{r} e_{ij}\tau_i \\ +2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j\tau_i + 2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j e_{ij} \end{array}}{t}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} t^2r\mu^2 + r\sum_{i=1}^{t} E(\tau_i^2) + r\sum\sum_{i\neq j} E\left(\tau_i\tau_j\right) + t^2\sum_{j=1}^{r}\beta_j^2 + \sum_{i=1}^{t}\sum_{j=1}^{r} E\left(e_{ij}^2\right) \\ +\sum_{i\neq g}\sum_{j\neq h} E\left(e_{ij}e_{gh}\right) + 2tr\mu\sum_{i=1}^{t} E(\tau_i) + 2t\mu\sum_{i=1}^{t}\sum_{j=1}^{r} E\left(e_{ij}\right) + 2\sum_{i=1}^{t}\sum_{j=1}^{r} E(e_{ij}\tau_i) \\ +2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j E(\tau_i) + 2t\sum_{i=1}^{t}\sum_{j=1}^{r}\beta_j E(e_{ij}) \end{array}}{t}$$

$$= \frac{t^2r\mu^2 + tr\sigma_\tau^2 + 0 + t^2\sum_{j=1}^{r}\beta_j^2 + tr\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0}{t}$$

$$E\left[\frac{\sum_{j=1}^{r} Y_{.j}^2}{t}\right] = tr\mu^2 + r\sigma_\tau^2 + t\sum_{j=1}^{r}\beta_j^2 + r\sigma^2$$

$$E(SSB) = E\left[\frac{\sum_{j=1}^{r} Y_{.j}^2}{t}\right] - E(C.F)$$

$$= tr\mu^2 + r\sigma_\tau^2 + t^2\sum_{j=1}^{r}\beta_j^2 + r\sigma^2 - tr\mu^2 - r\sigma_\tau^2 - \sigma^2$$

$$= t^2\sum_{j=1}^{r}\beta_j^2 + (r-1)\sigma^2$$

$$E(SSE) = E(TSS) - E(SST) - E(SSB)$$

$$= t\sum_{j=1}^{r}\beta_j^2 + r(t-1)\sigma_\tau^2 + (tr{-}1)\sigma^2 - r(t-1)\sigma_\tau^2 - (t-1)\sigma^2 - t\sum_{j=1}^{r}\beta_j^2 - (r-1)\sigma^2$$

$$= (tr - 1 - t + 1 - r + 1)\sigma^2 = (tr - t - r + 1)\sigma^2$$

$$= (t(r-1) - 1(r-1))\sigma^2 = (t-1)(r-1)\sigma^2$$

$$E(MSE) = E\left[\frac{SSE}{(t-1)(r-1)}\right]$$

$$= \frac{E(SSE)}{(t-1)(r-1)} = \frac{(t-1)(r-1)\sigma^2}{(t-1)(r-1)} = \sigma^2$$

$$E(MST) = E\left[\frac{SST}{t-1}\right]$$

$$= \frac{E(SST)}{t-1} = \frac{r(t-1)\sigma_\tau^2 + (t-1)\sigma^2}{t-1} = \sigma^2 + r\sigma_\tau^2$$

$$E(MSB) = E\left[\frac{SSB}{r-1}\right]$$

$$= \frac{E(SSB)}{r-1} = \frac{t\sum_{j=1}^{r}\beta_j^2 + (r-1)\sigma^2}{r-1} = \sigma^2 + \frac{t\sum_{j=1}^{r}\beta_j^2}{r-1}$$

Estimation of Missing Observations

Case I: One Missing Value

Blocks	T_1	T_2	…	T_i	…	T_t	Total
B_1	Y_{11}	Y_{21}	…	Y_{i1}	…	Y_{t1}	$Y_{.1}$
B_2	Y_{12}	Y_{22}	…	Y_{i2}	…	Y_{t2}	$Y_{.2}$
⋮	⋮	⋮		⋮		⋮	⋮
B_j	Y_{1j}	Y_{2j}	…	Y_{ij}	…	Y_{tj}	$Y_{.j}$
⋮	⋮	⋮	Y_{cd}	⋮		⋮	$Y'_{.d} + Y_{cd}$
B_r	Y_{1r}	Y_{2r}	…	Y_{ir}	…	Y_{tr}	$Y_{.1}$
Total	$Y_{1.}$	$Y_{2.}$	$Y'_{c.} + Y_{cd}$	$Y_{i.}$	…	$Y_{t.}$	$Y'_{..} + Y_{cd}$

$$C.F = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}\right)^2}{tr} = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd}\right)^2}{tr}$$

$$SSE = TSS - SST - SSB$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 + \hat{Y}_{cd}^2 - C.F - \frac{1}{r}\left[\sum_{i=1}^{t} Y_{i.}^2 + \left(Y'_{c.} + \hat{Y}_{cd}\right)^2\right] + C.F - \frac{1}{t}\left[\sum_{j=1}^{r} Y_{.j}^2 + \left(Y'_{.d} + \hat{Y}_{cd}\right)^2\right] + C.F$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 + \hat{Y}_{cd}^2 - \frac{\sum_{i=1}^{t} Y_{i.}^2}{r} - \frac{\left(Y'_{c.} + \hat{Y}_{cd}\right)^2}{r} - \frac{\sum_{j=1}^{r} Y_{.j}^2}{t} - \frac{\left(Y'_{.d} + \hat{Y}_{cd}\right)^2}{t} + \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd}\right)^2}{tr}$$

$$\frac{\partial SSE}{\partial \hat{Y}_{cd}} = 0$$

$$0 + 2\hat{Y}_{cd} - 0 - \frac{2\left(Y'_{c.} + \hat{Y}_{cd}\right)}{r} - 0 - \frac{2\left(Y'_{.d} + \hat{Y}_{cd}\right)}{t} + \frac{2\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd}\right)}{tr} = 0$$

$$\frac{2tr\hat{Y}_{cd} - 2t\left(Y'_{c.} + \hat{Y}_{cd}\right) - 2r\left(Y'_{.d} + \hat{Y}_{cd}\right) + 2\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd}\right)}{tr} = 0$$

$$\frac{2\left(tr\hat{Y}_{cd} - t\left(Y'_{c.} + \hat{Y}_{cd}\right) - r\left(Y'_{.d} + \hat{Y}_{cd}\right) + \left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd}\right)\right)}{tr} = 0$$

$$tr\hat{Y}_{cd} - tY'_{c.} - t\hat{Y}_{cd} - rY'_{.d} - r\hat{Y}_{cd} + \sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd} = 0$$

$$(tr - t - r + 1)\hat{Y}_{cd} - tY'_{c.} - rY'_{.d} + \sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} = 0$$

$$(tr - t - r + 1)\hat{Y}_{cd} = tY'_{c.} + rY'_{.d} - \sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij}$$

$$\hat{Y}_{cd} = \frac{tY'_{c.} + rY'_{.d} - \sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij}}{(t-1)(r-1)}$$

Case II: Two Missing Values in same treatment but different blocks

Blocks	T_1	T_2	...	T_i	...	T_t	Total
B_1	Y_{11}	Y_{21}	...	Y_{i1}	...	Y_{t1}	$Y_{.1}$
B_2	Y_{12}	Y_{22}	...	Y_{i2}	...	Y_{t2}	$Y_{.2}$
⋮	⋮	⋮	Y_{ce}	⋮		⋮	$Y'_{.e} + Y_{ce}$
B_j	Y_{1j}	Y_{2j}	...	Y_{ij}	...	Y_{tj}	$Y_{.j}$
⋮	⋮	⋮	Y_{cd}	⋮		⋮	$Y'_{.d} + Y_{cd}$
B_r	Y_{1r}	Y_{2r}	...	Y_{ir}	...	Y_{tr}	$Y_{.1}$
Total	$Y_{1.}$	$Y_{2.}$	$Y'_{c.} + Y_{cd} + Y_{ce}$	$Y_{i.}$	...	$Y_{t.}$	$Y'_{..} + Y_{cd} + Y_{ce}$

$$C.F = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}\right)^2}{tr} = \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd} + \hat{Y}_{ce}\right)^2}{tr}$$

$$SSE = TSS - SST - SSB$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 + \hat{Y}_{cd}^2 + \hat{Y}_{ce}^2 - C.F - \frac{1}{r}\left[\sum_{i=1}^{t} Y_{i.}^2 + \left(Y'_{c.} + \hat{Y}_{cd} + \hat{Y}_{ce}\right)^2\right] + C.F - \frac{1}{t}\left[\sum_{j=1}^{r} Y_{.j}^2 + \left(Y'_{.e} + \hat{Y}_{ce}\right)^2 + \left(Y'_{.d} + \hat{Y}_{cd}\right)^2\right] + C.F$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 + \hat{Y}_{cd}^2 + \hat{Y}_{ce}^2 - \frac{1}{r}\left[\sum_{i=1}^{t} Y_{i.}^2 + \left(Y'_{c.} + \hat{Y}_{cd} + \hat{Y}_{ce}\right)^2\right] - \frac{1}{t}\left[\sum_{j=1}^{r} Y_{.j}^2 + \left(Y'_{.e} + \hat{Y}_{ce}\right)^2 + \left(Y'_{.d} + \hat{Y}_{cd}\right)^2\right] + \frac{\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd} + \hat{Y}_{ce}\right)^2}{tr}$$

$$\frac{\partial SSE}{\partial \hat{Y}_{cd}} = 0$$

$$0 + 2\hat{Y}_{cd} + 0 - 0 - \frac{2\left(Y'_{c.} + \hat{Y}_{cd} + \hat{Y}_{ce}\right)}{r} - 0 - \frac{2\left(Y'_{.d} + \hat{Y}_{cd}\right)}{t} - 0 + \frac{2\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd} + \hat{Y}_{ce}\right)}{tr} = 0$$

$$\frac{2tr\hat{Y}_{cd} - 2t\left(Y'_{c.} + \hat{Y}_{cd} + \hat{Y}_{ce}\right) - 2r\left(Y'_{.d} + \hat{Y}_{cd}\right) + 2\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd} + \hat{Y}_{ce}\right)}{tr} = 0$$

$$\frac{2\left(tr\hat{Y}_{cd} - t\left(Y'_{c.} + \hat{Y}_{cd} + \hat{Y}_{ce}\right) - r\left(Y'_{.d} + \hat{Y}_{cd}\right) + \left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd} + \hat{Y}_{ce}\right)\right)}{tr} = 0$$

$$tr\hat{Y}_{cd} - tY'_{c.} - t\hat{Y}_{cd} - t\hat{Y}_{ce} - rY'_{.d} - r\hat{Y}_{cd} + \sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd} + \hat{Y}_{ce} = 0$$

$$(tr - t - r + 1)\hat{Y}_{cd} + (1 - t)\hat{Y}_{ce} - tY'_{c.} - rY'_{.d} + \sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} = 0$$

$$(tr - t - r + 1)\hat{Y}_{cd} + (1 - t)\hat{Y}_{ce} = tY'_{c.} + rY'_{.d} - \sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij}$$

Let $E = tr - t - r + 1, \quad F = 1 - t, \; Z_{cd} = tY'_{c.} + rY'_{.d} - \sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij}$

$$E\hat{Y}_{cd} + F\hat{Y}_{ce} = Z_{cd} \qquad (1)$$

$$\frac{\partial SSE}{\partial \hat{Y}_{ce}} = 0$$

$$0 + 0 + 2\hat{Y}_{ce} - 0 - \frac{2\left(Y'_{c.} + \hat{Y}_{cd} + \hat{Y}_{ce}\right)}{r} - 0 - \frac{2\left(Y'_{.e} + \hat{Y}_{ce}\right)}{t} - 0 + \frac{2\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd} + \hat{Y}_{ce}\right)}{tr} = 0$$

$$\frac{2tr\hat{Y}_{ce} - 2t\left(Y'_{c.} + \hat{Y}_{cd} + \hat{Y}_{ce}\right) - 2r\left(Y'_{.e} + \hat{Y}_{ce}\right) + 2\left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd} + \hat{Y}_{ce}\right)}{tr} = 0$$

$$\frac{2\left(tr\hat{Y}_{ce} - t\left(Y'_{c.} + \hat{Y}_{cd} + \hat{Y}_{ce}\right) - r\left(Y'_{.e} + \hat{Y}_{ce}\right) + \left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd} + \hat{Y}_{ce}\right)\right)}{tr} = 0$$

$$tr\hat{Y}_{ce} - t\left(Y'_{c.} + \hat{Y}_{cd} + \hat{Y}_{ce}\right) - r\left(Y'_{.e} + \hat{Y}_{ce}\right) + \left(\sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd} + \hat{Y}_{ce}\right) = 0$$

$$tr\hat{Y}_{ce} - tY'_{c.} - t\hat{Y}_{cd} - t\hat{Y}_{ce} - rY'_{.e} - r\hat{Y}_{ce} + \sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} + \hat{Y}_{cd} + \hat{Y}_{ce} = 0$$

$$(tr - t - r + 1)\hat{Y}_{ce} + (1 - t)\hat{Y}_{cd} - tY'_{c.} - rY'_{.e} + \sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij} = 0$$

$$(tr - t - r + 1)\hat{Y}_{ce} + (1 - t)\hat{Y}_{cd} = tY'_{c.} + rY'_{.e} - \sum_{i=1}^{t}\sum_{j=1}^{r} Y'_{ij}$$

$$E\hat{Y}_{ce} + F\hat{Y}_{cd} = Z_{ce} \qquad (2)$$

Multiply equation (1) by F and equation (2) by E and then subtract it

$$\begin{array}{c} EF\hat{Y}_{cd} + F^2\hat{Y}_{ce} = FZ_{cd} \\ \pm EF\hat{Y}_{cd} \pm E^2\hat{Y}_{ce} = \pm EZ_{ce} \\ \hline (F^2 - E^2)\hat{Y}_{ce} = FZ_{cd} - EZ_{ce} \end{array}$$

$$\hat{Y}_{ce} = \frac{FZ_{cd} - EZ_{ce}}{(F^2 - E^2)}$$

Multiply equation (1) by E and equation (2) by F and then subtract it

$$\begin{array}{c} EF\hat{Y}_{ce} + E^2\hat{Y}_{cd} = EZ_{cd} \\ \pm EF\hat{Y}_{ce} \pm F^2\hat{Y}_{cd} = \pm FZ_{ce} \\ \hline (E^2 - F^2)\hat{Y}_{cd} = EZ_{cd} - FZ_{ce} \end{array}$$

$$\hat{Y}_{cd} = \frac{EZ_{cd} - FZ_{ce}}{(E^2 - F^2)}$$

Exercise:

Estimate P-missing observations in different treatments and different blocks.

Estimate the two missing observations in different treatments but same block.

Efficiency of RCBD relative to CRD

RE (RCB, CR): the relative efficiency of the randomized complete block design compared to a completely randomized design. Did blocking increase the precision for comparing treatment means in a given experiment?

$$RE(RCB, CR) = \frac{MSE_{CR}}{MSE_{RCB}} = \frac{(r-1)MSB + r(t-1)MSE}{(rt-1)MSE}$$

Latin Square Design

Latin Square Designs are probably not used as much as they should be - they are very efficient designs. Latin square designs allow for two blocking factors. In other words, these designs are used to simultaneously control (or eliminate) **two sources of nuisance variability**. For instance, if you had a plot of land the fertility of this land might change in both directions, North -- South and East -- West due to soil or moisture gradients. So, both rows and columns can be used as blocking factors. However, you can use Latin squares in lots of other settings. As we shall see, Latin squares can be used as much as the RCBD in industrial experimentation as well as other experiments.

Whenever, you have more than one blocking factor a Latin square design will allow you to remove the variation for these two sources from the error variation. So, consider we had a plot of land, we might have blocked it in columns and rows, i.e. each row is a level of the row factor, and each column is a level of the column factor. We can remove the variation from our measured response in both directions if we consider both rows and columns as factors in our design.

The Latin Square Design gets its name from the fact that we can write it as a square with Latin letters to correspond to the treatments. The treatment factor levels are the Latin letters in the Latin square design. The number of rows and columns has to correspond to the number of treatment levels. So, if we have four treatments then we would need to have four rows and four columns in order to create a Latin square. This gives us a design where we have each of the treatments and in each row and in each column.

Experimental Layout

Latin square are always constructed by rotation e.g. in case of 4 treatments A,B,C,D we get,

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

Example:
A courier company is interested in deciding between five brand D,P,F,C and R of car for its next purchase of fleet cars.
1. The brands are all comparable in purchase price.
2. The company wants to carry out a study that will enable them to compare the brands w.r.t operating costs.
3. For this purpose they select five drivers (Rows)
4. In addition the study will be carried out over a five week period (Columns=Weeks)
5. Each week a driver is assigned to a car using randomization and a Latin square design.
6. The average cost per mile is recorded at the end of each week and is tabulated below:

Drivers	Week				
	1	2	3	4	5
1	5.83	6.22	7.67	9.43	6.57
	D	P	F	C	R
2	4.80	7.56	10.34	5.82	9.86
	P	D	C	R	F
3	7.43	11.29	7.01	10.48	9.27
	F	C	R	D	P
4	6.60	9.54	11.11	10.84	15.05
	R	F	D	P	C
5	11.24	6.34	11.30	12.58	16.04
	C	R	P	F	D

Statistical Model and Analysis

The linear statistical model for LSD is

$$Y_{ij(k)} = \mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)} \begin{cases} i = 1,2,\dots,p \\ j = 1,2,\dots,p \\ k = 1,2,\dots,p \end{cases}$$

Where

μ True mean effect
τ_i effect of ith row
β_j effect of jth row

γ_k effect of kth treatment

Formulation of hypotheses:

$$H_0: \gamma_k = 0$$
$$H_0': \tau_i = 0$$
$$H_0'': \beta_j = 0$$
$$H_1: \gamma_k \neq 0$$
$$H_0': \tau_i \neq 0$$
$$H_0'': \beta_j \neq 0$$

Level of significance:

$$\alpha = 0.05.0.01.0.001.0.10$$

Test Statistic:

$$F_1 = \frac{s_r^2}{s_e^2}, \quad F_2 = \frac{s_c^2}{s_e^2}, \quad F_3 = \frac{s_t^2}{s_e^2}$$

S.O.V	d.f	SS	MS	F
Rows	$p-1$	SSR	$s_r^2 = \frac{SSR}{p-1}$	$F_1 = \frac{s_r^2}{s_e^2}$
Columns	$p-1$	SSC	$s_c^2 = \frac{SSC}{p-1}$	$F_2 = \frac{s_c^2}{s_e^2}$
Treatment	$p-1$	SST	$s_t^2 = \frac{SST}{p-1}$	$F_3 = \frac{s_t^2}{s_e^2}$
Error	$(p-1)(p-2)$	SSE	$s_e^2 = \frac{SSE}{(p-1)(p-2)}$	
Total	p^2-1	TSS		

Where

$$C.F = \frac{Y_{\cdots}^2}{p^2}, \ SSR = \frac{1}{p}\sum R_i^2 - C.F, \ SSC = \frac{1}{p}\sum C_j^2 - C.F$$
$$SST = \frac{1}{p}\sum T_k^2 - C.F, \ SSE = TSS - SSR - SSC - SST$$

C.R

$F_1 \geq F_{\alpha(p-1,(p-1)(p-2))}$

$F_2 \geq F_{\alpha(p-1,(p-1)(p-2))}$

$F_3 \geq F_{\alpha(p-1,(p-1)(p-2))}$

Conclusion

If calculated value of F falls in the critical region then we reject null hypotheses.

Advantages of LSD

1. Greater power than the RBD when there are two external sources of variation.
2. Easy to analyze.

Disadvantages of LSD

1. The number of treatments, rows and columns must be the same.
2. Squares smaller than 5×5 are not practical because of the small number of degrees of freedom for error.

3. The effect of each treatment must be approximately the same across rows and columns.

Estimation of Model Prameters

$$S = \sum_{i=1}^{p}\sum_{j=1}^{p}\left(Y_{ij(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k\right)^2$$

$$\frac{\partial S}{\partial \hat{\mu}} = 2\sum_{i=1}^{p}\sum_{j=1}^{p}\left(Y_{ij(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\sum_{j=1}^{p}\left(Y_{ij(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k\right) = 0$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)} - p^2\hat{\mu} - p\sum_{i=1}^{p}\hat{\tau}_i - p\sum_{j=1}^{p}\hat{\beta}_j - p\sum_{k=1}^{p}\hat{\gamma}_k = 0$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)} = p^2\hat{\mu} + p\sum_{i=1}^{p}\hat{\tau}_i + p\sum_{j=1}^{p}\hat{\beta}_j + p\sum_{k=1}^{p}\hat{\gamma}_k$$

For unique solutions Put $\sum_{i=1}^{p}\hat{\tau}_i = 0, \sum_{j=1}^{p}\hat{\beta}_j = 0,\ \sum_{k=1}^{p}\hat{\gamma}_k = 0$

$$\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)} = p^2\hat{\mu} + 0 + 0 + 0$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)} = p^2\hat{\mu}$$

$$\hat{\mu} = \frac{\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}}{p^2} = \bar{\bar{Y}}$$

$$S = \sum_{j=1}^{p}\left(Y_{1j(k)} - \hat{\mu} - \hat{\tau}_1 - \hat{\beta}_j - \hat{\gamma}_k\right)^2 + \sum_{j=1}^{p}\left(Y_{2j(k)} - \hat{\mu} - \hat{\tau}_2 - \hat{\beta}_j - \hat{\gamma}_k\right)^2$$
$$+ \sum_{j=1}^{p}\left(Y_{3j(k)} - \hat{\mu} - \hat{\tau}_3 - \hat{\beta}_j - \hat{\gamma}_k\right)^2 + \cdots + \sum_{j=1}^{p}\left(Y_{pj(k)} - \hat{\mu} - \hat{\tau}_p - \hat{\beta}_j - \hat{\gamma}_k\right)^2$$

Differentiate w.r.t $\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3, \ldots \hat{\tau}_p$

$$\frac{\partial S}{\partial \hat{\tau}_1} = 2\sum_{j=1}^{p}\left(Y_{1j(k)} - \hat{\mu} - \hat{\tau}_1 - \hat{\beta}_j - \hat{\gamma}_k\right)(-1) = 0$$

$$-2\sum_{j=1}^{p}\left(Y_{1j(k)}-\hat{\mu}-\hat{\tau}_1-\hat{\beta}_j-\hat{\gamma}_k\right)=0$$

$$\sum_{j=1}^{p}Y_{1j(k)}-p\hat{\mu}-p\hat{\tau}_1-\sum_{j=1}^{p}\hat{\beta}_j-\sum_{k=1}^{p}\hat{\gamma}_k=0$$

$$\sum_{j=1}^{p}Y_{1j(k)}=p\hat{\mu}+p\hat{\tau}_1+\sum_{j=1}^{p}\hat{\beta}_j+\sum_{k=1}^{p}\hat{\gamma}_k$$

$$R_1=p\hat{\mu}+p\hat{\tau}_1+\sum_{j=1}^{p}\hat{\beta}_j+\sum_{k=1}^{p}\hat{\gamma}_k$$

$$\frac{\partial S}{\partial\hat{\tau}_2}=2\sum_{j=1}^{p}\left(Y_{2j(k)}-\hat{\mu}-\hat{\tau}_2-\hat{\beta}_j-\hat{\gamma}_k\right)(-1)=0$$

$$-2\sum_{j=1}^{p}\left(Y_{2j(k)}-\hat{\mu}-\hat{\tau}_2-\hat{\beta}_j-\hat{\gamma}_k\right)=0$$

$$\sum_{j=1}^{p}Y_{2j(k)}-p\hat{\mu}-p\hat{\tau}_2-\sum_{j=1}^{p}\hat{\beta}_j-\sum_{k=1}^{p}\hat{\gamma}_k=0$$

$$\sum_{j=1}^{p}Y_{2j(k)}=p\hat{\mu}+p\hat{\tau}_2+\sum_{j=1}^{p}\hat{\beta}_j+\sum_{k=1}^{p}\hat{\gamma}_k$$

$$R_2=p\hat{\mu}+p\hat{\tau}_2+\sum_{j=1}^{p}\hat{\beta}_j+\sum_{k=1}^{p}\hat{\gamma}_k$$

$$\frac{\partial S}{\partial\hat{\tau}_p}=2\sum_{j=1}^{p}\left(Y_{pj(k)}-\hat{\mu}-\hat{\tau}_p-\hat{\beta}_j-\hat{\gamma}_k\right)(-1)=0$$

$$-2\sum_{j=1}^{p}\left(Y_{pj(k)}-\hat{\mu}-\hat{\tau}_p-\hat{\beta}_j-\hat{\gamma}_k\right)=0$$

$$\sum_{j=1}^{p}Y_{pj(k)}-p\hat{\mu}-p\hat{\tau}_p-\sum_{j=1}^{p}\hat{\beta}_j-\sum_{k=1}^{p}\hat{\gamma}_k=0$$

$$\sum_{j=1}^{p}Y_{pj(k)}=p\hat{\mu}+p\hat{\tau}_p+\sum_{j=1}^{p}\hat{\beta}_j+\sum_{k=1}^{p}\hat{\gamma}_k$$

$$R_p = p\hat{\mu} + p\hat{\tau}_p + \sum_{j=1}^{p} \hat{\beta}_j + \sum_{k=1}^{p} \hat{\gamma}_k$$

For unique solutions Put $\sum_{j=1}^{p} \hat{\beta}_j = 0, \sum_{k=1}^{p} \hat{\gamma}_k = 0$

From above equations, we get

$$R_1 = p\hat{\mu} + p\hat{\tau}_1$$
$$\hat{\tau}_1 = \frac{R_1}{p} - \frac{p\hat{\mu}}{p}$$
$$\hat{\tau}_1 = \frac{R_1}{p} - \bar{\bar{Y}}$$

Similarly

$$\hat{\tau}_2 = \frac{R_2}{p} - \bar{\bar{Y}}$$
$$\vdots \qquad \vdots$$
$$\hat{\tau}_p = \frac{R_p}{p} - \bar{\bar{Y}}$$

$$S = \sum_{i=1}^{p} \left(Y_{i1(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_1 - \hat{\gamma}_k\right)^2 + \sum_{i=1}^{p} \left(Y_{i2(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_2 - \hat{\gamma}_k\right)^2 + \sum_{i=1}^{p} \left(Y_{i3(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_3 - \hat{\gamma}_k\right)^2 + \cdots + \sum_{i=1}^{p} \left(Y_{ip(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_p - \hat{\gamma}_k\right)^2$$

Differentiate w.r.t $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \ldots \hat{\beta}_p$

$$\frac{\partial S}{\partial \hat{\beta}_1} = 2\sum_{i=1}^{p} \left(Y_{i1(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_1 - \hat{\gamma}_k\right)(-1) = 0$$

$$-2\sum_{i=1}^{p} \left(Y_{i1(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_1 - \hat{\gamma}_k\right) = 0$$

$$\sum_{i=1}^{p} Y_{i1(k)} - p\hat{\mu} - \sum_{i=1}^{p} \hat{\tau}_i - p\hat{\beta}_1 - \sum_{k=1}^{p} \hat{\gamma}_k = 0$$

$$\sum_{i=1}^{p} Y_{i1(k)} = p\hat{\mu} + \sum_{i=1}^{p} \hat{\tau}_i + p\hat{\beta}_1 + \sum_{k=1}^{p} \hat{\gamma}_k$$

$$C_1 = p\hat{\mu} + \sum_{i=1}^{p} \hat{\tau}_i + p\hat{\beta}_1 + \sum_{k=1}^{p} \hat{\gamma}_k$$

$$\frac{\partial S}{\partial \hat{\beta}_2} = 2\sum_{i=1}^{p}\left(Y_{i2(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_2 - \hat{\gamma}_k\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\left(Y_{i2(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_2 - \hat{\gamma}_k\right) = 0$$

$$\sum_{i=1}^{p} Y_{i2(k)} - p\hat{\mu} - \sum_{i=1}^{p}\hat{\tau}_i - p\hat{\beta}_2 - \sum_{k=1}^{p}\hat{\gamma}_k = 0$$

$$\sum_{i=1}^{p} Y_{i2(k)} = p\hat{\mu} + \sum_{i=1}^{p}\hat{\tau}_i + p\hat{\beta}_2 + \sum_{k=1}^{p}\hat{\gamma}_k$$

$$C_2 = p\hat{\mu} + \sum_{i=1}^{p}\hat{\tau}_i + p\hat{\beta}_2 + \sum_{k=1}^{p}\hat{\gamma}_k$$

$$\frac{\partial S}{\partial \hat{\beta}_p} = 2\sum_{i=1}^{p}\left(Y_{ip(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_p - \hat{\gamma}_k\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\left(Y_{ip(k)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_p - \hat{\gamma}_k\right) = 0$$

$$\sum_{i=1}^{p} Y_{ip(k)} - p\hat{\mu} - \sum_{i=1}^{p}\hat{\tau}_i - p\hat{\beta}_p - \sum_{k=1}^{p}\hat{\gamma}_k = 0$$

$$\sum_{i=1}^{p} Y_{ip(k)} = p\hat{\mu} + \sum_{i=1}^{p}\hat{\tau}_i + p\hat{\beta}_p + \sum_{k=1}^{p}\hat{\gamma}_k$$

$$C_p = p\hat{\mu} + \sum_{i=1}^{p}\hat{\tau}_i + p\hat{\beta}_p + \sum_{k=1}^{p}\hat{\gamma}_k$$

For unique solutions Put $\sum_{k=1}^{p}\hat{\gamma}_k = 0, \sum_{i=1}^{p}\hat{\tau}_i = 0$

From above equations, we get

$$C_1 = p\hat{\mu} + p\hat{\beta}_1$$

$$\hat{\beta}_1 = \frac{C_1}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\beta}_1 = \frac{C_1}{p} - \bar{\bar{Y}}$$

Similarly

$$\hat{\beta}_2 = \frac{C_2}{p} - \bar{\bar{Y}}$$
$$\vdots \qquad \vdots$$
$$\hat{\beta}_p = \frac{C_p}{p} - \bar{\bar{Y}}$$

$$S = \sum_{i=1}^{p}\left(Y_{ij(1)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_1\right)^2 + \sum_{i=1}^{p}\left(Y_{ij(2)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_2\right)^2 + \sum_{i=1}^{p}\left(Y_{ij(3)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_3\right)^2 + \cdots + \sum_{i=1}^{p}\left(Y_{ij(p)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_p\right)^2$$

Differentiate w.r.t $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \dots, \hat{\gamma}_k$

$$\frac{\partial S}{\partial \hat{\gamma}_1} = 2\sum_{i=1}^{p}\left(Y_{ij(1)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_1\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\left(Y_{ij(1)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_1\right) = 0$$

$$\sum_{i=1}^{p} Y_{ij(1)} - p\hat{\mu} - \sum_{i=1}^{p}\hat{\tau}_i - \sum_{j=1}^{p}\hat{\beta}_j - p\hat{\gamma}_1 = 0$$

$$\sum_{i=1}^{p} Y_{ij(1)} = p\hat{\mu} + \sum_{i=1}^{p}\hat{\tau}_i + \sum_{j=1}^{p}\hat{\beta}_j + p\hat{\gamma}_1$$

$$T_1 = p\hat{\mu} + \sum_{i=1}^{p}\hat{\tau}_i + \sum_{j=1}^{p}\hat{\beta}_j + p\hat{\gamma}_1$$

$$\frac{\partial S}{\partial \hat{\gamma}_2} = 2\sum_{i=1}^{p}\left(Y_{ij(2)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_2\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\left(Y_{ij(2)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_2\right) = 0$$

$$\sum_{i=1}^{p} Y_{ij(2)} - p\hat{\mu} - \sum_{i=1}^{p}\hat{\tau}_i - \sum_{j=1}^{p}\hat{\beta}_j - p\hat{\gamma}_2 = 0$$

$$\sum_{i=1}^{p} Y_{ij(2)} = p\hat{\mu} + \sum_{i=1}^{p}\hat{\tau}_i + \sum_{j=1}^{p}\hat{\beta}_j + p\hat{\gamma}_2$$

$$T_2 = p\hat{\mu} + \sum_{i=1}^{p} \hat{\tau}_i + \sum_{j=1}^{p} \hat{\beta}_j + p\hat{\gamma}_2$$

$$\frac{\partial S}{\partial \hat{\gamma}_p} = 2\sum_{i=1}^{p} \left(Y_{ij(p)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_p\right)(-1) = 0$$

$$-2\sum_{i=1}^{p} \left(Y_{ij(p)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_1\right) = 0$$

$$\sum_{i=1}^{p} Y_{ij(1)} - p\hat{\mu} - \sum_{i=1}^{p} \hat{\tau}_i - \sum_{j=1}^{p} \hat{\beta}_j - p\hat{\gamma}_p = 0$$

$$\sum_{i=1}^{p} Y_{ij(p)} = p\hat{\mu} + \sum_{i=1}^{p} \hat{\tau}_i + \sum_{j=1}^{p} \hat{\beta}_j + p\hat{\gamma}_p$$

$$T_p = p\hat{\mu} + \sum_{i=1}^{p} \hat{\tau}_i + \sum_{j=1}^{p} \hat{\beta}_j + p\hat{\gamma}_p$$

For unique solutions Put $\sum_{k=1}^{p} \hat{\gamma}_k = 0, \sum_{j=1}^{p} \hat{\beta}_j = 0$

From above equations, we get

$$T_1 = p\hat{\mu} + p\hat{\gamma}_1$$

$$\hat{\gamma}_1 = \frac{T_1}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\gamma}_1 = \frac{T_1}{p} - \bar{\bar{Y}}$$

Similarly

$$\hat{\gamma}_2 = \frac{T_2}{p} - \bar{\bar{Y}}$$

$$\vdots \qquad \vdots$$

$$\hat{\gamma}_p = \frac{T_p}{p} - \bar{\bar{Y}}$$

Expected Mean Square Error

Fixed Effect Model

Assumptions:

The effect of treatments, rows and columns are fixed and we assume that

1. $\sum_{i=1}^{p} \hat{\tau}_i = 0$
2. $\sum_{i=1}^{p} \hat{\beta}_j = 0$

3. $\sum_{i=1}^{p} \hat{\gamma}_k = 0$
4. $E\left(e_{ij(k)}\right) = 0$
5. $E\left(e_{ij(k)} e_{gh(l)}\right) = 0$

$$Y_{ij(k)} = \mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}$$

$$SSE = TSS - SSR - SSC - SST$$

$$TSS = \sum_{i=1}^{p} \sum_{j=1}^{p} Y_{ij(k)}^2 - C.F$$

$$C.F = \frac{\left(\sum_{i=1}^{p} \sum_{j=1}^{p} Y_{ij(k)}\right)^2}{p^2} = \frac{\left(\sum_{i=1}^{p} \sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p} \tau_i + p\sum_{j=1}^{p} \beta_j + p\sum_{k=1}^{p} \gamma_k + \sum_{i=1}^{p} \sum_{j=1}^{p} e_{ij(k)}\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + 0 + 0 + 0 + \sum_{i=1}^{p} \sum_{j=1}^{p} e_{ij(k)}\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + \sum_{i=1}^{p} \sum_{j=1}^{p} e_{ij(k)}\right)^2}{p^2}$$

$$= \frac{p^4\mu^2 + \sum_{i=1}^{p} \sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g} \sum_{j\neq h} e_{ij(k)} e_{gh(l)} + 2p^2\mu \sum_{i=1}^{p} \sum_{j=1}^{p} e_{ij(k)}}{p^2}$$

Apply expectation on both sides

$$= \frac{p^4\mu^2 + \sum_{i=1}^{p} \sum_{j=1}^{p} E(e_{ij(k)}^2) + \sum_{i\neq g} \sum_{j\neq h} E(e_{ij(k)} e_{gh(l)}) + 2p^2\mu \sum_{i=1}^{p} \sum_{j=1}^{p} E(e_{ij(k)})}{p^2}$$

$$E(C.F) = \frac{p^4\mu^2 + p^2\sigma^2}{p^2} = p^2\mu^2 + \sigma^2$$

$$\sum_{i=1}^{p} \sum_{j=1}^{p} Y_{ij(k)}^2 = \sum_{i=1}^{p} \sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)^2$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{p}\left(\mu^2 + \tau_i^2 + \beta_j^2 + \gamma_k^2 + e_{ij(k)}^2 + 2\mu\tau_i + 2\mu\beta_j + 2\mu\gamma_k + 2\mu e_{ij(k)} + 2\tau_i\beta_j + 2\tau_i\gamma_k \right.$$
$$\left. + 2\tau_i e_{ij(k)} + 2\beta_j\gamma_k + 2\beta_j e_{ij(k)} + 2\gamma_k e_{ij(k)}\right)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j$$

$$+ 2p\mu\sum_{k=1}^{p}\gamma_k + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2\sum_{i=1}^{p}\tau_i\sum_{j=1}^{p}\beta_j + 2\sum_{i=1}^{p}\tau_i\sum_{k=1}^{p}\gamma_k$$

$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{j=1}^{p}\beta_j\sum_{k=1}^{p}\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 0 + 0 + 0 + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 0$$

$$+ 0 + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 0 + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k)}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)}$$

$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k)}$$

Apply expectation on both sides

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)})$$

$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k E(e_{ij(k)})$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2 + 0 + 0 + 0 + 0$$

$$E\left(\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}^2\right) = p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2$$

$$E(TSS) = E\left(\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}^2\right) - E(C.F)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2 - p^2\mu^2 - \sigma^2$$

$$E(TSS) = p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + (p^2 - 1)\sigma^2$$

$$SSR = \frac{\sum_{i=1}^{p}R_i^2}{p} - C.F$$

$$R_i = \sum_{j=1}^{p} Y_{ij(k)} = \sum_{j=1}^{p} \left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + p\tau_i + \sum_{j=1}^{p} \beta_j + \sum_{k=1}^{p} \gamma_k + \sum_{j=1}^{p} e_{ij(k)}$$

$$= p\mu + p\tau_i + 0 + 0 + \sum_{j=1}^{p} e_{ij(k)}$$

$$= p\mu + p\tau_i + \sum_{j=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{i=1}^{p} R_i^2}{p} = \frac{\sum_{i=1}^{p}\left(p\mu + p\tau_i + \sum_{j=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{i=1}^{p}\left(p^2\mu^2 + p^2\tau_i^2 + \sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{j\neq h} e_{ij(k)}e_{ih(l)} + 2p^2\mu\tau_i + 2p\mu\sum_{j=1}^{p} e_{ij(k)} + 2p\tau_i\sum_{j=1}^{p} e_{ij(k)}\right)}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 0 + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)}}{p}$$

Apply expectation on both sides

$$= \frac{p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k)}e_{gh(l)}) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i E(e_{ij(k)})}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p^2\sigma^2 + 0 + 0 + 0}{p}$$

$$E\left(\frac{\sum_{i=1}^{p} R_i^2}{p}\right) = p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sigma^2$$

$$E(SSR) = E\left(\frac{\sum_{i=1}^{p} R_i^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sigma^2 - p^2\mu^2 - \sigma^2$$

$$= p\sum_{i=1}^{p}\tau_i^2 + (p-1)\sigma^2$$

$$SSC = \frac{\sum_{j=1}^{p} C_j^2}{p} - C.F$$

$$C_j = \sum_{i=1}^{p} Y_{ij(k)} = \sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + 0 + p\beta_j + 0 + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + p\beta_j + \sum_{i=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{j=1}^{p} C_j^2}{p} = \frac{\sum_{j=1}^{p}\left(p\mu + p\beta_j + \sum_{i=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{j=1}^{p}\left(p^2\mu^2 + p^2\beta_j^2 + \sum_{i=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g} e_{ij(k)}e_{gj(l)} + 2p^2\mu\beta_j + 2p\mu\sum_{i=1}^{p} e_{ij(k)} + 2p\beta_j\sum_{i=1}^{p} e_{ij(k)}\right)}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 0 + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)}}{p}$$

Apply expectation on both sides

$$= \frac{p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k)}e_{gh(l)}) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j E(e_{ij(k)})}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + p^2\sigma^2 + 0 + 0 + 0}{p}$$

$$E\left(\frac{\sum_{j=1}^{p} C_j^2}{p}\right) = p^2\mu^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sigma^2$$

$$E(SSC) = E\left(\frac{\sum_{j=1}^{p} C_j^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sigma^2 - p^2\mu^2 - \sigma^2$$

$$= p\sum_{j=1}^{p}\beta_j^2 + (p-1)\sigma^2$$

$$SSC = \frac{\sum_{j=1}^{p} C_j^2}{p} - C.F$$

$$T_k = \sum_{i=1}^{p} Y_{ij(k)} = \sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + 0 + 0 + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{k=1}^{p} T_k^2}{p} = \frac{\sum_{k=1}^{p}\left(p\mu + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{k=1}^{p}\left(p^2\mu^2 + p^2\gamma_k^2 + \sum_{i=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g} e_{ij(k)}e_{gh(k)} + 2p^2\mu\gamma_k + 2p\mu\sum_{i=1}^{p} e_{ij(k)} + 2p\gamma_k\sum_{i=1}^{p} e_{ij(k)}\right)}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k)}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 0 + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k)}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k)}}{p}$$

Apply expectation on both sides

$$= \frac{p^3\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k)}e_{gh(l)}) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k E(e_{ij(k)})}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2 + 0 + 0 + 0}{p}$$

$$E\left(\frac{\sum_{k=1}^{p} T_k^2}{p}\right) = p^2\mu^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sigma^2$$

$$E(SST) = E\left(\frac{\sum_{k=1}^{p} T_k^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sigma^2 - p^2\mu^2 - \sigma^2$$

$$= p\sum_{k=1}^{p}\gamma_k^2 + (p-1)\sigma^2$$

$$E(SSE) = E(TSS) - E(SSR) - E(SSC) - E(SST)$$

$$= p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + (p^2-1)\sigma^2 - p\sum_{i=1}^{p}\tau_i^2 - (p-1)\sigma^2 - p\sum_{j=1}^{p}\beta_j^2 - (p-1)\sigma^2 - p\sum_{k=1}^{p}\gamma_k^2 - (p-1)\sigma^2$$

$$= (p^2-1)\sigma^2 - (p-1)\sigma^2 - (p-1)\sigma^2 - (p-1)\sigma^2$$

$$= (p^2 - 1 - p + 1 - p + 1 - p + 1)\sigma^2$$

$$= (p^2 - 3p + 2)\sigma^2$$

$$= (p^2 - 2p - p + 2)\sigma^2$$

$$= (p(p-2) - 1(p-2))\sigma^2$$

$$= (p-1)(p-2)\sigma^2$$

$$E(MSE) = E\left[\frac{SSE}{(p-1)(p-2)}\right] = \frac{E(SSE)}{(p-1)(p-2)}$$
$$= \frac{(p-1)(p-2)\sigma^2}{(p-1)(p-2)} = \sigma^2$$
$$E(MSR) = E\left[\frac{SSR}{p-1}\right] = \frac{E(SSR)}{p-1}$$
$$= \frac{p\sum_{i=1}^{p}\tau_i^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{i=1}^{p}\tau_i^2$$
$$E(MSC) = E\left[\frac{SSC}{p-1}\right] = \frac{E(SSC)}{p-1}$$
$$= \frac{p\sum_{j=1}^{p}\beta_j^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{j=1}^{p}\beta_j^2$$
$$E(MST) = E\left[\frac{SST}{p-1}\right] = \frac{E(SST)}{p-1}$$
$$= \frac{p\sum_{k=1}^{p}\gamma_k^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{k=1}^{p}\gamma_k^2$$

Random Effect Model

Assumptions:

1. $E\left(e_{ij(k)}\right) = 0$
2. $E\left(e_{ij(k)}e_{gh(l)}\right) = 0$
3. $\tau_i \sim iidN(0, \sigma_\tau^2)$
4. $\beta_j \sim iidN(0, \sigma_\beta^2)$
5. $\gamma_k \sim iidN(0, \sigma_\gamma^2)$
6. $E\left(\tau_i\tau_j\right) = 0$
7. $E\left(\beta_i\beta_j\right) = 0$
8. $E(\gamma_k\gamma_l) = 0$
9. $E\left(\tau_i\beta_j\right) = 0$
10. $E(\tau_i\gamma_k) = 0$
11. $E\left(\beta_j\gamma_k\right) = 0$
12. $E\left(\tau_i e_{ij(k)}\right) = 0$
13. $E\left(\beta_j e_{ij(k)}\right) = 0$
14. $E\left(\gamma_k e_{ij(k)}\right) = 0$

$$Y_{ij(k)} = \mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}$$

$$SSE = TSS - SSR - SSC - SST$$

$$TSS = \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 - C.F$$

$$C.F = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}\right)^2}{p^2} = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p}\tau_i + p\sum_{j=1}^{p}\beta_j + p\sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}\right)^2}{p^2}$$

$$= \frac{\begin{array}{c} p^4\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + p^2\sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum\sum_{k\neq l}\gamma_k\gamma_l \\ + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^3\mu\sum_{i=1}^{p}\tau_i + 2p^3\mu\sum_{j=1}^{p}\beta_j + 2p^3\mu\sum_{k=1}^{p}\gamma_k \\ +2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p^2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2p^2\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} \\ +2p^2\sum_{i=1}^{p}\sum_{k=1}^{p}\beta_j\gamma_k + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k)} \end{array}}{p^2}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^4\mu^2 + p^2\sum_{i=1}^{p}E(\tau_i^2) + \sum\sum_{i\neq j}E(\tau_i\tau_j) + p^2\sum_{j=1}^{p}E(\beta_j^2) + \sum\sum_{i\neq j}E(\beta_i\beta_j) + p^2\sum_{k=1}^{p}E(\gamma_k^2) + \sum\sum_{k\neq l}E(\gamma_k\gamma_l) \\ + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) + 2p^3\mu\sum_{i=1}^{p}E(\tau_i) + 2p^3\mu\sum_{j=1}^{p}E(\beta_j) + 2p^3\mu\sum_{k=1}^{p}E(\gamma_k) \\ +2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2p^2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i\beta_j) + 2p^2\sum_{i=1}^{p}\sum_{k=1}^{p}E(\tau_i\gamma_k) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i e_{ij(k)}) \\ +2p^2\sum_{i=1}^{p}\sum_{k=1}^{p}E(\beta_j\gamma_k) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j e_{ij(k)}) + +2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\gamma_k e_{ij(k)}) \end{array}}{p^2}$$

$$= \frac{p^4\mu^2 + p^3\sigma_\tau^2 + 0 + p^3\sigma_\beta^2 + 0 + p^3\sigma_\gamma^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p^2}$$

$$= \frac{p^4\mu^2 + p^3\sigma_\tau^2 + p^3\sigma_\beta^2 + p^3\sigma_\gamma^2 + p^2\sigma^2}{p^2}$$

$$E(C.F) = p^2\mu^2 + p\sigma_\tau^2 + p\sigma_\beta^2 + p\sigma_\gamma^2 + \sigma^2$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 = \sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)^2$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p}\Big(\mu^2 + \tau_i^2 + \beta_j^2 + \gamma_k^2 + e_{ij(k)}^2 + 2\mu\tau_i + 2\mu\beta_j + 2\mu\gamma_k + 2\mu e_{ij(k)} + 2\tau_i\beta_j + 2\tau_i\gamma_k$$
$$+ 2\tau_i e_{ij(k)} + 2\beta_j\gamma_k + 2\beta_j e_{ij(k)} + 2\gamma_k e_{ij(k)}\Big)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k$$
$$+ 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2\sum_{i=1}^{p}\tau_i\sum_{j=1}^{p}\beta_j + 2\sum_{i=1}^{p}\tau_i\sum_{k=1}^{p}\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{j=1}^{p}\beta_j\sum_{k=1}^{p}\gamma_k$$
$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

Apply expectation on both sides

$$= p^2\mu^2 + p\sum_{i=1}^{p}E(\tau_i^2) + p\sum_{j=1}^{p}E(\beta_j^2) + p\sum_{k=1}^{p}E(\gamma_k^2) + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + 2p\mu\sum_{i=1}^{p}E(\tau_i) + 2p\mu\sum_{j=1}^{p}E(\beta_j)$$
$$+ 2p\mu\sum_{k=1}^{p}E(\gamma_k) + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i\beta_j) + 2\sum_{i=1}^{p}\sum_{k=1}^{p}E(\tau_i\gamma_k)$$
$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i e_{ij(k)}) + 2\sum_{j=1}^{p}\sum_{k=1}^{p}E(\beta_j\gamma_k) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{k=1}^{p}E(\gamma_k e_{ij(k)})$$
$$= p^2\mu^2 + p^2\sigma_\tau^2 + p^2\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$

$$E\left(\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}^2\right) = p^2\mu^2 + p^2\sigma_\tau^2 + p^2\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma^2$$

$$E(TSS) = E\left(\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}^2\right) - E(C.F)$$
$$= p^2\mu^2 + p^2\sigma_\tau^2 + p^2\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\beta^2 - p\sigma_\gamma^2 - \sigma^2$$
$$= p(p-1)\sigma_\tau^2 + p(p-1)\sigma_\beta^2 + p(p-1)\sigma_\gamma^2 + (p^2-1)\sigma^2$$

$$SSR = \frac{\sum_{i=1}^{p}R_i^2}{p} - C.F$$

$$R_i = \sum_{j=1}^{p}Y_{ij(k)} = \sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$
$$= p\mu + p\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p}e_{ij(k)}$$

$$\frac{\sum_{i=1}^{p}R_i^2}{p} = \frac{\sum_{i=1}^{p}\left(p\mu + p\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p}e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{i=1}^{p}\begin{pmatrix} p^2\mu^2 + p^2\tau_i^2 + \sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + \sum_{k=1}^{p}\gamma_k^2 + \sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{j=1}^{p}e_{ij(k)}^2 + \sum_{j\neq h}e_{ij(k)}e_{ih(l)} \\ +2p^2\mu\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{j=1}^{p}e_{ij(k)} + 2p\tau_i\sum_{j=1}^{p}\beta_j + 2p\tau_i\sum_{k=1}^{p}\gamma_k \\ +2p\tau_i\sum_{j=1}^{p}e_{ij(k)} + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\beta_j\gamma_k + 2\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{j=1}^{p}\gamma_k e_{ij(k)} \end{pmatrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + p\sum_{k=1}^{p}\gamma_k^2 + p\sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 \\ + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\beta_j\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} \\ +2\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k)} \end{matrix}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p} E(\tau_i^2) + p\sum_{j=1}^{p} E\left(\beta_j^2\right) + p\sum\sum_{i\neq j} E(\beta_i\beta_j) + p\sum_{k=1}^{p} E\left(\gamma_k^2\right) + p\sum\sum_{k\neq l} E(\gamma_k\gamma_l) \\ +\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h} E\left(e_{ij(k)}e_{gh(l)}\right) + +2p^2\mu\sum_{i=1}^{p} E(\tau_i) + 2p^2\mu\sum_{j=1}^{p} E(\beta_j) \\ +2p^2\mu\sum_{k=1}^{p} E(\gamma_k) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i\beta_j) + 2p\sum_{i=1}^{p}\sum_{k=1}^{p} E(\tau_i\gamma_k) \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i e_{ij(k)}) + 2\sum_{j=1}^{p}\sum_{k=1}^{p} E(\beta_j\gamma_k) + 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\beta_j\, e_{ij(k)}) + +2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\gamma_k e_{ij(k)}) \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^3\sigma_\tau^2 + p^2\sigma_\beta^2 + 0 + p^2\sigma_\gamma^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^3\sigma_\tau^2 + p^2\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma^2$$

$$E\left(\frac{\sum_{i=1}^{p} R_i^2}{p}\right) = p^2\mu^2 + p^2\sigma_\tau^2 + p\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma^2$$

$$E(SSR) = E\left(\frac{\sum_{i=1}^{p} R_i^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\beta^2 - p\sigma_\gamma^2 - \sigma^2$$

$$= p(p-1)\sigma_\tau^2 + (p-1)\sigma^2$$

$$SSC = \frac{\sum_{j=1}^{p} C_j^2}{p} - C.F$$

$$C_j = \sum_{i=1}^{p} Y_{ij(k)} = \sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{j=1}^{p} C_j^2}{p} = \frac{\sum_{j=1}^{p}\left(p\mu + \sum_{i=1}^{p}\tau_i + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{j=1}^{p}\left(\begin{array}{c} p^2\mu^2 + \sum_{i=1}^{p}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + p^2\beta_j^2 + \sum_{k=1}^{p}\gamma_k^2 + \sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g} e_{ij(k)}e_{gj(l)} \\ 2p\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\tau_i\beta_j + 2\sum_{k=1}^{p}\tau_i\gamma_k \\ +2\sum_{i=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{k=1}^{p}\beta_j\gamma_k + 2p\sum_{i=1}^{p}\beta_j\, e_{ij(k)} + 2\sum_{i=1}^{p}\gamma_k e_{ij(k)} \end{array}\right)}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum\sum_{i\neq j}\tau_i\tau_j + p^2\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 \\ +\sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j \\ +2\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{k=1}^{p}\sum_{j=1}^{p}\beta_j\gamma_k + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\, e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p\sum_{i=1}^{p} E(\tau_i^2) + p\sum\sum_{i\neq j} E(\tau_i\tau_j) + p^2\sum_{j=1}^{p} E(\beta_j^2) + p\sum_{k=1}^{p} E(\gamma_k^2) + p\sum\sum_{k\neq l} E(\gamma_k\gamma_l) + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) \\ +\sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k)}e_{gh(l)}) + 2p^2\mu\sum_{i=1}^{p} E(\tau_i) + 2p\mu\sum_{j=1}^{p} E(\beta_j) + 2p^2\mu\sum_{k=1}^{p} E(\gamma_k) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}) \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i\beta_j) + 2\sum_{i=1}^{p}\sum_{k=1}^{p} E(\tau_i\gamma_k) + 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i e_{ij(k)}) + 2p\sum_{k=1}^{p}\sum_{j=1}^{p} E(\beta_j\gamma_k) \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p} E(\beta_j\, e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\gamma_k e_{ij(k)}) \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + 0 + p^3\sigma_\beta^2 + p^2\sigma_\gamma^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + p^3\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p^2\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma^2$$

$$E\left(\frac{\sum_{j=1}^{p} C_j^2}{p}\right) = p^2\mu^2 + p\sigma_\tau^2 + p^2\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma^2$$

$$E(SSC) = E\left(\frac{\sum_{j=1}^{p} C_j^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p^2\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\beta^2 - p\sigma_\gamma^2 - \sigma^2$$

$$= p(p-1)\sigma_\beta^2 + (p-1)\sigma^2$$

$$SST = \frac{\sum_{k=1}^{p} T_k^2}{p} - C.F$$

$$T_k = \sum_{i=1}^{p} Y_{ij(k)} = \sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{k=1}^{p} T_k^2}{p} = \frac{\sum_{k=1}^{p}\left(p\mu + \sum_{i=1}^{p}\tau_i + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{k=1}^{p}\begin{pmatrix} p^2\mu^2 + \sum_{i=1}^{p}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + \sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + p^2\gamma_k^2 + \sum_{i=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} \\ 2p\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\gamma_k + 2p\mu\sum_{i=1}^{p} e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2p\sum_{i=1}^{p}\tau_i\gamma_k \\ +2\sum_{i=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{j=1}^{p}\beta_j\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2p\sum_{i=1}^{p}\gamma_k e_{ij(k)} \end{pmatrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum\sum_{i\neq j}\tau_i\tau_j + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 \\ + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j \\ +2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{k=1}^{p}\sum_{j=1}^{p}\beta_j\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)} \end{matrix}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{matrix} p^3\mu^2 + p\sum_{i=1}^{p}E(\tau_i^2) + p\sum\sum_{i\neq j}E(\tau_i\tau_j) + p\sum_{j=1}^{p}E(\beta_j^2) + p\sum\sum_{i\neq j}E(\beta_i\beta_j) + p^2\sum_{k=1}^{p}E(\gamma_k^2) \\ + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) + 2p^2\mu\sum_{i=1}^{p}E(\tau_i) + 2p^2\mu\sum_{j=1}^{p}E(\beta_j) + 2p^2\mu\sum_{k=1}^{p}E(\gamma_k) \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i\beta_j) + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}E(\tau_i\gamma_k) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i e_{ij(k)}) \\ +2p\sum_{k=1}^{p}\sum_{j=1}^{p}E(\beta_j\gamma_k) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\gamma_k e_{ij(k)}) \end{matrix}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + 0 + p^2\sigma_\beta^2 + 0 + p^3\sigma_\gamma^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + p^2\sigma_\beta^2 + p^3\sigma_\gamma^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p\sigma_\beta^2 + p^2\sigma_\gamma^2 + p\sigma^2$$

$$E\left(\frac{\sum_{k=1}^{p} T_k^2}{p}\right) = p^2\mu^2 + p\sigma_\tau^2 + p\sigma_\beta^2 + p^2\sigma_\gamma^2 + p\sigma^2$$

$$E(SST) = E\left(\frac{\sum_{k=1}^{p} T_k^2}{p}\right) - E(C.F)$$
$$= p^2\mu^2 + p\sigma_\tau^2 + p\sigma_\beta^2 + p^2\sigma_\gamma^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\beta^2 - p\sigma_\gamma^2 - \sigma^2$$
$$= p(p-1)\sigma_\gamma^2 + (p-1)\sigma^2$$
$$E(SSE) = E(TSS) - E(SSR) - E(SSC) - E(SST)$$
$$= p(p-1)\sigma_\tau^2 + p(p-1)\sigma_\beta^2 + p(p-1)\sigma_\gamma^2 + (p^2-1)\sigma^2 - p(p-1)\sigma_\tau^2 - (p-1)\sigma^2$$
$$- p(p-1)\sigma_\beta^2 - (p-1)\sigma^2 - p(p-1)\sigma_\gamma^2 - (p-1)\sigma^2$$
$$= (p^2 - 1 - p + 1 - p + 1 - p + 1)\sigma^2$$
$$= (p^2 - 3p + 2)\sigma^2$$
$$= (p-2)(p-1)\sigma^2$$
$$E(MSE) = \frac{E(SSE)}{(p-1)(p-2)} = \frac{(p-2)(p-1)\sigma^2}{(p-1)(p-2)} = \sigma^2$$
$$E(MSR) = \frac{E(SSR)}{p-1} = \frac{p(p-1)\sigma_\tau^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + p\sigma_\tau^2$$
$$E(MSC) = \frac{E(SSC)}{p-1} = \frac{p(p-1)\sigma_\beta^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + p\sigma_\beta^2$$
$$E(MST) = \frac{E(SST)}{p-1} = \frac{p(p-1)\sigma_\gamma^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + p\sigma_\gamma^2$$

Mixed Effect Model

Case I: In this model the effect of τ_i is random and effect of β_j and γ_k is fixed

Assumptions:

1. $E(e_{ij(k)}) = 0$
2. $E(e_{ij(k)}e_{gh(l)}) = 0$
3. $e_{ij(k)} \sim iidN(0, \sigma^2)$
4. $\tau_i \sim iidN(0, \sigma_\tau^2)$
5. $E(\tau_i\tau_j) = 0$
6. $E(\tau_i e_{ij(k)}) = 0$
7. $\sum_{j=1}^{p} \beta_j = 0$
8. $\sum_{k=1}^{p} \gamma_k = 0$

$$Y_{ij(k)} = \mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}$$
$$SSE = TSS - SSR - SSC - SST$$
$$TSS = \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 - C.F$$
$$C.F = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}\right)^2}{p^2} = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p}\tau_i + p\sum_{j=1}^{p}\beta_j + p\sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p}\tau_i + 0 + 0 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p}\tau_i + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\right)^2}{p^2}$$

$$= \frac{\begin{array}{c} p^4\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} + 2p^3\mu\sum_{i=1}^{p}\tau_i \\ +2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} \end{array}}{p^2}$$

Apply expectation of both sides

$$= \frac{\begin{array}{c} p^4\mu^2 + p^2\sum_{i=1}^{p}E(\tau_i^2) + \sum\sum_{i\neq j}E(\tau_i\tau_j) + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) \\ +2p^3\mu\sum_{i=1}^{p}E(\tau_i) + 2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i e_{ij(k)}) \end{array}}{p^2}$$

$$= \frac{p^4\mu^2 + p^3\sigma_\tau^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0}{p^2}$$

$$E(C.F) = p^2\mu^2 + p\sigma_\tau^2 + \sigma^2$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}^2 = \sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)^2$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p}\Big(\mu^2 + \tau_i^2 + \beta_j^2 + \gamma_k^2 + e_{ij(k)}^2 + 2\mu\tau_i + 2\mu\beta_j + 2\mu\gamma_k + 2\mu e_{ij(k)} + 2\tau_i\beta_j + 2\tau_i\gamma_k + 2\tau_i e_{ij(k)} + 2\beta_j\gamma_k + 2\beta_j e_{ij(k)} + 2\gamma_k e_{ij(k)}\Big)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2\sum_{i=1}^{p}\tau_i\sum_{j=1}^{p}\beta_j + 2\sum_{i=1}^{p}\tau_i\sum_{k=1}^{p}\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{j=1}^{p}\beta_j\sum_{k=1}^{p}\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i + 0 + 0 + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 0 + 0 + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 0 + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

Apply expectation on both sides

$$= p^2\mu^2 + p\sum_{i=1}^{p} E(\tau_i^2) + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) + 2p\mu\sum_{i=1}^{p} E(\tau_i) + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)})$$

$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k E(e_{ij(k)})$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2$$

$$E(TSS) = E\left[\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij}^2\right] - E(C.F)$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - \sigma^2$$

$$= p(p-1)\sigma_\tau^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + (p^2-1)\sigma^2$$

$$SSR = \frac{\sum_{i=1}^{p} R_i^2}{p} - C.F$$

$$R_i = \sum_{j=1}^{p} Y_{ij(k)} = \sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + p\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p} e_{ij(k)}$$

$$= p\mu + p\tau_i + 0 + 0 + \sum_{j=1}^{p} e_{ij(k)}$$

$$= p\mu + p\tau_i + \sum_{j=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{i=1}^{p} R_i^2}{p} = \frac{\sum_{i=1}^{p}\left(p\mu + p\tau_i + \sum_{j=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{i=1}^{p}\left(p^2\mu^2 + p^2\tau_i^2 + \sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{j\neq h} e_{ij(k)}e_{ih(l)} + 2p^2\mu\tau_i + 2p\mu\sum_{j=1}^{p} e_{ij(k)} + 2p\tau_i\sum_{j=1}^{p} e_{ij(k)}\right)}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p} E(\tau_i^2) + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k)}e_{gh(l)}) + 2p^2\mu\sum_{i=1}^{p} E(\tau_i) \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i e_{ij(k)}) \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^3\sigma_\tau^2 + p^2\sigma^2 + 0 + 0 + 0 + 0}{p}$$

$$=\frac{p^3\mu^2+p^3\sigma_\tau^2+p^2\sigma^2}{p}$$
$$=p^2\mu^2+p^2\sigma_\tau^2+p\sigma^2$$
$$E\left(\frac{\sum_{i=1}^{p}R_i^2}{p}\right)=p^2\mu^2+p^2\sigma_\tau^2+p\sigma^2$$
$$E(SSR)=E\left(\frac{\sum_{i=1}^{p}R_i^2}{p}\right)-E(C.F)$$
$$=p^2\mu^2+p^2\sigma_\tau^2+p\sigma^2-p^2\mu^2-p\sigma_\tau^2-\sigma^2$$
$$=p(p-1)\sigma_\tau^2+(p-1)\sigma^2$$
$$SSC=\frac{\sum_{j=1}^{p}C_j^2}{p}-C.F$$
$$C_j=\sum_{i=1}^{p}Y_{ij(k)}=\sum_{i=1}^{p}\left(\mu+\tau_i+\beta_j+\gamma_k+e_{ij(k)}\right)$$
$$=p\mu+\sum_{i=1}^{p}\tau_i+p\beta_j+\sum_{k=1}^{p}\gamma_k+\sum_{i=1}^{p}e_{ij(k)}$$
$$=p\mu+\sum_{i=1}^{p}\tau_i+p\beta_j+0+\sum_{i=1}^{p}e_{ij(k)}$$
$$=p\mu+\sum_{i=1}^{p}\tau_i+p\beta_j+\sum_{i=1}^{p}e_{ij(k)}$$
$$\frac{\sum_{j=1}^{p}C_j^2}{p}=\frac{\sum_{j=1}^{p}\left(p\mu+\sum_{i=1}^{p}\tau_i+p\beta_j+\sum_{i=1}^{p}e_{ij(k)}\right)^2}{p}$$
$$=\frac{\sum_{j=1}^{p}\left(\begin{array}{c}p^2\mu^2+\sum_{i=1}^{p}\tau_i^2+\sum\sum_{i\neq j}\tau_i\tau_j+p^2\beta_j^2+\sum_{i=1}^{p}e_{ij(k)}^2+\sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)}+2p\mu\sum_{i=1}^{p}\tau_i+2p^2\mu\beta_j\\+2p\mu\sum_{i=1}^{p}e_{ij(k)}+2p\sum_{i=1}^{p}\tau_i\beta_j+2\sum_{i=1}^{p}\tau_ie_{ij(k)}+2p\beta_j\sum_{i=1}^{p}e_{ij(k)}\end{array}\right)}{p}$$
$$=\frac{\begin{array}{c}p^3\mu^2+p\sum_{i=1}^{p}\tau_i^2+p\sum\sum_{i\neq j}\tau_i\tau_j+p^2\sum_{j=1}^{p}\beta_j^2+\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2+\sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)}+2p^2\mu\sum_{i=1}^{p}\tau_i\\+2p^2\mu\sum_{j=1}^{p}\beta_j+2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}+2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j+2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_ie_{ij(k)}+2p\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\beta_j\end{array}}{p}$$
$$=\frac{\begin{array}{c}p^3\mu^2+p\sum_{i=1}^{p}\tau_i^2+p\sum\sum_{i\neq j}\tau_i\tau_j+p^2\sum_{j=1}^{p}\beta_j^2+\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2+\sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)}+2p^2\mu\sum_{i=1}^{p}\tau_i\\+0+2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}+2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j+2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_ie_{ij(k)}+2p\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\beta_j\end{array}}{p}$$
$$=\frac{\begin{array}{c}p^3\mu^2+p\sum_{i=1}^{p}\tau_i^2+p\sum\sum_{i\neq j}\tau_i\tau_j+p^2\sum_{j=1}^{p}\beta_j^2+\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2+\sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)}+2p^2\mu\sum_{i=1}^{p}\tau_i\\+2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}+2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j+2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_ie_{ij(k)}+2p\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\beta_j\end{array}}{p}$$

Apply expectation on both sides

$$=\frac{\begin{array}{c}p^3\mu^2+p\sum_{i=1}^{p}E(\tau_i^2)+p\sum\sum_{i\neq j}E(\tau_i\tau_j)+p^2\sum_{j=1}^{p}\beta_j^2+\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2)+\sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)})\\+2p^2\mu\sum_{i=1}^{p}E(\tau_i)+2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)})+2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i)\beta_j\\+2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_ie_{ij(k)})+2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)})\beta_j\end{array}}{p}$$
$$=\frac{p^3\mu^2+p^2\sigma_\tau^2+0+p^2\sum_{j=1}^{p}\beta_j^2+p^2\sigma^2+0+0+0+0+0+0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + p^2\sum_{j=1}^{p}\beta_j^2 + p^2\sigma^2}{p}$$

$$E\left(\frac{\sum_{j=1}^{p} C_j^2}{p}\right) = p^2\mu^2 + p\sigma_\tau^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sigma^2$$

$$E(SSC) = E\left(\frac{\sum_{j=1}^{p} C_j^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - \sigma^2$$

$$= p\sum_{j=1}^{p}\beta_j^2 + (p-1)\sigma^2$$

$$SST = \frac{\sum_{k=1}^{p} T_k^2}{p} - C.F$$

$$T_k = \sum_{i=1}^{p} Y_{ij(k)} = \sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + 0 + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{k=1}^{p} T_k^2}{p} = \frac{\sum_{k=1}^{p}\left(p\mu + \sum_{i=1}^{p}\tau_i + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{k=1}^{p}\begin{pmatrix} p^2\mu^2 + \sum_{i=1}^{p}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + p^2\gamma_k^2 + \sum_{i=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\gamma_k \\ +2p\mu\sum_{i=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\tau_i\gamma_k + 2\sum_{i=1}^{p}\tau_i e_{ij(k)} + 2p\gamma_k\sum_{i=1}^{p} e_{ij(k)} \end{pmatrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum\sum_{i\neq j}\tau_i\tau_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i \\ +2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p} e_{ij(k)}\gamma_k \end{matrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum\sum_{i\neq j}\tau_i\tau_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i \\ +0 + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p} e_{ij(k)}\gamma_k \end{matrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum\sum_{i\neq j}\tau_i\tau_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p} e_{ij(k)}\gamma_k \end{matrix}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{gathered}p^3\mu^2 + p\sum_{i=1}^{p} E(\tau_i^2) + p\sum\sum_{i\neq j} E(\tau_i\tau_j) + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k)}e_{gh(l)}) \\ +2p^2\mu\sum_{i=1}^{p} E(\tau_i) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{k=1}^{p} E(\tau_i)\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p} E\big(\tau_i e_{ij(k)}\big) \\ +2p\sum_{i=1}^{p}\sum_{k=1}^{p} E(e_{ij(k)})\gamma_k\end{gathered}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + 0 + p^2\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2}{p}$$

$$E\left(\frac{\sum_{k=1}^{p} T_k^2}{p}\right) = p^2\mu^2 + p\sigma_\tau^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sigma^2$$

$$E(SST) = E\left(\frac{\sum_{k=1}^{p} T_k^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - \sigma^2$$

$$= p\sum_{k=1}^{p}\gamma_k^2 + (p-1)\sigma^2$$

$$E(SSE) = E(TSS) - E(SSR) - E(SSC) - E(SST)$$

$$= p(p-1)\sigma_\tau^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + (p^2-1)\sigma^2 - p(p-1)\sigma_\tau^2 - (p-1)\sigma^2 - p\sum_{j=1}^{p}\beta_j^2 - (p-1)\sigma^2 - p\sum_{k=1}^{p}\gamma_k^2 - (p-1)\sigma^2$$

$$= (p^2 - 1 - p + 1 - p + 1 - p + 1)\sigma^2 = (p^2 - 3p + 2)\sigma^2$$

$$= (p-2)(p-1)\sigma^2$$

$$E(MSE) = \frac{E(SSE)}{(p-1)(p-2)} = \frac{(p-2)(p-1)\sigma^2}{(p-1)(p-2)} = \sigma^2$$

$$E(MSR) = \frac{E(SSR)}{p-1} = \frac{p(p-1)\sigma_\tau^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + p\sigma_\tau^2$$

$$E(MSC) = \frac{E(SSC)}{p-1} = \frac{p\sum_{j=1}^{p}\beta_j^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{j=1}^{p}\beta_j^2$$

$$E(MST) = \frac{E(SST)}{p-1} = \frac{p\sum_{k=1}^{p}\gamma_k^2 - (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{k=1}^{p}\gamma_k^2$$

Case II: In this model the effect of β_j is random and effect of τ_i and γ_k is fixed

Assumptions:

1. $E\big(e_{ij(k)}\big) = 0$
2. $E\big(e_{ij(k)}e_{gh(l)}\big) = 0$
3. $e_{ij(k)} \sim iidN(0, \sigma^2)$
4. $\beta_j \sim iidN(0, \sigma_\beta^2)$
5. $E\big(\beta_i\beta_j\big) = 0$

6. $E(\beta_j e_{ij(k)}) = 0$
7. $\sum_{i=1}^{p} \tau_i = 0$
8. $\sum_{k=1}^{p} \gamma_k = 0$

$$Y_{ij(k)} = \mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}$$

$$SSE = TSS - SSR - SSC - SST$$

$$TSS = \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 - C.F$$

$$C.F = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}\right)^2}{p^2} = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p}\tau_i + p\sum_{j=1}^{p}\beta_j + p\sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + 0 + p\sum_{j=1}^{p}\beta_j + 0 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + p\sum_{j=1}^{p}\beta_j + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}\right)^2}{p^2}$$

$$= \frac{\begin{matrix} p^4\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^3\mu\sum_{j=1}^{p}\beta_j \\ +2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} \end{matrix}}{p^2}$$

Apply expectation of both sides

$$= \frac{\begin{matrix} p^4\mu^2 + p^2\sum_{j=1}^{p}E(\beta_j^2) + \sum\sum_{i\neq j}E(\beta_i\beta_j) + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) \\ +2p^3\mu\sum_{j=1}^{p}E(\beta_j) + 2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j e_{ij(k)}) \end{matrix}}{p^2}$$

$$= \frac{p^4\mu^2 + p^3\sigma_\beta^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0}{p^2}$$

$$E(C.F) = p^2\mu^2 + p\sigma_\beta^2 + \sigma^2$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 = \sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)^2$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p}\big(\mu^2 + \tau_i^2 + \beta_j^2 + \gamma_k^2 + e_{ij(k)}^2 + 2\mu\tau_i + 2\mu\beta_j + 2\mu\gamma_k + 2\mu e_{ij(k)} + 2\tau_i\beta_j + 2\tau_i\gamma_k$$
$$+ 2\tau_i e_{ij(k)} + 2\beta_j\gamma_k + 2\beta_j e_{ij(k)} + 2\gamma_k e_{ij(k)}\big)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k$$
$$+ 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2\sum_{i=1}^{p}\tau_i\sum_{j=1}^{p}\beta_j + 2\sum_{i=1}^{p}\tau_i\sum_{k=1}^{p}\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{j=1}^{p}\beta_j\sum_{k=1}^{p}\gamma_k$$
$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$
$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 0 + 2p\mu\sum_{j=1}^{p}\beta_j + 0 + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 0 + 0$$
$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 0 + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$
$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 2p\mu\sum_{j=1}^{p}\beta_j + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)}$$
$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

Apply expectation on both sides

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}E(\beta_j^2) + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + 2p\mu\sum_{j=1}^{p}E(\beta_j) + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)})$$
$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k E(e_{ij(k)})$$
$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p^2\sigma_\beta^2 + p\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0$$
$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p^2\sigma_\beta^2 + p\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2$$
$$E(TSS) = E\left(\sum_{I=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}^2\right) - E(C.F)$$
$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p^2\sigma_\beta^2 + p\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2 - p^2\mu^2 - p\sigma_\beta^2 - \sigma^2$$
$$E(TSS) = p\sum_{i=1}^{p}\tau_i^2 + p(p-1)\sigma_\beta^2 + p\sum_{k=1}^{p}\gamma_k^2 + (p^2-1)\sigma^2$$
$$SSR = \frac{\sum_{i=1}^{p}R_i^2}{p} - C.F$$
$$R_i = \sum_{j=1}^{p}Y_{ij(k)} = \sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$
$$= p\mu + p\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p}e_{ij(k)}$$

$$= p\mu + p\tau_i + \sum_{j=1}^{p} \beta_j + 0 + \sum_{j=1}^{p} e_{ij(k)}$$

$$= p\mu + p\tau_i + \sum_{j=1}^{p} \beta_j + \sum_{j=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{i=1}^{p} R_i^2}{p} = \frac{\sum_{i=1}^{p}\left(p\mu + p\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{j=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{i=1}^{p}\left(\begin{array}{c} p^2\mu^2 + p^2\tau_i^2 + \sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_j\beta_j + \sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{j\neq h} e_{ij(k)}e_{ih(l)} + 2p^2\mu\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{j=1}^{p} e_{ij(k)} \\ +2p\sum_{j=1}^{p}\tau_i\beta_j + 2p\tau_i\sum_{j=1}^{p} e_{ij(k)} + 2\sum_{j=1}^{p}\beta_j\, e_{ij(k)} \end{array}\right)}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_j\beta_j + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\sum_{j=1}^{p}\beta_j \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\, e_{ij(k)} \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_j\beta_j + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 0 + 2p^2\mu\sum_{j=1}^{p}\beta_j \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\, e_{ij(k)} \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_j\beta_j + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{j=1}^{p}\beta_j \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\, e_{ij(k)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}E(\beta_j^2) + p\sum\sum_{i\neq j}E(\beta_j\beta_j) + \sum_{i=1}^{p}\sum_{j=1}^{p}E\left(e_{ij(k)}^2\right) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) + 2p^2\mu\sum_{j=1}^{p}E(\beta_j) \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i E(\beta_j) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j\, e_{ij(k)}) \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p^2\sigma_\beta^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p^2\sigma_\beta^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sigma_\beta^2 + p\sigma^2$$

$$E\left(\frac{\sum_{i=1}^{p} R_i^2}{p}\right) = p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sigma_\beta^2 + p\sigma^2$$

$$E(SSR) = E\left(\frac{\sum_{i=1}^{p} R_i^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sigma_\beta^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\beta^2 - \sigma^2$$

$$= p\sum_{i=1}^{p}\tau_i^2 + (p-1)\sigma^2$$

$$SSC = \frac{\sum_{j=1}^{p} C_j^2}{p} - C.F$$

$$C_j = \sum_{i=1}^{p} Y_{ij(k)} = \sum_{i=1}^{p} \left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + \sum_{i=1}^{p} \tau_i + p\beta_j + \sum_{k=1}^{p} \gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + 0 + p\beta_j + 0 + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + p\beta_j + \sum_{i=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{j=1}^{p} C_j^2}{p} = \frac{\sum_{j=1}^{p}\left(p\mu + p\beta_j + \sum_{i=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{j=1}^{p}\left(p^2\mu^2 + p^2\beta_j^2 + \sum_{i=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g} e_{ij(k)}e_{gj(l)} + 2p^2\mu\beta_j + 2p\mu\sum_{i=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\beta_j\, e_{ij(k)}\right)}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\, e_{ij(k)}}{p}$$

Apply expectation on both sides

$$= \frac{p^3\mu^2 + p^2\sum_{j=1}^{p} E\left(\beta_j^2\right) + \sum_{i=1}^{p}\sum_{j=1}^{p} E\left(e_{ij(k)}^2\right) + \sum_{i\neq g}\sum_{j\neq h} E\left(e_{ij(k)}e_{gh(l)}\right) + 2p^2\mu\sum_{j=1}^{p} E(\beta_j) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E\left(e_{ij(k)}\right) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p} E\left(\beta_j\, e_{ij(k)}\right)}{p}$$

$$= \frac{p^3\mu^2 + p^3\sigma_\beta^2 + p^2\sigma^2 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^3\sigma_\beta^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p^2\sigma_\beta^2 + p\sigma^2$$

$$E\left(\frac{\sum_{j=1}^{p} C_j^2}{p}\right) = p^2\mu^2 + p^2\sigma_\beta^2 + p\sigma^2$$

$$E(SSC) = E\left(\frac{\sum_{j=1}^{p} C_j^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p^2\sigma_\beta^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\beta^2 - \sigma^2$$

$$= p(p-1)\sigma_\beta^2 + (p-1)\sigma^2$$

$$T_k = \sum_{i=1}^{p} Y_{ij(k)} = \sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + 0 + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + \sum_{j=1}^{p} \beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{k=1}^{p} T_k^2}{p} = \frac{\sum_{k=1}^{p}\left(p\mu + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{k=1}^{p}\begin{pmatrix} p^2\mu^2 + \sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + p^2\gamma_k^2 + \sum_{i=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\gamma_k \\ +2p\mu\sum_{i=1}^{p} e_{ij(k)} + 2p\sum_{j=1}^{p}\beta_j\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2p\sum_{i=1}^{p}\gamma_k e_{ij(k)}\end{pmatrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{k=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\sum_{k=1}^{p}\gamma_k \\ +2p\mu\sum_{i=1}^{p}\sum_{k=1}^{p} e_{ij(k)} + 2p\sum_{j=1}^{p}\sum_{k=1}^{p}\beta_j\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2p\sum_{k=1}^{p}\sum_{i=1}^{p}\gamma_k e_{ij(k)}\end{matrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{k=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{j=1}^{p}\beta_j + 0 \\ +2p\mu\sum_{i=1}^{p}\sum_{k=1}^{p} e_{ij(k)} + 2p\sum_{j=1}^{p}\sum_{k=1}^{p}\beta_j\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2p\sum_{k=1}^{p}\sum_{i=1}^{p}\gamma_k e_{ij(k)}\end{matrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{k=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{j=1}^{p}\beta_j \\ +2p\mu\sum_{i=1}^{p}\sum_{k=1}^{p} e_{ij(k)} + 2p\sum_{j=1}^{p}\sum_{k=1}^{p}\beta_j\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2p\sum_{k=1}^{p}\sum_{i=1}^{p}\gamma_k e_{ij(k)}\end{matrix}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{matrix} p^3\mu^2 + p\sum_{j=1}^{p}E(\beta_j^2) + p\sum\sum_{i\neq j}E(\beta_i\beta_j) + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{k=1}^{p} E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k)}e_{gh(l)}) + 2p^2\mu\sum_{j=1}^{p}E(\beta_j) \\ +2p\mu\sum_{i=1}^{p}\sum_{k=1}^{p} E(e_{ij(k)}) + 2p\sum_{j=1}^{p}\sum_{k=1}^{p}E(\beta_j)\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j e_{ij(k)}) + 2p\sum_{k=1}^{p}\sum_{i=1}^{p}E(\gamma_k e_{ij(k)})\end{matrix}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\beta^2 + 0 + p^2\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\beta^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sigma_\beta^2 + p\sigma^2$$

$$E\left(\frac{\sum_{k=1}^{p} T_k^2}{p}\right) = p^2\mu^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sigma_\beta^2 + p\sigma^2$$

$$E(SST) = E\left(\frac{\sum_{k=1}^{p} T_k^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sigma_\beta^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\beta^2 - \sigma^2$$

$$= p\sum_{k=1}^{p}\gamma_k^2 + (p-1)\sigma^2$$

$$E(SSE) = E(TSS) - E(SSR) - E(SSC) - E(SST)$$

$$= p\sum_{i=1}^{p}\tau_i^2 + p(p-1)\sigma_\beta^2 + p\sum_{k=1}^{p}\gamma_k^2 + (p^2-1)\sigma^2 - p\sum_{i=1}^{p}\tau_i^2 - (p-1)\sigma^2 - p(p-1)\sigma_\beta^2 - (p-1)\sigma^2 - p\sum_{k=1}^{p}\gamma_k^2 - (p-1)\sigma^2$$

$$= (p^2-1-p+1-p+1-p+1)\sigma^2$$

$$= (p^2-3p+2)\sigma^2 = (p-2)(p-1)\sigma^2$$

$$E(MSE) = \frac{E(SSE)}{(p-1)(p-2)} = \frac{(p-1)(p-2)\sigma^2}{(p-1)(p-2)} = \sigma^2$$

$$E(MSR) = \frac{E(SSR)}{p-1} = \frac{p\sum_{i=1}^{p}\tau_i^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{i=1}^{p}\tau_i^2$$

$$E(MSC) = \frac{E(SSC)}{p-1} = \frac{p(p-1)\sigma_\beta^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + p\sigma_\beta^2$$

$$E(MST) = \frac{E(SST)}{p-1} = \frac{p\sum_{k=1}^{p}\gamma_k^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{k=1}^{p}\gamma_k^2$$

Case III: In this model the effect of γ_k is random and effect of τ_i and β_j is fixed

Assumptions:

1. $E\left(e_{ij(k)}\right) = 0$
2. $E\left(e_{ij(k)}e_{gh(l)}\right) = 0$
3. $e_{ij(k)} \sim iidN(0, \sigma^2)$
4. $\gamma_k \sim iidN(0, \sigma_\gamma^2)$
5. $E(\gamma_k\gamma_l) = 0$
6. $E\left(\gamma_k e_{ij(k)}\right) = 0$
7. $\sum_{j=1}^{p}\beta_j = 0$
8. $\sum_{i=1}^{p}\tau_i = 0$

$$Y_{ij(k)} = \mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}$$

$$SSE = TSS - SSR - SSC - SST$$

$$TSS = \sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}^2 - C.F$$

$$C.F = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}\right)^2}{p^2} = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p}\tau_i + p\sum_{j=1}^{p}\beta_j + p\sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + 0 + 0 + p\sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + p\sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\right)^2}{p^2}$$

$$= \frac{\begin{array}{c} p^4\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} + 2p^3\mu\sum_{k=1}^{p}\gamma_k \\ +2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)} \end{array}}{p^2}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^4\mu^2 + p^2\sum_{k=1}^{p} E(\gamma_k^2) + \sum\sum_{k\neq l} E(\gamma_k\gamma_l) + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k)}e_{gh(l)}) \\ +2p^3\mu\sum_{k=1}^{p} E(\gamma_k) + 2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{k=1}^{p} E(\gamma_k e_{ij(k)}) \end{array}}{p^2}$$

$$= \frac{p^4\mu^2 + p^3\sigma_\gamma^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0}{p^2}$$

$$= \frac{p^4\mu^2 + p^3\sigma_\gamma^2 + p^2\sigma^2}{p^2}$$

$$E(C.F) = p^2\mu^2 + p\sigma_\gamma^2 + \sigma^2$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 = \sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)^2$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p}\Big(\mu^2 + \tau_i^2 + \beta_j^2 + \gamma_k^2 + e_{ij(k)}^2 + 2\mu\tau_i + 2\mu\beta_j + 2\mu\gamma_k + 2\mu e_{ij(k)} + 2\tau_i\beta_j + 2\tau_i\gamma_k + 2\tau_i e_{ij(k)} + 2\beta_j\gamma_k + 2\beta_j e_{ij(k)} + 2\gamma_k e_{ij(k)}\Big)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2\sum_{i=1}^{p}\tau_i\sum_{j=1}^{p}\beta_j + 2\sum_{i=1}^{p}\tau_i\sum_{k=1}^{p}\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{j=1}^{p}\beta_j\sum_{k=1}^{p}\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + 0 + 0 + 2p\mu\sum_{k=1}^{p}\gamma_k + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 0 + 0 + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 0 + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + 2p\mu\sum_{k=1}^{p}\gamma_k + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

Apply expectation on both sides

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}E(\gamma_k^2) + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) + 2p\mu\sum_{k=1}^{p}E(\gamma_k) + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k E(e_{ij(k)})$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p^2\sigma_\gamma^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p^2\sigma_\gamma^2 + p^2\sigma^2$$

$$E(TSS) = E\left(\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}^2\right) - E(C.F)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p^2\sigma_\gamma^2 + p^2\sigma^2 - p^2\mu^2 - p\sigma_\gamma^2 - \sigma^2$$

$$= p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p(p-1)\sigma_\gamma^2 + (p^2-1)\sigma^2$$

$$SSR = \frac{\sum_{i=1}^{p}R_i^2}{p} - C.F$$

$$R_i = \sum_{j=1}^{p}Y_{ij(k)} = \sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + p\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p}e_{ij(k)}$$

$$= p\mu + p\tau_i + 0 + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p}e_{ij(k)}$$

$$= p\mu + p\tau_i + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p}e_{ij(k)}$$

$$\frac{\sum_{i=1}^{p}R_i^2}{p} = \frac{\sum_{i=1}^{p}\left(p\mu + p\tau_i + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p}e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{i=1}^{p}\begin{pmatrix} p^2\mu^2 + p^2\tau_i^2 + \sum_{k=1}^{p}\gamma_k^2 + \sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{j=1}^{p}e_{ij(k)}^2 + \sum_{j\neq h}e_{ij(k)}e_{ih(l)} + 2p^2\mu\tau_i + 2p\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{j=1}^{p}e_{ij(k)} \\ +2p\sum_{k=1}^{p}\tau_i\gamma_k + 2p\tau_i\sum_{j=1}^{p}e_{ij(k)} + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\gamma_k\, e_{ij(k)} \end{pmatrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i \\ +2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\gamma_k\, e_{ij(k)} \end{matrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} + 0 \\ +2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\gamma_k\, e_{ij(k)} \end{matrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} \\ +2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\gamma_k\, e_{ij(k)} \end{matrix}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{k=1}^{p}E(\gamma_k^2) + p\sum\sum_{k\neq l}E(\gamma_k\gamma_l) + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) \\ +2p^2\mu\sum_{k=1}^{p}E(\gamma_k) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i E(\gamma_k) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i E(e_{ij(k)}) \\ +2\sum_{j=1}^{p}\sum_{k=1}^{p}\gamma_k\, E(e_{ij(k)}) \end{matrix}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p^2\sigma_\gamma^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p^2\sigma_\gamma^2 + p^2\sigma^2}{p}$$

$$E\left(\frac{\sum_{i=1}^{p}R_i^2}{p}\right) = p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sigma_\gamma^2 + p\sigma^2$$

$$E(SSR) = E\left(\frac{\sum_{i=1}^{p}R_i^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sigma_\gamma^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\gamma^2 - \sigma^2$$

$$= p\sum_{i=1}^{p}\tau_i^2 + (p-1)\sigma^2$$

$$SSC = \frac{\sum_{j=1}^{p}C_j^2}{p} - C.F$$

$$C_j = \sum_{i=1}^{p}Y_{ij(k)} = \sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}e_{ij(k)}$$

$$= p\mu + 0 + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}e_{ij(k)}$$

$$= p\mu + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}e_{ij(k)}$$

$$\frac{\sum_{j=1}^{p}C_j^2}{p} = \frac{\sum_{j=1}^{p}\left(p\mu + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{j=1}^{p}\begin{pmatrix} p^2\mu^2 + p^2\beta_j^2 + \sum_{k=1}^{p}\gamma_k^2 + \sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}e_{ij(k)}^2 + \sum_{i\neq g}e_{ij(k)}e_{gj(l)} + 2p^2\mu\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k \\ +2p\mu\sum_{i=1}^{p}e_{ij(k)} + 2p\beta_j\sum_{k=1}^{p}\gamma_k + 2p\sum_{i=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k)} \end{pmatrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{j=1}^{p}\beta_j \\ +2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2p\sum_{k=1}^{p}\sum_{j=1}^{p}\beta_j\gamma_k + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k)} \end{matrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} + 0 \\ +2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2p\sum_{k=1}^{p}\sum_{j=1}^{p}\beta_j\gamma_k + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k)} \end{matrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} \\ +2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2p\sum_{k=1}^{p}\sum_{j=1}^{p}\beta_j\gamma_k + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k)} \end{matrix}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}E(\gamma_k^2) + p\sum\sum_{k\neq l}E(\gamma_k\gamma_l) + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) \\ +2p^2\mu\sum_{k=1}^{p}E(\gamma_k) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2p\sum_{k=1}^{p}\sum_{j=1}^{p}\beta_j E(\gamma_k) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j E(e_{ij(k)}) \\ +2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\gamma_k e_{ij(k)}) \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + p^2\sigma_\gamma^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + p^2\sigma_\gamma^2 + p^2\sigma^2}{p}$$

$$E\left(\frac{\sum_{j=1}^{p}C_j^2}{p}\right) = p^2\mu^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sigma_\gamma^2 + p\sigma^2$$

$$E(SSC) = E\left(\frac{\sum_{j=1}^{p}C_j^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sigma_\gamma^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\gamma^2 - \sigma^2$$

$$= p\sum_{j=1}^{p}\beta_j^2 + (p-1)\sigma^2$$

$$T_k = \sum_{i=1}^{p}Y_{ij(k)} = \sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p}e_{ij(k)}$$

$$= p\mu + 0 + 0 + p\gamma_k + \sum_{i=1}^{p}e_{ij(k)}$$

$$= p\mu + p\gamma_k + \sum_{i=1}^{p}e_{ij(k)}$$

$$\frac{\sum_{k=1}^{p}T_k^2}{p} = \frac{\sum_{k=1}^{p}\left(p\mu + p\gamma_k + \sum_{i=1}^{p}e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{k=1}^{p}\left(p^2\mu^2 + p^2\gamma_k^2 + \sum_{i=1}^{p}e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} + 2p^2\mu\gamma_k + 2p\mu\sum_{i=1}^{p}e_{ij(k)} + 2p\sum_{i=1}^{p}\gamma_k e_{ij(k)}\right)}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{k=1}^{p}e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{k=1}^{p}e_{ij(k)} \\ +2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{k=1}^{p}e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{k=1}^{p}e_{ij(k)} \\ +2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)} \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{k=1}^{p}E(\gamma_k^2) + \sum_{i=1}^{p}\sum_{k=1}^{p}E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) + 2p^2\mu\sum_{k=1}^{p}E(\gamma_k) \\ +2p\mu\sum_{i=1}^{p}\sum_{k=1}^{p}E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}E(\gamma_k e_{ij(k)}) \end{array}}{p}$$

$$=\frac{p^3\mu^2+p^3\sigma_\gamma^2+p^2\sigma^2+0+0+0+0}{p}$$

$$=\frac{p^3\mu^2+p^3\sigma_\gamma^2+p^2\sigma^2}{p}$$

$$E\left(\frac{\sum_{k=1}^{p}T_k^2}{p}\right)=p^2\mu^2+p^2\sigma_\gamma^2+p\sigma^2$$

$$E(SST)=E\left(\frac{\sum_{k=1}^{p}T_k^2}{p}\right)-E(C.F)$$

$$=p^2\mu^2+p^2\sigma_\gamma^2+p\sigma^2-p^2\mu^2-p\sigma_\gamma^2-\sigma^2$$

$$=p(p-1)\sigma_\gamma^2+(p-1)\sigma^2$$

$$E(SSE)=E(TSS)-E(SSR)-E(SSC)-E(SST)$$

$$=p\sum_{i=1}^{p}\tau_i^2+p\sum_{j=1}^{p}\beta_j^2+p(p-1)\sigma_\gamma^2+(p^2-1)\sigma^2-p\sum_{i=1}^{p}\tau_i^2-(p-1)\sigma^2-p\sum_{j=1}^{p}\beta_j^2$$

$$-(p-1)\sigma^2-p(p-1)\sigma_\gamma^2-(p-1)\sigma^2$$

$$=(p^2-1-p+1-p+1-p+1)\sigma^2$$

$$=(p^2-3p+2)\sigma^2=(p-1)(p-2)\sigma^2$$

$$E(MSE)=\frac{E(SSE)}{(p-1)(p-2)}=\frac{(p-1)(p-2)\sigma^2}{(p-1)(p-2)}=\sigma^2$$

$$E(MSR)=\frac{E(SSR)}{p-1}=\frac{p\sum_{i=1}^{p}\tau_i^2+(p-1)\sigma^2}{p-1}=\sigma^2+\frac{p}{p-1}\sum_{i=1}^{p}\tau_i^2$$

$$E(MSC)=\frac{E(SSC)}{p-1}=\frac{p\sum_{j=1}^{p}\beta_j^2+(p-1)\sigma^2}{p-1}=\sigma^2+\frac{p}{p-1}\sum_{j=1}^{p}\beta_j^2$$

$$E(MST)=\frac{E(SST)}{p-1}=\frac{p(p-1)\sigma_\gamma^2+(p-1)\sigma^2}{p-1}=\sigma^2+p\sigma_\gamma^2$$

Case IV: In this model the effect of $\boldsymbol{\tau_i}$ and $\boldsymbol{\beta_j}$ is random and effect of $\boldsymbol{\gamma_k}$ is fixed

Assumptions:

1. $E\left(e_{ij(k)}\right)=0$
2. $E\left(e_{ij(k)}e_{gh(l)}\right)=0$
3. $e_{ij(k)}\sim iidN(0,\sigma^2)$
4. $\tau_i\sim iidN(0,\sigma_\tau^2)$
5. $E\left(\tau_i\tau_j\right)=0$
6. $E\left(\tau_i e_{ij(k)}\right)=0$
7. $\beta_j\sim iidN(0,\sigma_\beta^2)$
8. $E\left(\beta_i\beta_j\right)=0$
9. $E\left(\beta_j e_{ij(k)}\right)=0$
10. $E\left(\tau_i\beta_j\right)=0$

11. $\sum_{k=1}^{p}\gamma_k = 0$

$$Y_{ij(k)} = \mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}$$
$$SSE = TSS - SSR - SSC - SST$$
$$TSS = \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 - C.F$$
$$C.F = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}\right)^2}{p^2} = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)\right)^2}{p^2}$$
$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p}\tau_i + p\sum_{j=1}^{p}\beta_j + p\sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}\right)^2}{p^2}$$
$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p}\tau_i + p\sum_{j=1}^{p}\beta_j + 0 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}\right)^2}{p^2}$$
$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p}\tau_i + p\sum_{j=1}^{p}\beta_j + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}\right)^2}{p^2}$$
$$= \frac{\begin{array}{c} p^4\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + p^2\sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 \\ + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^3\mu\sum_{i=1}^{p}\tau_i + 2p^3\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} \end{array}}{p^2}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^4\mu^2 + p^2\sum_{i=1}^{p}E(\tau_i^2) + \sum\sum_{i\neq j}E(\tau_i\tau_j) + p^2\sum_{j=1}^{p}E(\beta_j^2) + \sum\sum_{i\neq j}E(\beta_i\beta_j) \\ + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) + 2p^3\mu\sum_{i=1}^{p}E(\tau_i) + 2p^3\mu\sum_{j=1}^{p}E(\beta_j) \\ +2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i\beta_j) \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j e_{ij(k)}) \end{array}}{p^2}$$

$$= \frac{p^4\mu^2 + p^3\sigma_\tau^2 + 0 + p^3\sigma_\beta^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p^2}$$
$$= \frac{p^4\mu^2 + p^3\sigma_\tau^2 + p^3\sigma_\beta^2 + p^2\sigma^2}{p^2}$$
$$E(C.F) = p^2\mu^2 + p\sigma_\tau^2 + p\sigma_\beta^2 + \sigma^2$$
$$\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 = \sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)^2$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{p}\Big(\mu^2 + \tau_i^2 + \beta_j^2 + \gamma_k^2 + e_{ij(k)}^2 + 2\mu\tau_i + 2\mu\beta_j + 2\mu\gamma_k + 2\mu e_{ij(k)} + 2\tau_i\beta_j + 2\tau_i\gamma_k$$
$$+ 2\tau_i e_{ij(k)} + 2\beta_j\gamma_k + 2\beta_j e_{ij(k)} + 2\gamma_k e_{ij(k)}\Big)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k$$
$$+ 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2\sum_{i=1}^{p}\tau_i\sum_{j=1}^{p}\beta_j + 2\sum_{i=1}^{p}\tau_i\sum_{k=1}^{p}\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{j=1}^{p}\beta_j\sum_{k=1}^{p}\gamma_k$$
$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 0 + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}$$
$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\,\tau_i + 0 + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 0 + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}$$
$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\,\tau_i + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

Apply expectation on both sides

$$= p^2\mu^2 + p\sum_{i=1}^{p}E(\tau_i^2) + p\sum_{j=1}^{p}E(\beta_j^2) + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + 2p\mu\sum_{i=1}^{p}E(\tau_i) + 2p\mu\sum_{j=1}^{p}E(\beta_j)$$
$$+ 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j\,\tau_i) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j e_{ij(k)})$$
$$+ 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k E(e_{ij(k)})$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p^2\sigma_\beta^2 + p\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p^2\sigma_\beta^2 + p\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2$$

$$E(TSS) = E\left(\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}^2\right) - E(C.F)$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p^2\sigma_\beta^2 + p\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\beta^2 - \sigma^2$$

$$= p(p-1)\sigma_\tau^2 + p(p-1)\sigma_\beta^2 + p\sum_{k=1}^{p}\gamma_k^2 + (p^2-1)\sigma^2$$

$$SSR = \frac{\sum_{i=1}^{p}R_i^2}{p} - C.F$$

$$R_i = \sum_{j=1}^{p}Y_{ij(k)} = \sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + p\tau_i + \sum_{j=1}^{p} \beta_j + \sum_{k=1}^{p} \gamma_k + \sum_{j=1}^{p} e_{ij(k)}$$

$$= p\mu + p\tau_i + \sum_{j=1}^{p} \beta_j + 0 + \sum_{j=1}^{p} e_{ij(k)}$$

$$= p\mu + p\tau_i + \sum_{j=1}^{p} \beta_j + \sum_{j=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{i=1}^{p} R_i^2}{p} = \frac{\sum_{i=1}^{p}\left(p\mu + p\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{j=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{i=1}^{p}\begin{pmatrix} p^2\mu^2 + p^2\tau_i^2 + \sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + \sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{j\neq h} e_{ij(k)}e_{ih(l)} + 2p^2\mu\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j \\ +2p\mu\sum_{j=1}^{p} e_{ij(k)} + 2p\tau_i\sum_{j=1}^{p}\beta_j + 2p\tau_i\sum_{j=1}^{p} e_{ij(k)} + 2\sum_{j=1}^{p}\beta_j\, e_{ij(k)} \end{pmatrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} \\ +2p^2\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\, e_{ij(k)} \end{matrix}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{i=1}^{p} E(\tau_i^2) + p\sum_{j=1}^{p} E(\beta_j^2) + p\sum\sum_{i\neq j} E(\beta_i\beta_j) + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k)}e_{gh(l)}) \\ +2p^2\mu\sum_{i=1}^{p} E(\tau_i) + 2p^2\mu\sum_{j=1}^{p} E(\beta_j) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i\beta_j) \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\beta_j\, e_{ij(k)}) \end{matrix}}{p}$$

$$= \frac{p^3\mu^2 + p^3\sigma_\tau^2 + p^2\sigma_\beta^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^3\sigma_\tau^2 + p^2\sigma_\beta^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p\sigma_\beta^2 + p\sigma^2$$

$$E(SSR) = E\left(\frac{\sum_{i=1}^{p} R_i^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p\sigma_\beta^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\beta^2 - \sigma^2$$

$$= p(p-1)\sigma_\tau^2 + (p-1)\sigma^2$$

$$SSC = \frac{\sum_{j=1}^{p} C_j^2}{p} - C.F$$

$$C_j = \sum_{i=1}^{p} Y_{ij(k)} = \sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + p\beta_j + 0 + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + \sum_{i=1}^{p} \tau_i + p\beta_j + \sum_{i=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{j=1}^{p} C_j^2}{p} = \frac{\sum_{j=1}^{p}\left(p\mu + \sum_{i=1}^{p}\tau_i + p\beta_j + \sum_{i=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{j=1}^{p}\begin{pmatrix} p^2\mu^2 + \sum_{i=1}^{p}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + p^2\beta_j^2 + \sum_{i=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g} e_{ij(k)}e_{gj(l)} + 2p\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\beta_j \\ +2p\mu\sum_{i=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\tau_i\beta_j + 2\sum_{i=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{i=1}^{p}\beta_j\, e_{ij(k)} \end{pmatrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum\sum_{i\neq j}\tau_i\tau_j + p^2\sum_{j=1}^{p}\beta_j^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i \\ +2p^2\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\, e_{ij(k)} \end{matrix}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{matrix} p^3\mu^2 + p\sum_{i=1}^{p}E(\tau_i^2) + p\sum\sum_{i\neq j}E(\tau_i\tau_j) + p^2\sum_{j=1}^{p}E(\beta_j^2) + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) \\ +\sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) + 2p^2\mu\sum_{i=1}^{p}E(\tau_i) + 2p^2\mu\sum_{j=1}^{p}E(\beta_j) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i\beta_j) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j\, e_{ij(k)}) \end{matrix}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + 0 + p^3\sigma_\beta^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + p^3\sigma_\beta^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p^2\sigma_\beta^2 + p\sigma^2$$

$$E(SSC) = E\left(\frac{\sum_{j=1}^{p} C_j^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p^2\sigma_\beta^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\beta^2 - \sigma^2$$

$$= p(p-1)\sigma_\beta^2 + (p-1)\sigma^2$$

$$SST = \frac{\sum_{k=1}^{p} T_k^2}{p} - C.F$$

$$T_k = \sum_{i=1}^{p} Y_{ij(k)} = \sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{k=1}^{p} T_k^2}{p} = \frac{\sum_{k=1}^{p}\left(p\mu + \sum_{i=1}^{p}\tau_i + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{k=1}^{p}\begin{pmatrix} p^2\mu^2 + \sum_{i=1}^{p}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + \sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + p^2\gamma_k^2 + \sum_{i=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} \\ +2p\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\gamma_k + 2p\mu\sum_{i=1}^{p} e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2p\sum_{i=1}^{p}\tau_i\gamma_k \\ +2\sum_{i=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{j=1}^{p}\beta_j\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\, e_{ij(k)} + 2p\sum_{i=1}^{p}\gamma_k e_{ij(k)} \end{pmatrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum\sum_{i\neq j}\tau_i\tau_j + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 \\ +\sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j \\ +2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{k=1}^{p}\sum_{j=1}^{p}\beta_j\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\, e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)} \end{matrix}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum\sum_{i\neq j}\tau_i\tau_j + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 \\ +\sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\sum_{j=1}^{p}\beta_j + 0 + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j \\ +2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{k=1}^{p}\sum_{j=1}^{p}\beta_j\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)} \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum\sum_{i\neq j}\tau_i\tau_j + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 \\ +\sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j \\ +2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{k=1}^{p}\sum_{j=1}^{p}\beta_j\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p\sum_{i=1}^{p}E(\tau_i^2) + p\sum\sum_{i\neq j}E(\tau_i\tau_j) + p\sum_{j=1}^{p}E(\beta_j^2) + p\sum\sum_{i\neq j}E(\beta_i\beta_j) + p^2\sum_{k=1}^{p}\gamma_k^2 \\ +\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) + 2p^2\mu\sum_{i=1}^{p}E(\tau_i) + 2p^2\mu\sum_{j=1}^{p}E(\beta_j) \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i\beta_j) + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k E(\tau_i) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i e_{ij(k)}) \\ +2p\sum_{k=1}^{p}\sum_{j=1}^{p}\gamma_k E(\beta_j) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k E(e_{ij(k)}) \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + 0 + p^2\sigma_\beta^2 + 0 + p^2\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + p^2\sigma_\beta^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p\sigma_\beta^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sigma^2$$

$$E(SST) = E\left(\frac{\sum_{k=1}^{p}T_k^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p\sigma_\beta^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\beta^2 - \sigma^2$$

$$= p\sum_{k=1}^{p}\gamma_k^2 + (p-1)\sigma^2$$

$$E(SSE) = E(TSS) - E(SSR) - E(SSC) - E(SST)$$

$$= p(p-1)\sigma_\tau^2 + p(p-1)\sigma_\beta^2 + p\sum_{k=1}^{p}\gamma_k^2 + (p^2-1)\sigma^2 - p(p-1)\sigma_\tau^2 - (p-1)\sigma^2 - p(p-1)\sigma_\beta^2 - (p-1)\sigma^2 - p\sum_{k=1}^{p}\gamma_k^2 - (p-1)\sigma^2$$

$$= (p^2 - 1 - p + 1 - p + 1 - p + 1)\sigma^2$$

$$= (p^2 - 3p + 2)\sigma^2 = (p-1)(p-2)\sigma^2$$

$$E(MSE) = \frac{E(SSE)}{(p-1)(p-2)} = \frac{(p-1)(p-2)\sigma^2}{(p-1)(p-2)} = \sigma^2$$

$$E(MSR) = \frac{E(SSR)}{p-1} = \frac{p(p-1)\sigma_\tau^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + p\sigma_\tau^2$$

$$E(MSC) = \frac{E(SSC)}{p-1} = \frac{p(p-1)\sigma_\beta^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + p\sigma_\beta^2$$

$$E(MST) = \frac{E(SST)}{p-1} = \frac{p\sum_{k=1}^{p}\gamma_k^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{k=1}^{p}\gamma_k^2$$

Case V: In this model the effect of τ_i and γ_k is random and effect of β_j is fixed

Assumptions:

1. $E(e_{ij(k)}) = 0$
2. $E(e_{ij(k)}e_{gh(l)}) = 0$
3. $e_{ij(k)} \sim iidN(0, \sigma^2)$
4. $\tau_i \sim iidN(0, \sigma_\tau^2)$
5. $E(\tau_i\tau_j) = 0$
6. $E(\tau_i e_{ij(k)}) = 0$
7. $\gamma_k \sim iidN(0, \sigma_\gamma^2)$
8. $E(\gamma_k\gamma_l) = 0$
9. $E(\gamma_k e_{ij(k)}) = 0$
10. $E(\tau_i\gamma_k) = 0$
11. $\sum_{j=1}^{p}\beta_j = 0$

$$Y_{ij(k)} = \mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}$$
$$SSE = TSS - SSR - SSC - SST$$
$$TSS = \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 - C.F$$
$$C.F = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}\right)^2}{p^2} = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)\right)^2}{p^2}$$
$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p}\tau_i + p\sum_{j=1}^{p}\beta_j + p\sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}\right)^2}{p^2}$$
$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p}\tau_i + 0 + p\sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}\right)^2}{p^2}$$
$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p}\tau_i + p\sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}\right)^2}{p^2}$$
$$= \frac{\begin{array}{c} p^4\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 \\ + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^3\mu\sum_{i=1}^{p}\tau_i + 2p^3\mu\sum_{k=1}^{p}\gamma_k + 2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)} \end{array}}{p^2}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^4\mu^2 + p^2\sum_{i=1}^{p} E(\tau_i^2) + \sum\sum_{i\neq j} E(\tau_i\tau_j) + p^2\sum_{k=1}^{p} E(\gamma_k^2) + \sum\sum_{k\neq l} E(\gamma_k\gamma_l) \\ + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k)}e_{gh(l)}) + 2p^3\mu\sum_{i=1}^{p} E(\tau_i) + 2p^3\mu\sum_{k=1}^{p} E(\gamma_k) \\ +2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{k=1}^{p} E(\tau_i\gamma_k) \\ +2p\sum_{i=1}^{p}\sum_{k=1}^{p} E(\gamma_k e_{ij(k)}) \end{array}}{p^2}$$

$$= \frac{p^4\mu^2 + p^3\sigma_\tau^2 + 0 + p^3\sigma_\gamma^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p^2}$$

$$= \frac{p^4\mu^2 + p^3\sigma_\tau^2 + p^3\sigma_\gamma^2 + p^2\sigma^2}{p^2}$$

$$E(C.F) = p^2\mu^2 + p\sigma_\tau^2 + p\sigma_\gamma^2 + \sigma^2$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 = \sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)^2$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p}\Big(\mu^2 + \tau_i^2 + \beta_j^2 + \gamma_k^2 + e_{ij(k)}^2 + 2\mu\tau_i + 2\mu\beta_j + 2\mu\gamma_k + 2\mu e_{ij(k)} + 2\tau_i\beta_j + 2\tau_i\gamma_k + 2\tau_i e_{ij(k)} + 2\beta_j\gamma_k + 2\beta_j e_{ij(k)} + 2\gamma_k e_{ij(k)}\Big)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2\sum_{i=1}^{p}\tau_i\sum_{j=1}^{p}\beta_j + 2\sum_{i=1}^{p}\tau_i\sum_{k=1}^{p}\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{j=1}^{p}\beta_j\sum_{k=1}^{p}\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i + 0 + 2p\mu\sum_{k=1}^{p}\gamma_k + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 0 + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k\tau_i + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 0 + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i + 2p\mu\sum_{k=1}^{p}\gamma_k + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k\tau_i + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

Apply expectation on both sides

$$= p^2\mu^2 + p\sum_{i=1}^{p} E(\tau_i^2) + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p} E(\gamma_k^2) + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) + 2p\mu\sum_{i=1}^{p} E(\tau_i) + 2p\mu\sum_{k=1}^{p} E(\gamma_k) + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{k=1}^{p} E(\gamma_k\tau_i) + 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{k=1}^{p} E(\gamma_k e_{ij(k)})$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p\sum_{j=1}^{p}\beta_j^2 + p^2\sigma_\gamma^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p\sum_{j=1}^{p}\beta_j^2 + p^2\sigma_\gamma^2 + p^2\sigma^2$$

$$E(TSS) = E\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2\right) - E(C.F)$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p\sum_{j=1}^{p}\beta_j^2 + p^2\sigma_\gamma^2 + p^2\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\gamma^2 - \sigma^2$$

$$= p(p-1)\sigma_\tau^2 + p\sum_{j=1}^{p}\beta_j^2 + p(p-1)\sigma_\gamma^2 + (p^2-1)\sigma^2$$

$$SSR = \frac{\sum_{i=1}^{p} R_i^2}{p} - C.F$$

$$R_i = \sum_{j=1}^{p} Y_{ij(k)} = \sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + p\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p} e_{ij(k)}$$

$$= p\mu + p\tau_i + 0 + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p} e_{ij(k)}$$

$$= p\mu + p\tau_i + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{i=1}^{p} R_i^2}{p} = \frac{\sum_{i=1}^{p}\left(p\mu + p\tau_i + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{i=1}^{p}\left(\begin{array}{c} p^2\mu^2 + p^2\tau_i^2 + \sum_{k=1}^{p}\gamma_k^2 + \sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{j\neq h} e_{ij(k)}e_{ih(l)} + 2p^2\mu\tau_i + 2p\mu\sum_{k=1}^{p}\gamma_k \\ +2p\mu\sum_{j=1}^{p} e_{ij(k)} + 2p\tau_i\sum_{k=1}^{p}\gamma_k + 2p\tau_i\sum_{j=1}^{p} e_{ij(k)} + 2\sum_{k=1}^{p}\gamma_k\, e_{ij(k)} \end{array}\right)}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} \\ +2p^2\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k\, e_{ij(k)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p}E(\tau_i^2) + p\sum_{k=1}^{p}E(\gamma_k^2) + p\sum\sum_{k\neq l}E(\gamma_k\gamma_l) + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) \\ +2p^2\mu\sum_{i=1}^{p}E(\tau_i) + 2p^2\mu\sum_{k=1}^{p}E(\gamma_k) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}E(\tau_i\gamma_k) \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{k=1}^{p}E(\gamma_k\, e_{ij(k)}) \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^3\sigma_\tau^2 + p^2\sigma_\gamma^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$=\frac{p^3\mu^2+p^3\sigma_\tau^2+p^2\sigma_\gamma^2+p^2\sigma^2}{p}$$

$$=p^2\mu^2+p^2\sigma_\tau^2+p\sigma_\gamma^2+p\sigma^2$$

$$E(SSR)=E\left(\frac{\sum_{i=1}^{p}R_i^2}{p}\right)-E(C.F)$$

$$=p^2\mu^2+p^2\sigma_\tau^2+p\sigma_\gamma^2+p\sigma^2-p^2\mu^2-p\sigma_\tau^2-p\sigma_\gamma^2-\sigma^2$$

$$=p(p-1)\sigma_\tau^2+(p-1)\sigma^2$$

$$SSC=\frac{\sum_{j=1}^{p}C_j^2}{p}-C.F$$

$$C_j=\sum_{i=1}^{p}Y_{ij(k)}=\sum_{i=1}^{p}\left(\mu+\tau_i+\beta_j+\gamma_k+e_{ij(k)}\right)$$

$$=p\mu+\sum_{i=1}^{p}\tau_i+p\beta_j+\sum_{k=1}^{p}\gamma_k+\sum_{i=1}^{p}e_{ij(k)}$$

$$=p\mu+\sum_{i=1}^{p}\tau_i+p\beta_j+\sum_{k=1}^{p}\gamma_k+\sum_{i=1}^{p}e_{ij(k)}$$

$$=p\mu+\sum_{i=1}^{p}\tau_i+p\beta_j+\sum_{k=1}^{p}\gamma_k+\sum_{i=1}^{p}e_{ij(k)}$$

$$\frac{\sum_{j=1}^{p}C_j^2}{p}=\frac{\sum_{j=1}^{p}\left(p\mu+\sum_{i=1}^{p}\tau_i+p\beta_j+\sum_{k=1}^{p}\gamma_k+\sum_{i=1}^{p}e_{ij(k)}\right)^2}{p}$$

$$=\frac{\sum_{j=1}^{p}\begin{pmatrix}p^2\mu^2+\sum_{i=1}^{p}\tau_i^2+\sum\sum_{i\neq j}\tau_i\tau_j+p^2\beta_j^2+\sum_{k=1}^{p}\gamma_k^2+\sum\sum_{k\neq l}\gamma_k\gamma_l+\sum_{i=1}^{p}e_{ij(k)}^2+\sum_{i\neq g}e_{ij(k)}e_{gj(l)}\\+2p\mu\sum_{i=1}^{p}\tau_i+2p^2\mu\beta_j+2p\mu\sum_{k=1}^{p}\gamma_k+2p\mu\sum_{i=1}^{p}e_{ij(k)}+2p\sum_{i=1}^{p}\tau_i\beta_j+2\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k\\+2\sum_{i=1}^{p}\tau_ie_{ij(k)}+2p\sum_{k=1}^{p}\beta_j\gamma_k+2p\sum_{i=1}^{p}\beta_j\,e_{ij(k)}+2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_ke_{ij(k)}\end{pmatrix}}{p}$$

$$=\frac{\begin{gathered}p^3\mu^2+p\sum_{i=1}^{p}\tau_i^2+p\sum\sum_{i\neq j}\tau_i\tau_j+p^2\sum_{j=1}^{p}\beta_j^2+p\sum_{k=1}^{p}\gamma_k^2+p\sum\sum_{k\neq l}\gamma_k\gamma_l+\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2\\+\sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)}+2p^2\mu\sum_{i=1}^{p}\tau_i+2p^2\mu\sum_{j=1}^{p}\beta_j+2p^2\mu\sum_{k=1}^{p}\gamma_k+2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\\+2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j+2\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k+2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_ie_{ij(k)}+2p\sum_{k=1}^{p}\sum_{j=1}^{p}\beta_j\gamma_k\\+2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\,e_{ij(k)}+2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_ke_{ij(k)}\end{gathered}}{p}$$

Apply expectation on both sides

$$=\frac{\begin{gathered}p^3\mu^2+p\sum_{i=1}^{p}E(\tau_i^2)+p\sum\sum_{i\neq j}E(\tau_i\tau_j)+p^2\sum_{j=1}^{p}\beta_j^2+p\sum_{k=1}^{p}E(\gamma_k^2)+p\sum\sum_{k\neq l}E(\gamma_k\gamma_l)+\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2)\\+\sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)})+2p^2\mu\sum_{i=1}^{p}E(\tau_i)+2p^2\mu\sum_{j=1}^{p}\beta_j+2p^2\mu\sum_{k=1}^{p}E(\gamma_k)+2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)})\\+2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i)\beta_j+2\sum_{i=1}^{p}\sum_{k=1}^{p}E(\tau_i\gamma_k)+2\sum_{i=1}^{p}\sum_{j=1}^{p}(\tau_ie_{ij(k)})+2p\sum_{k=1}^{p}\sum_{j=1}^{p}\beta_jE(\gamma_k)\\+2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\,E(e_{ij(k)})+2\sum_{i=1}^{p}\sum_{k=1}^{p}E(\gamma_ke_{ij(k)})\end{gathered}}{p}$$

$$=\frac{p^3\mu^2+p^2\sigma_\tau^2+0+p^2\sum_{j=1}^{p}\beta_j^2+p^2\sigma_\gamma^2+0+p^2\sigma^2+0+0+0+0+0+0+0+0+0+0+0}{p}$$

$$=\frac{p^3\mu^2+p^2\sigma_\tau^2+p^2\sum_{j=1}^{p}\beta_j^2+p^2\sigma_\gamma^2+p^2\sigma^2}{p}$$

$$=p^2\mu^2+p\sigma_\tau^2+p\sum_{j=1}^{p}\beta_j^2+p\sigma_\gamma^2+p\sigma^2$$

$$E(SSC)=E\left(\frac{\sum_{j=1}^{p}C_j^2}{p}\right)-E(C.F)$$

$$=p^2\mu^2+p\sigma_\tau^2+p\sum_{j=1}^{p}\beta_j^2+p\sigma_\gamma^2+p\sigma^2-p^2\mu^2-p\sigma_\tau^2-p\sigma_\gamma^2-\sigma^2$$

$$=p\sum_{j=1}^{p}\beta_j^2+(p-1)\sigma^2$$

$$SST=\frac{\sum_{k=1}^{p}T_k^2}{p}-C.F$$

$$T_k=\sum_{i=1}^{p}Y_{ij(k)}=\sum_{i=1}^{p}\left(\mu+\tau_i+\beta_j+\gamma_k+e_{ij(k)}\right)$$

$$=p\mu+\sum_{i=1}^{p}\tau_i+\sum_{j=1}^{p}\beta_j+p\gamma_k+\sum_{i=1}^{p}e_{ij(k)}$$

$$=p\mu+\sum_{i=1}^{p}\tau_i+0+p\gamma_k+\sum_{i=1}^{p}e_{ij(k)}$$

$$=p\mu+\sum_{i=1}^{p}\tau_i+p\gamma_k+\sum_{i=1}^{p}e_{ij(k)}$$

$$\frac{\sum_{k=1}^{p}T_k^2}{p}=\frac{\sum_{k=1}^{p}\left(p\mu+\sum_{i=1}^{p}\tau_i+p\gamma_k+\sum_{i=1}^{p}e_{ij(k)}\right)^2}{p}$$

$$=\frac{\sum_{k=1}^{p}\left(\begin{array}{c}p^2\mu^2+\sum_{i=1}^{p}\tau_i^2+\sum\sum_{i\neq j}\tau_i\tau_j+p^2\gamma_k^2+\sum_{i=1}^{p}e_{ij(k)}^2+\sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)}+2p\mu\sum_{i=1}^{p}\tau_i+2p^2\mu\gamma_k\\+2p\mu\sum_{i=1}^{p}e_{ij(k)}+2p\sum_{i=1}^{p}\tau_i\gamma_k+2\sum_{i=1}^{p}\tau_ie_{ij(k)}+2p\sum_{i=1}^{p}\gamma_ke_{ij(k)}\end{array}\right)}{p}$$

$$=\frac{\begin{array}{c}p^3\mu^2+p\sum_{i=1}^{p}\tau_i^2+p\sum\sum_{i\neq j}\tau_i\tau_j+p^2\sum_{k=1}^{p}\gamma_k^2+\sum_{i=1}^{p}\sum_{k=1}^{p}e_{ij(k)}^2+\sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)}\\+2p^2\mu\sum_{i=1}^{p}\tau_i+2p^2\mu\sum_{k=1}^{p}\gamma_k+2p\mu\sum_{i=1}^{p}\sum_{k=1}^{p}e_{ij(k)}+2p\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k\\+2\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_ie_{ij(k)}+2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_ke_{ij(k)}\end{array}}{p}$$

Apply expectation on both sides

$$=\frac{\begin{array}{c}p^3\mu^2+p\sum_{i=1}^{p}E(\tau_i^2)+p\sum\sum_{i\neq j}E(\tau_i\tau_j)+p^2\sum_{k=1}^{p}E(\gamma_k^2)+\sum_{i=1}^{p}\sum_{k=1}^{p}E(e_{ij(k)}^2)+\sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)})\\+2p^2\mu\sum_{i=1}^{p}E(\tau_i)+2p^2\mu\sum_{k=1}^{p}E(\gamma_k)+2p\mu\sum_{i=1}^{p}\sum_{k=1}^{p}E(e_{ij(k)})+2p\sum_{i=1}^{p}\sum_{k=1}^{p}E(\tau_i\gamma_k)\\+2\sum_{i=1}^{p}\sum_{k=1}^{p}E(\tau_ie_{ij(k)})+2p\sum_{i=1}^{p}\sum_{k=1}^{p}E(\gamma_ke_{ij(k)})\end{array}}{p}$$

$$=\frac{p^3\mu^2+p^2\sigma_\tau^2+0+p^3\sigma_\gamma^2+p^2\sigma^2+0+0+0+0+0+0+0}{p}$$

$$=\frac{p^3\mu^2+p^2\sigma_\tau^2+p^3\sigma_\gamma^2+p^2\sigma^2}{p}$$

$$=p^2\mu^2+p\sigma_\tau^2+p^2\sigma_\gamma^2+p\sigma^2$$

$$E(SST)=E\left(\frac{\sum_{k=1}^{p}T_k^2}{p}\right)-E(C.F)$$

$$=p^2\mu^2+p\sigma_\tau^2+p^2\sigma_\gamma^2+p\sigma^2-p^2\mu^2-p\sigma_\tau^2-p\sigma_\gamma^2-\sigma^2$$

$$= p(p-1)\sigma_\gamma^2 + (p-1)\sigma^2$$

$$E(SSE) = E(TSS) - E(SSR) - E(SSC) - E(SST)$$

$$= p(p-1)\sigma_\tau^2 + p\sum_{j=1}^{p}\beta_j^2 + p(p-1)\sigma_\gamma^2 + (p^2-1)\sigma^2 - p(p-1)\sigma_\tau^2 - (p-1)\sigma^2$$

$$- p\sum_{j=1}^{p}\beta_j^2 - (p-1)\sigma^2 - p(p-1)\sigma_\gamma^2 + (p-1)\sigma^2$$

$$= (p^2 - 1 - p + 1 - p + 1 - p + 1)\sigma^2$$

$$= (p^2 - 3p + 2)\sigma^2 = (p-1)(p-2)\sigma^2$$

$$E(MSE) = \frac{E(SSE)}{(p-1)(p-2)} = \frac{(p-1)(p-2)\sigma^2}{(p-1)(p-2)} = \sigma^2$$

$$E(MSR) = \frac{E(SSR)}{p-1} = \frac{p(p-1)\sigma_\tau^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + p\sigma_\tau^2$$

$$E(MSC) = \frac{E(SSC)}{p-1} = \frac{p\sum_{j=1}^{p}\beta_j^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{j=1}^{p}\beta_j^2$$

$$E(MST) = \frac{E(SST)}{p-1} = \frac{p(p-1)\sigma_\gamma^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + p\sigma_\gamma^2$$

Case VI: In this model the effect of $\boldsymbol{\beta_j}$ and $\boldsymbol{\gamma_k}$ is random and effect of $\boldsymbol{\tau_i}$ is fixed

Assumptions:

1. $E\left(e_{ij(k)}\right) = 0$
2. $E\left(e_{ij(k)}e_{gh(l)}\right) = 0$
3. $e_{ij(k)} \sim iidN(0, \sigma^2)$
4. $\beta_j \sim iidN(0, \sigma_\beta^2)$
5. $E\left(\beta_i\beta_j\right) = 0$
6. $E\left(\beta_j e_{ij(k)}\right) = 0$
7. $\gamma_k \sim iidN(0, \sigma_\gamma^2)$
8. $E(\gamma_k\gamma_l) = 0$
9. $E\left(\gamma_k e_{ij(k)}\right) = 0$
10. $E\left(\beta_j\gamma_k\right) = 0$
11. $\sum_{i=1}^{p}\tau_i = 0$

$$Y_{ij(k)} = \mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}$$

$$SSE = TSS - SSR - SSC - SST$$

$$TSS = \sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}^2 - C.F$$

$$C.F=\frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}\right)^2}{p^2}=\frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu+\tau_i+\beta_j+\gamma_k+e_{ij(k)}\right)\right)^2}{p^2}$$

$$=\frac{\left(p^2\mu+p\sum_{i=1}^{p}\tau_i+p\sum_{j=1}^{p}\beta_j+p\sum_{k=1}^{p}\gamma_k+\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\right)^2}{p^2}$$

$$=\frac{\left(p^2\mu+0+p\sum_{j=1}^{p}\beta_j+p\sum_{k=1}^{p}\gamma_k+\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\right)^2}{p^2}$$

$$=\frac{\left(p^2\mu+p\sum_{j=1}^{p}\beta_j+p\sum_{k=1}^{p}\gamma_k+\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\right)^2}{p^2}$$

$$=\frac{\begin{array}{c}p^4\mu^2+p^2\sum_{j=1}^{p}\beta_j^2+\sum\sum_{i\neq j}\beta_i\beta_j+p^2\sum_{k=1}^{p}\gamma_k^2+\sum\sum_{k\neq l}\gamma_k\gamma_l+\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2\\+\sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)}+2p^3\mu\sum_{j=1}^{p}\beta_j+2p^3\mu\sum_{k=1}^{p}\gamma_k+2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}\\+2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_je_{ij(k)}+2p\sum_{j=1}^{p}\sum_{k=1}^{p}\beta_j\gamma_k+2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_ke_{ij(k)}\end{array}}{p^2}$$

Apply expectation on both sides

$$=\frac{\begin{array}{c}p^4\mu^2+p^2\sum_{j=1}^{p}E(\beta_j^2)+\sum\sum_{i\neq j}E(\beta_i\beta_j)+p^2\sum_{k=1}^{p}E(\gamma_k^2)+\sum\sum_{k\neq l}E(\gamma_k\gamma_l)\\+\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2)+\sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)})+2p^3\mu\sum_{j=1}^{p}E(\beta_j)+2p^3\mu\sum_{k=1}^{p}E(\gamma_k)\\+2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)})+2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_je_{ij(k)})+2p\sum_{j=1}^{p}\sum_{k=1}^{p}E(\beta_j\gamma_k)\\+2p\sum_{i=1}^{p}\sum_{k=1}^{p}E(\gamma_ke_{ij(k)})\end{array}}{p^2}$$

$$=\frac{p^4\mu^2+p^3\sigma_\beta^2+0+p^3\sigma_\gamma^2+0+p^2\sigma^2+0+0+0+0+0+0+0}{p^2}$$

$$=\frac{p^4\mu^2+p^3\sigma_\beta^2+p^3\sigma_\gamma^2+p^2\sigma^2}{p^2}$$

$$E(C.F)=p^2\mu^2+p\sigma_\beta^2+p\sigma_\gamma^2+\sigma^2$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}^2=\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu+\tau_i+\beta_j+\gamma_k+e_{ij(k)}\right)^2$$

$$=\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu^2+\tau_i^2+\beta_j^2+\gamma_k^2+e_{ij(k)}^2+2\mu\tau_i+2\mu\beta_j+2\mu\gamma_k+2\mu e_{ij(k)}+2\tau_i\beta_j+2\tau_i\gamma_k\right.$$
$$\left.+2\tau_ie_{ij(k)}+2\beta_j\gamma_k+2\beta_je_{ij(k)}+2\gamma_ke_{ij(k)}\right)$$

$$=p^2\mu^2+p\sum_{i=1}^{p}\tau_i^2+p\sum_{j=1}^{p}\beta_j^2+p\sum_{k=1}^{p}\gamma_k^2+\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2+2p\mu\sum_{i=1}^{p}\tau_i+2p\mu\sum_{j=1}^{p}\beta_j+2p\mu\sum_{k=1}^{p}\gamma_k$$
$$+2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}+2\sum_{i=1}^{p}\tau_i\sum_{j=1}^{p}\beta_j+2\sum_{i=1}^{p}\tau_i\sum_{k=1}^{p}\gamma_k+2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_ie_{ij(k)}+2\sum_{j=1}^{p}\beta_j\sum_{k=1}^{p}\gamma_k$$
$$+2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_je_{ij(k)}+2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_ke_{ij(k)}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 0 + 2p\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} + 0$$

$$+ 0 + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{j=1}^{p}\beta_j\sum_{k=1}^{p}\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 + 2p\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k + 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}$$

$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2\sum_{j=1}^{p}\beta_j\sum_{k=1}^{p}\gamma_k + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}$$

Apply expectation on both sides

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}E(\beta_j^2) + p\sum_{k=1}^{p}E(\gamma_k^2) + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + 2p\mu\sum_{j=1}^{p}E(\beta_j) + 2p\mu\sum_{k=1}^{p}E(\gamma_k)$$

$$+ 2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i e_{ij(k)}) + 2\sum_{j=1}^{p}\sum_{k=1}^{p}E(\gamma_k\,\beta_j) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j e_{ij(k)})$$

$$+ 2\sum_{i=1}^{p}\sum_{k=1}^{p}E(\gamma_k e_{ij(k)})$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p^2\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p^2\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma^2$$

$$E(TSS) = E\left(\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}^2\right) - E(C.F)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p^2\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma^2 - p^2\mu^2 - p\sigma_\beta^2 - p\sigma_\gamma^2 - \sigma^2$$

$$= p\sum_{i=1}^{p}\tau_i^2 + p(p-1)\sigma_\beta^2 + p(p-1)\sigma_\gamma^2 + (p^2-1)\sigma^2$$

$$SSR = \frac{\sum_{i=1}^{p}R_i^2}{p} - C.F$$

$$R_i = \sum_{j=1}^{p}Y_{ij(k)} = \sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + p\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p}e_{ij(k)}$$

$$\frac{\sum_{i=1}^{p}R_i^2}{p} = \frac{\sum_{i=1}^{p}\left(p\mu + p\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{j=1}^{p}e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{i=1}^{p}\begin{pmatrix} p^2\mu^2 + p^2\tau_i^2 + \sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + \sum_{k=1}^{p}\gamma_k^2 + \sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{j=1}^{p}e_{ij(k)}^2 + \sum_{j\neq h}e_{ij(k)}e_{ih(l)} \\ +2p^2\mu\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{j=1}^{p}e_{ij(k)} + 2p\tau_i\sum_{j=1}^{p}\beta_j + 2p\tau_i\sum_{k=1}^{p}\gamma_k \\ +2p\tau_i\sum_{j=1}^{p}e_{ij(k)} + 2\sum_{j=1}^{p}\beta_j\gamma_k + 2\sum_{j=1}^{p}\beta_j\, e_{ij(k)} + 2\sum_{k=1}^{p}\gamma_k\, e_{ij(k)} \end{pmatrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + p\sum_{k=1}^{p}\gamma_k^2 + p\sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)}^2 \\ +\sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k)} \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2p\sum_{k=1}^{p}\sum_{i=1}^{p}\tau_i\gamma_k + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k)} + 2p\sum_{j=1}^{p}\beta_j\gamma_k \\ +2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\, e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k\, e_{ij(k)} \end{matrix}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}E(\beta_j^2) + p\sum\sum_{i\neq j}E(\beta_i\beta_j) + p\sum_{k=1}^{p}E(\gamma_k^2) + p\sum\sum_{k\neq l}E(\gamma_k\gamma_l) \\ +\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) + 2p^2\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\sum_{j=1}^{p}E(\beta_j) + 2p^2\mu\sum_{k=1}^{p}E(\gamma_k) \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_iE(\beta_j) + 2p\sum_{k=1}^{p}\sum_{i=1}^{p}\tau_iE(\gamma_k) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_iE(e_{ij(k)}) \\ +2p\sum_{j=1}^{p}E(\beta_j\gamma_k) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j\, e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{k=1}^{p}E(\gamma_k\, e_{ij(k)}) \end{matrix}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p^2\sigma_\beta^2 + 0 + p^2\sigma_\gamma^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p^2\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma^2$$

$$E(SSR) = E\left(\frac{\sum_{i=1}^{p}R_i^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\beta^2 - p\sigma_\gamma^2 - \sigma^2$$

$$= p\sum_{i=1}^{p}\tau_i^2 + (p-1)\sigma^2$$

$$SSC = \frac{\sum_{j=1}^{p}C_j^2}{p} - C.F$$

$$C_j = \sum_{i=1}^{p}Y_{ij(k)} = \sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}e_{ij(k)}$$

$$= p\mu + 0 + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}e_{ij(k)}$$

$$= p\mu + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p}e_{ij(k)}$$

$$\frac{\sum_{j=1}^{p} C_j^2}{p} = \frac{\sum_{j=1}^{p}\left(p\mu + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{i=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{j=1}^{p}\begin{pmatrix} p^2\mu^2 + p^2\beta_j^2 + \sum_{k=1}^{p}\gamma_k^2 + \sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g} e_{ij(k)}e_{gj(l)} + 2p^2\mu\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k \\ +2p\mu\sum_{i=1}^{p} e_{ij(k)} + 2p\sum_{k=1}^{p}\beta_j\gamma_k + 2p\sum_{i=1}^{p}\beta_j\, e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)}\end{pmatrix}}{p}$$

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p^2\mu\sum_{j=1}^{p}\beta_j \\ +2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k)} + 2p\sum_{k=1}^{p}\sum_{j=1}^{p}\beta_j\gamma_k + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\, e_{ij(k)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k)}\end{matrix}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{matrix} p^3\mu^2 + p^2\sum_{j=1}^{p} E(\beta_j^2) + p\sum_{k=1}^{p} E(\gamma_k^2) + p\sum\sum_{k\neq l} E(\gamma_k\gamma_l) + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}^2) \\ +\sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k)}e_{gh(l)}) + 2p^2\mu\sum_{j=1}^{p} E(\beta_j) + 2p^2\mu\sum_{k=1}^{p} E(\gamma_k) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k)}) \\ +2p\sum_{k=1}^{p}\sum_{j=1}^{p} E(\beta_j\gamma_k) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p} E(\beta_j\, e_{ij(k)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\gamma_k e_{ij(k)})\end{matrix}}{p}$$

$$= \frac{p^3\mu^2 + p^3\sigma_\beta^2 + p^2\sigma_\gamma^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^3\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p^2\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma^2$$

$$E(SSC) = E\left(\frac{\sum_{j=1}^{p} C_j^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p^2\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\beta^2 - p\sigma_\gamma^2 - \sigma^2$$

$$= p(p-1)\sigma_\beta^2 + (p-1)\sigma^2$$

$$SST = \frac{\sum_{k=1}^{p} T_k^2}{p} - C.F$$

$$T_k = \sum_{i=1}^{p} Y_{ij(k)} = \sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + e_{ij(k)}\right)$$

$$= p\mu + \sum_{i=1}^{p}\tau_i + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + 0 + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$= p\mu + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}$$

$$\frac{\sum_{k=1}^{p} T_k^2}{p} = \frac{\sum_{k=1}^{p}\left(p\mu + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{i=1}^{p} e_{ij(k)}\right)^2}{p}$$

$$= \frac{\sum_{k=1}^{p}\begin{pmatrix} p^2\mu^2 + \sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + p^2\gamma_k^2 + \sum_{i=1}^{p} e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k)}e_{gh(l)} + 2p\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\gamma_k \\ +2p\mu\sum_{i=1}^{p} e_{ij(k)} + 2p\sum_{j=1}^{p}\beta_j\gamma_k + 2\sum_{j=1}^{p}\beta_j e_{ij(k)} + 2p\sum_{i=1}^{p}\gamma_k e_{ij(k)}\end{pmatrix}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{k=1}^{p}e_{ij(k)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k)}e_{gh(l)} \\ +2p^2\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{i=1}^{p}\sum_{k=1}^{p}e_{ij(k)} + 2p\sum_{j=1}^{p}\sum_{k=1}^{p}\beta_j\gamma_k \\ +2\sum_{j=1}^{p}\sum_{k=1}^{p}\beta_j e_{ij(k)} + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p\sum_{j=1}^{p}E(\beta_j^2) + p\sum\sum_{i\neq j}E(\beta_i\beta_j) + p^2\sum_{k=1}^{p}E(\gamma_k^2) + \sum_{i=1}^{p}\sum_{k=1}^{p}E(e_{ij(k)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k)}e_{gh(l)}) \\ +2p^2\mu\sum_{j=1}^{p}E(\beta_j) + 2p^2\mu\sum_{k=1}^{p}E(\gamma_k) + 2p\mu\sum_{i=1}^{p}\sum_{k=1}^{p}E(e_{ij(k)}) + 2p\sum_{j=1}^{p}\sum_{k=1}^{p}E(\beta_j\gamma_k) \\ +2\sum_{j=1}^{p}\sum_{k=1}^{p}E(\beta_j e_{ij(k)}) + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}E(\gamma_k e_{ij(k)}) \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\beta^2 + 0 + p^3\sigma_\gamma^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\beta^2 + p^3\sigma_\gamma^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p\sigma_\beta^2 + p^2\sigma_\gamma^2 + p\sigma^2$$

$$E(SST) = E\left(\frac{\sum_{k=1}^{p}T_k^2}{p}\right) - E(C.F)$$

$$= p^2\mu^2 + p\sigma_\beta^2 + p^2\sigma_\gamma^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\beta^2 - p\sigma_\gamma^2 - \sigma^2$$

$$= p(p-1)\sigma_\gamma^2 + (p-1)\sigma^2$$

$$E(SSE) = E(TSS) - E(SSR) - E(SSC) - E(SST)$$

$$= p\sum_{i=1}^{p}\tau_i^2 + p(p-1)\sigma_\beta^2 + p(p-1)\sigma_\gamma^2 + (p^2-1)\sigma^2 - p\sum_{i=1}^{p}\tau_i^2 - (p-1)\sigma^2$$
$$- p(p-1)\sigma_\beta^2 - (p-1)\sigma^2 - p(p-1)\sigma_\gamma^2 - (p-1)\sigma^2$$

$$= (p^2 - 1 - p + 1 - p + 1 - p + 1)\sigma^2$$

$$= (p^2 - 3p + 2)\sigma^2 = (p-1)(p-2)\sigma^2$$

$$E(MSE) = \frac{E(SSE)}{(p-1)(p-2)} = \frac{(p-1)(p-2)\sigma^2}{(p-1)(p-2)} = \sigma^2$$

$$E(MSR) = \frac{E(SSR)}{p-1} = \frac{p\sum_{i=1}^{p}\tau_i^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{i=1}^{p}\tau_i^2$$

$$E(MSC) = \frac{E(SSC)}{p-1} = \frac{p(p-1)\sigma_\beta^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + p\sigma_\beta^2$$

$$E(MST) = \frac{E(SST)}{p-1} = \frac{p(p-1)\sigma_\gamma^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + p\sigma_\gamma^2$$

Estimation of missing values

Case I: One missing value in one treatment

Rows	**Columns**						
	1	**2**	...	***j***	...	***p***	**Total**
1	$Y_{11(A)}$	$Y_{21(B)}$	...	$Y_{1j(D)}$	...	$Y_{1p(Z)}$	R_1

2	$Y_{12(E)}$	$Y_{22(D)}$	…	$Y_{2j(M)}$	…	$Y_{2p(X)}$	R_2
⋮	⋮	⋮		⋮		⋮	⋮
$\boldsymbol{i}$	$Y_{i1(F)}$	$Y_{i2(G)}$	…	$Y_{ij(k)}$	…	$Y_{ip(O)}$	R_i
⋮	⋮	⋮	$Y_{ab(D)}$	⋮		⋮	$R'_a + Y_{ab(D)}$
$\boldsymbol{p}$	$Y_{p1(Z)}$	$Y_{p2(N)}$	…	$Y_{pj(L)}$	…	$Y_{pp(Y)}$	R_p
Total	C_1	C_2	$C'_b + Y_{ab(D)}$	C_j	…	C_p	$G' + Y_{ab(D)}$

Let $Y_{ab(D)}$ is missing and $G = \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}$

Effected Total:

$$R_a = R'_a + Y_{ab(D)}$$
$$C_b = C'_b + Y_{ab(D)}$$
$$T_D = T'_D + Y_{ab(D)}$$
$$G = G' + Y_{ab(D)}$$
$$C.F = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)}\right)^2}{p^2}$$
$$SSE = TSS - SSR - SSC - SST$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 + Y_{ab(D)}^2 - C.F - \frac{1}{p}\left[\sum_{i=1}^{p} R_i^2 + \left(R'_a + Y_{ab(D)}\right)^2\right] + C.F$$
$$- \frac{1}{p}\left[\sum_{j=1}^{p} C_j^2 + \left(C'_b + Y_{ab(D)}\right)^2\right] + C.F - \frac{1}{p}\left[\sum_{k=1}^{p} T_k^2 + \left(T'_D + Y_{ab(D)}\right)^2\right] + C.F$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 + Y_{ab(D)}^2 - \frac{1}{p}\left[\sum_{i=1}^{p} R_i^2 + \left(R'_a + Y_{ab(D)}\right)^2\right] - \frac{1}{p}\left[\sum_{j=1}^{p} C_j^2 + \left(C'_b + Y_{ab(D)}\right)^2\right]$$
$$- \frac{1}{p}\left[\sum_{k=1}^{p} T_k^2 + \left(T'_D + Y_{ab(D)}\right)^2\right] + 2C.F$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 + Y_{ab(D)}^2 - \frac{1}{p}\left[\sum_{i=1}^{p} R_i^2 + \left(R'_a + Y_{ab(D)}\right)^2\right] - \frac{1}{p}\left[\sum_{j=1}^{p} C_j^2 + \left(C'_b + Y_{ab(D)}\right)^2\right]$$
$$- \frac{1}{p}\left[\sum_{k=1}^{p} T_k^2 + \left(T'_D + Y_{ab(D)}\right)^2\right] + \frac{2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)}\right)^2}{p^2}$$

$$\frac{\partial SSE}{\partial Y_{ab(D)}} = 0$$

$$0 + 2Y_{ab(D)} - \frac{2\left(R'_a + Y_{ab(D)}\right)}{p} - 0 - \frac{2\left(C'_b + Y_{ab(D)}\right)}{p} - 0 - \frac{2\left(T'_D + Y_{ab(D)}\right)}{p} + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)}\right)}{p^2} = 0$$

$$2Y_{ab(D)} - \frac{2\left(R'_a + Y_{ab(D)}\right)}{p} - \frac{2\left(C'_b + Y_{ab(D)}\right)}{p} - \frac{2\left(T'_D + Y_{ab(D)}\right)}{p} + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)}\right)}{p^2} = 0$$

$$\frac{2p^2 Y_{ab(D)} - 2p\left(R'_a + Y_{ab(D)}\right) - 2p\left(C'_b + Y_{ab(D)}\right) - 2p\left(T'_D + Y_{ab(D)}\right) + 4\left(G' + Y_{ab(D)}\right)}{p^2} = 0$$

$$\frac{2\left(p^2Y_{ab(D)} - p\left(R'_a + Y_{ab(D)}\right) - p\left(C'_b + Y_{ab(D)}\right) - p\left(T'_D + Y_{ab(D)}\right) + 2\left(G' + Y_{ab(D)}\right)\right)}{p^2} = 0$$

$$p^2Y_{ab(D)} - pR'_a - pY_{ab(D)} - pC'_b - pY_{ab(D)} - pT'_D - pY_{ab(D)} + 2G' + 2Y_{ab(D)} = 0$$

$$(p^2 - 3p + 2)Y_{ab(D)} - p(R'_a + C'_b + T'_D) + 2G' = 0$$

$$(p-1)(p-2)Y_{ab(D)} = p(R'_a + C'_b + T'_D) - 2G'$$

$$Y_{ab(D)} = \frac{p(R'_a + C'_b + T'_D) - 2G'}{(p-1)(p-2)}$$

Case II: Two missing Observations

(a) Missing in different treatment but same row and different columns

Rows	Columns						
	1	**2**	…	***j***	…	***p***	**Total**
1	$Y_{11(A)}$	$Y_{21(B)}$	…	$Y_{1j(D)}$	…	$Y_{1p(Z)}$	R_1
2	$Y_{12(E)}$	$Y_{22(D)}$	…	$Y_{2j(M)}$	…	$Y_{2p(X)}$	R_2
⋮	⋮	⋮	$Y_{ab(D)}$	⋮	$Y_{ac(E)}$	⋮	$R'_a + Y_{ab(D)} + Y_{ac(E)}$
i	$Y_{i1(F)}$	$Y_{i2(G)}$	…	$Y_{ij(k)}$	…	$Y_{ip(O)}$	R_i
⋮	⋮	⋮		⋮		⋮	⋮
p	$Y_{p1(Z)}$	$Y_{p2(N)}$	…	$Y_{pj(L)}$	…	$Y_{pp(Y)}$	R_p
Total	C_1	C_2	$C'_b + Y_{ab(D)}$	C_j	$C'_c + Y_{ac(E)}$	C_p	$G' + Y_{ab(D)} + Y_{ac(E)}$

Effected Total:

$$R_a = R'_a + Y_{ab(D)} + Y_{ac(E)}$$

$$C_b = C'_b + Y_{ab(D)}$$

$$C_c = C'_c + Y_{ac(E)}$$

$$T_D = T'_D + Y_{ab(D)}$$

$$T_E = T'_E + Y_{ac(E)}$$

$$G = G' + Y_{ab(D)} + Y_{ac(E)}$$

$$C.F = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ac(E)}\right)^2}{p^2}$$

$$SSE = TSS - SSR - SSC - SST$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 + Y_{ab(D)}^2 + Y_{ac(E)}^2 - C.F - \frac{1}{p}\left[\sum_{i=1}^{p} R_i^2 + \left(R'_a + Y_{ab(D)} + Y_{ac(E)}\right)^2\right] + C.F$$

$$-\frac{1}{p}\left[\sum_{j=1}^{p} C_j^2 + \left(C'_b + Y_{ab(D)}\right)^2 + \left(C'_c + Y_{ac(E)}\right)^2\right] + C.F$$

$$-\frac{1}{p}\left[\sum_{k=1}^{p} T_k^2 + \left(T'_D + Y_{ab(D)}\right)^2 + \left(T'_E + Y_{ab(E)}\right)^2\right] + C.F$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 + Y_{ab(D)}^2 + Y_{ac(E)}^2 - \frac{1}{p}\left[\sum_{i=1}^{p} R_i^2 + \left(R'_a + Y_{ab(D)} + Y_{ac(E)}\right)^2\right] - \frac{1}{p}\left[\sum_{j=1}^{p} C_j^2 + \left(C'_b + Y_{ab(D)}\right)^2 + \left(C'_c + Y_{ac(E)}\right)^2\right]$$

$$-\frac{1}{p}\left[\sum_{k=1}^{p} T_k^2 + \left(T'_D + Y_{ab(D)}\right)^2 + \left(T'_E + Y_{ab(E)}\right)^2\right] + \frac{2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ac(E)}\right)^2}{p^2}$$

$$\frac{\partial SSE}{\partial Y_{ab(D)}} = 0$$

$$0 + 2Y_{ab(D)} + 0 - 0 - \frac{2\left(R'_a + Y_{ab(D)} + Y_{ac(E)}\right)}{p} - 0 - \frac{2\left(C'_b + Y_{ab(D)}\right)}{p} - 0 - 0$$

$$-\frac{2\left(T'_D + Y_{ab(D)}\right)}{p} - 0 + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ac(E)}\right)}{p^2} = 0$$

$$2Y_{ab(D)} - \frac{2\left(R'_a + Y_{ab(D)} + Y_{ac(E)}\right)}{p} - \frac{2\left(C'_b + Y_{ab(D)}\right)}{p} - \frac{2\left(T'_D + Y_{ab(D)}\right)}{p} + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ac(E)}\right)}{p^2} = 0$$

$$\frac{2p^2Y_{ab(D)} - 2p\left(R'_a + Y_{ab(D)} + Y_{ac(E)}\right) - 2p\left(C'_b + Y_{ab(D)}\right) - 2p\left(T'_D + Y_{ab(D)}\right) + 4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ac(E)}\right)}{p^2} = 0$$

$$\frac{2\left(p^2Y_{ab(D)} - p\left(R'_a + Y_{ab(D)} + Y_{ac(E)}\right) - p\left(C'_b + Y_{ab(D)}\right) - p\left(T'_D + Y_{ab(D)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ac(E)}\right)\right)}{p^2} = 0$$

$$p^2Y_{ab(D)} - p\left(R'_a + Y_{ab(D)} + Y_{ac(E)}\right) - p\left(C'_b + Y_{ab(D)}\right) - p\left(T'_D + Y_{ab(D)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ac(E)}\right) = 0$$

$$p^2Y_{ab(D)} - pR'_a - pY_{ab(D)} - pY_{ac(E)} - pC'_b - pY_{ab(D)} - pT'_D - pY_{ab(D)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + 2Y_{ab(D)} + 2Y_{ac(E)} = 0$$

$$(p^2 - 3p + 2)Y_{ab(D)} - p(R'_a + C'_b + T'_D) + (2 - p)Y_{ac(E)} + 2G' = 0$$

$$(p^2 - 3p + 2)Y_{ab(D)} + (2 - p)Y_{ac(E)} = p(R'_a + C'_b + T'_D) - 2G'$$

Let $F = p^2 - 3p + 2,\quad E = 2 - p,\quad Z_{ab(D)} = p(R'_a + C'_b + T'_D) - 2G'$

$$FY_{ab(D)} + EY_{ac(E)} = Z_{ab(D)} \qquad (1)$$

$$\frac{\partial SSE}{\partial Y_{ac(E)}} = 0$$

$$0 + 2Y_{ac(E)} + 0 - 0 - \frac{2\left(R'_a + Y_{ab(D)} + Y_{ac(E)}\right)}{p} - 0 - \frac{2\left(C'_c + Y_{ac(E)}\right)}{p} - 0 - 0$$

$$-\frac{2\left(T'_E + Y_{ac(E)}\right)}{p} - 0 + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ac(E)}\right)}{p^2} = 0$$

$$2Y_{ac(E)} - \frac{2\left(R'_a + Y_{ab(D)} + Y_{ac(E)}\right)}{p} - \frac{2\left(C'_c + Y_{ac(E)}\right)}{p} - \frac{2\left(T'_E + Y_{ac(E)}\right)}{p} + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ac(E)}\right)}{p^2} = 0$$

$$\frac{2p^2Y_{ac(E)} - 2p\left(R'_a + Y_{ab(D)} + Y_{ac(E)}\right) - 2p\left(C'_c + Y_{ac(E)}\right) - 2p\left(T'_E + Y_{ac(E)}\right) + 4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ac(E)}\right)}{p^2} = 0$$

$$\frac{2\left(p^2Y_{ac(E)} - p\left(R'_a + Y_{ab(D)} + Y_{ac(E)}\right) - p\left(C'_c + Y_{ac(E)}\right) - p\left(T'_E + Y_{ac(E)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ac(E)}\right)\right)}{p^2} = 0$$

$$p^2Y_{ac(E)} - p\left(R'_a + Y_{ab(D)} + Y_{ac(E)}\right) - p\left(C'_c + Y_{ac(E)}\right) - p\left(T'_E + Y_{ac(E)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ac(E)}\right) = 0$$

$$p^2Y_{ac(E)} - pR'_a - pY_{ab(D)} - pY_{ac(E)} - pC'_c - pY_{ac(E)} - pT'_E - pY_{ac(E)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + 2Y_{ab(D)} + 2Y_{ac(E)} = 0$$

$$(p^2 - 3p + 2)Y_{ac(E)} - p(R'_a + C'_c + T'_E) + (2 - p)Y_{ab(D)} + 2G' = 0$$

$$(p^2 - 3p + 2)Y_{ac(E)} + (2 - p)Y_{ab(D)} = p(R'_a + C'_c + T'_E) - 2G'$$

Let $F = p^2 - 3p + 2,\quad E = 2 - p,\quad Z_{ac(E)} = p(R'_a + C'_c + T'_E) - 2G'$

$$FY_{ac(E)} + EY_{ab(D)} = Z_{ac(E)} \qquad (2)$$

Multiply (1) by E and (2) by F and then subtract it

$$\begin{array}{r} EFY_{ab(D)} + E^2Y_{ac(E)} = EZ_{ab(D)} \\ \pm EFY_{ab(D)} \pm F^2Y_{ac(E)} = \pm FZ_{ac(E)} \\ \hline (E^2 - F^2)Y_{ac(E)} = EZ_{ab(D)} - FZ_{ac(E)} \end{array}$$

$$Y_{ac(E)} = \frac{EZ_{ab(D)} - FZ_{ac(E)}}{E^2 - F^2}$$

Multiply (1) by F and (2) by E and then subtract it

$$\begin{array}{r} EFY_{ac(E)} + F^2Y_{ab(D)} = FZ_{ab(D)} \\ \pm EFY_{ac(E)} \pm E^2Y_{ab(D)} = \pm EZ_{ac(E)} \\ \hline (F^2 - E^2)Y_{ab(D)} = FZ_{ab(D)} - EZ_{ac(E)} \end{array}$$

$$Y_{ab(D)} = \frac{FZ_{ab(D)} - EZ_{ac(E)}}{F^2 - E^2}$$

(b) Missing in different treatment but same column and different rows

Rows	Columns						
	1	**2**	…	j	…	p	**Total**
1	$Y_{11(A)}$	$Y_{21(B)}$	…	$Y_{1j(D)}$	…	$Y_{1p(Z)}$	R_1
2	$Y_{12(E)}$	$Y_{22(D)}$	…	$Y_{2j(M)}$	…	$Y_{2p(X)}$	R_2
⋮	⋮	⋮	$Y_{ab(D)}$	⋮		⋮	$R_a' + Y_{ab(D)}$
i	$Y_{i1(F)}$	$Y_{i2(G)}$	…	$Y_{ij(k)}$	…	$Y_{ip(O)}$	R_i
⋮	⋮	⋮	$Y_{cb(E)}$	⋮		⋮	$R_c' + Y_{cb(E)}$
p	$Y_{p1(Z)}$	$Y_{p2(N)}$	…	$Y_{pj(L)}$	…	$Y_{pp(Y)}$	R_p
Total	C_1	C_2	$C_b' + Y_{ab(D)} + Y_{cb(E)}$	C_j	…	C_p	$G' + Y_{ab(D)} + Y_{cb(E)}$

Effected Total:

$$R_a = R_a' + Y_{ab(D)}$$

$$R_c = R_c' + Y_{cb(E)}$$

$$C_b = C_b' + Y_{ab(D)} + Y_{cb(E)}$$

$$T_D = T_D' + Y_{ab(D)}$$

$$T_E = T_E' + Y_{cb(E)}$$

$$G = G' + Y_{ab(D)} + Y_{cb(E)}$$

$$C.F = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}' + Y_{ab(D)} + Y_{cb(E)}\right)^2}{p^2}$$

$$SSE = TSS - SSR - SSC - SST$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 + Y_{ab(D)}^2 + Y_{cb(E)}^2 - C.F - \frac{1}{p}\left[\sum_{i=1}^{p} R_i^2 + \left(R_a' + Y_{ab(D)}\right)^2 + \left(R_c' + Y_{cb(E)}\right)^2\right]$$

$$+ C.F - \frac{1}{p}\left[\sum_{j=1}^{p} C_j^2 + \left(C_b' + Y_{ab(D)} + Y_{cb(E)}\right)^2\right] + C.F$$

$$- \frac{1}{p}\left[\sum_{k=1}^{p} T_k^2 + \left(T_D' + Y_{ab(D)}\right)^2 + \left(T_E' + Y_{cb(E)}\right)^2\right] + C.F$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 + Y_{ab(D)}^2 + Y_{cb(E)}^2 - \frac{1}{p}\left[\sum_{i=1}^{p} R_i^2 + \left(R'_a + Y_{ab(D)}\right)^2 + \left(R'_c + Y_{cb(E)}\right)^2\right] - \frac{1}{p}\left[\sum_{j=1}^{p} C_j^2 + \left(C'_b + Y_{ab(D)} + Y_{cb(E)}\right)^2\right] - \frac{1}{p}\left[\sum_{k=1}^{p} T_k^2 + \left(T'_D + Y_{ab(D)}\right)^2 + \left(T'_E + Y_{cb(E)}\right)^2\right] + \frac{2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{cb(E)}\right)^2}{p^2}$$

$$\frac{\partial SSE}{\partial Y_{ab(D)}} = 0$$

$$0 + 2Y_{ab(D)} + 0 - 0 - \frac{2\left(R'_a + Y_{ab(D)}\right)}{p} - 0 - \frac{2\left(C'_b + Y_{ab(D)} + Y_{cb(E)}\right)}{p} - 0 - 0 - \frac{2\left(T'_D + Y_{ab(D)}\right)}{p} - 0 + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{cb(E)}\right)}{p^2} = 0$$

$$2Y_{ab(D)} - \frac{2\left(R'_a + Y_{ab(D)}\right)}{p} - \frac{2\left(C'_b + Y_{ab(D)} + Y_{cb(E)}\right)}{p} - \frac{2\left(T'_D + Y_{ab(D)}\right)}{p} + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{cb(E)}\right)}{p^2} = 0$$

$$\frac{2p^2Y_{ab(D)} - 2p\left(R'_a + Y_{ab(D)}\right) - 2p\left(C'_b + Y_{ab(D)} + Y_{cb(E)}\right) - 2p\left(T'_D + Y_{ab(D)}\right) + 4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{cb(E)}\right)}{p^2} = 0$$

$$\frac{2\left(p^2Y_{ab(D)} - p\left(R'_a + Y_{ab(D)}\right) - p\left(C'_b + Y_{ab(D)} + Y_{cb(E)}\right) - p\left(T'_D + Y_{ab(D)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{cb(E)}\right)\right)}{p^2} = 0$$

$$p^2Y_{ab(D)} - p\left(R'_a + Y_{ab(D)}\right) - p\left(C'_b + Y_{ab(D)} + Y_{cb(E)}\right) - p\left(T'_D + Y_{ab(D)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{cb(E)}\right) = 0$$

$$p^2Y_{ab(D)} - pR'_a - pY_{ab(D)} - pC'_b - pY_{ab(D)} - pY_{cb(E)} - pT'_D - pY_{ab(D)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + 2Y_{ab(D)} + 2Y_{cb(E)} = 0$$

$$(p^2 - 3p + 2)Y_{ab(D)} - p(R'_a + C'_b + T'_D) + (2 - p)Y_{cb(E)} + 2G' = 0$$

$$(p^2 - 3p + 2)Y_{ab(D)} + (2 - p)Y_{cb(E)} = p(R'_a + C'_b + T'_D) - 2G'$$

Let $N = p^2 - 3p + 2$, $M = 2 - p$, $Z_{ab(D)} = p(R'_a + C'_b + T'_D) - 2G'$

$$NY_{ab(D)} + MY_{cb(E)} = Z_{ab(D)} \qquad (1)$$

$$\frac{\partial SSE}{\partial Y_{cb(E)}} = 0$$

$$0 + 2Y_{cb(E)} + 0 - 0 - \frac{2\left(R'_c + Y_{cb(E)}\right)}{p} - 0 - \frac{2\left(C'_b + Y_{ab(D)} + Y_{cb(E)}\right)}{p} - 0 - 0 - \frac{2\left(T'_E + Y_{cb(E)}\right)}{p} - 0 + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{cb(E)}\right)}{p^2} = 0$$

$$2Y_{cb(E)} - \frac{2\left(R'_c + Y_{cb(E)}\right)}{p} - \frac{2\left(C'_b + Y_{ab(D)} + Y_{cb(E)}\right)}{p} - \frac{2\left(T'_E + Y_{cb(E)}\right)}{p} + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{cb(E)}\right)}{p^2} = 0$$

$$\frac{2p^2Y_{cb(E)} - 2p\left(R'_c + Y_{cb(E)}\right) - 2p\left(C'_b + Y_{ab(D)} + Y_{cb(E)}\right) - 2p\left(T'_E + Y_{cb(E)}\right) + 4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{cb(E)}\right)}{p^2} = 0$$

$$\frac{2\left(p^2Y_{cb(E)} - p\left(R'_c + Y_{cb(E)}\right) - p\left(C'_b + Y_{ab(D)} + Y_{cb(E)}\right) - p\left(T'_E + Y_{cb(E)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{cb(E)}\right)\right)}{p^2} = 0$$

$$p^2Y_{cb(E)} - p\left(R'_c + Y_{cb(E)}\right) - p\left(C'_b + Y_{ab(D)} + Y_{cb(E)}\right) - p\left(T'_E + Y_{cb(E)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{cb(E)}\right) = 0$$

$$p^2Y_{cb(E)} - pR'_c - pY_{cb(E)} - pC'_b - pY_{ab(D)} - pY_{cb(E)} - pT'_E - pY_{cb(E)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + 2Y_{ab(D)} + 2Y_{cb(E)} = 0$$

$$(p^2 - 3p + 2)Y_{cb(E)} - p(R'_c + C'_b + T'_E) + (2 - p)Y_{ab(D)} + 2G' = 0$$

$$(p^2 - 3p + 2)Y_{cb(E)} + (2 - p)Y_{ab(D)} = p(R'_c + C'_b + T'_E) - 2G'$$

Let $N = p^2 - 3p + 2$, $M = 2 - p$, $Z_{cb(E)} = p(R'_c + C'_b + T'_E) - 2G'$

$$NY_{cb(E)} + MY_{ab(D)} = Z_{cb(E)} \qquad (2)$$

Multiply (1) by M and (2) by N and then subtract it

$$MNY_{ab(D)} + M^2Y_{cb(E)} = MZ_{ab(D)}$$
$$\pm MNY_{ab(D)} \pm N^2Y_{cb(E)} = \pm NZ_{cb(E)}$$
$$(M^2 - N^2)Y_{cb(E)} = MZ_{ab(D)} - NZ_{cb(E)}$$

$$Y_{cb(E)} = \frac{MZ_{ab(D)} - NZ_{cb(E)}}{M^2 - N^2}$$

Multiply (1) by N and (2) by M and then subtract it

$$MNY_{cb(E)} + N^2Y_{ab(D)} = NZ_{ab(D)}$$
$$\pm MNY_{cb(E)} \pm M^2Y_{ab(D)} = \pm MZ_{cb(E)}$$
$$(N^2 - M^2)Y_{ab(D)} = NZ_{ab(D)} - MZ_{cb(E)}$$

$$Y_{ab(D)} = \frac{NZ_{ab(D)} - MZ_{cb(E)}}{N^2 - M^2}$$

(c) Missing in different rows and different columns but same treatments

Rows	**Columns**						
	1	**2**	…	$\boldsymbol{j}$	…	$\boldsymbol{p}$	**Total**
1	$Y_{11(A)}$	$Y_{21(B)}$	…	$Y_{1j(D)}$	…	$Y_{1p(Z)}$	R_1
2	$Y_{12(E)}$	$Y_{22(D)}$	…	$Y_{2j(M)}$	…	$Y_{2p(X)}$	R_2
⋮	⋮	⋮	$Y_{ab(D)}$	⋮		⋮	$R_a' + Y_{ab(D)}$
$\boldsymbol{i}$	$Y_{i1(F)}$	$Y_{i2(G)}$	…	$Y_{ij(k)}$	…	$Y_{ip(O)}$	R_i
⋮	⋮	⋮		⋮	$Y_{ce(D)}$	⋮	$R_c' + Y_{ce(D)}$
$\boldsymbol{p}$	$Y_{p1(Z)}$	$Y_{p2(N)}$	…	$Y_{pj(L)}$	…	$Y_{pp(Y)}$	R_p
Total	C_1	C_2	$C_b' + Y_{ab(D)}$	C_j	$C_e' + Y_{ce(D)}$	C_p	$G' + Y_{ab(D)} + Y_{ce(D)}$

Effected Total:

$$R_a = R_a' + Y_{ab(D)}$$
$$R_c = R_c' + Y_{ce(D)}$$
$$C_b = C_b' + Y_{ab(D)}$$
$$C_e = C_e' + Y_{ce(D)}$$
$$T_D = T_D' + Y_{ab(D)} + Y_{ce(D)}$$
$$G = G' + Y_{ab(D)} + Y_{ce(D)}$$

$$C.F = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}' + Y_{ab(D)} + Y_{ce(D)}\right)^2}{p^2}$$

$$SSE = TSS - SSR - SSC - SST$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 + Y_{ab(D)}^2 + Y_{ce(D)}^2 - C.F - \frac{1}{p}\left[\sum_{i=1}^{p} R_i^2 + \left(R_a' + Y_{ab(D)}\right)^2 + \left(R_c' + Y_{ce(D)}\right)^2\right]$$
$$+ C.F - \frac{1}{p}\left[\sum_{j=1}^{p} C_j^2 + \left(C_b' + Y_{ab(D)}\right)^2 + \left(C_e' + Y_{ce(D)}\right)^2\right] + C.F$$
$$- \frac{1}{p}\left[\sum_{k=1}^{p} T_k^2 + \left(T_D' + Y_{ab(D)} + Y_{ce(D)}\right)^2\right] + C.F$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 + Y_{ab(D)}^2 + Y_{ce(D)}^2 - \frac{1}{p}\left[\sum_{i=1}^{p} R_i^2 + \left(R'_a + Y_{ab(D)}\right)^2 + \left(R'_c + Y_{ce(D)}\right)^2\right]$$
$$- \frac{1}{p}\left[\sum_{j=1}^{p} C_j^2 + \left(C'_b + Y_{ab(D)}\right)^2 + \left(C'_e + Y_{ce(D)}\right)^2\right] - \frac{1}{p}\left[\sum_{k=1}^{p} T_k^2 + \left(T'_D + Y_{ab(D)} + Y_{ce(D)}\right)^2\right]$$
$$+ \frac{2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(D)}\right)^2}{p^2}$$

$$\frac{\partial SSE}{\partial Y_{ab(D)}} = 0$$

$$0 + 2Y_{ab(D)} + 0 - 0 - \frac{2\left(R'_a + Y_{ab(D)}\right)}{p} - 0 - 0 - \frac{2\left(C'_b + Y_{ab(D)}\right)}{p} - 0 - 0$$
$$- \frac{2\left(T'_D + Y_{ab(D)} + Y_{ce(D)}\right)}{p} + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(D)}\right)}{p^2} = 0$$

$$2Y_{ab(D)} - \frac{2\left(R'_a + Y_{ab(D)}\right)}{p} - \frac{2\left(C'_b + Y_{ab(D)}\right)}{p} - \frac{2\left(T'_D + Y_{ab(D)} + Y_{ce(D)}\right)}{p} + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(D)}\right)}{p^2} = 0$$

$$\frac{2p^2Y_{ab(D)} - 2p\left(R'_a + Y_{ab(D)}\right) - 2p\left(C'_b + Y_{ab(D)}\right) - 2p\left(T'_D + Y_{ab(D)} + Y_{ce(D)}\right) + 4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(D)}\right)}{p^2} = 0$$

$$\frac{2\left(p^2Y_{ab(D)} - p\left(R'_a + Y_{ab(D)}\right) - p\left(C'_b + Y_{ab(D)}\right) - p\left(T'_D + Y_{ab(D)} + Y_{ce(D)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(D)}\right)\right)}{p^2} = 0$$

$$p^2Y_{ab(D)} - p\left(R'_a + Y_{ab(D)}\right) - p\left(C'_b + Y_{ab(D)}\right) - p\left(T'_D + Y_{ab(D)} + Y_{ce(D)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(D)}\right) = 0$$

$$p^2Y_{ab(D)} - pR'_a - pY_{ab(D)} - pC'_b - pY_{ab(D)} - pT'_D - pY_{ab(D)} - pY_{ce(D)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + 2Y_{ab(D)} + 2Y_{ce(D)} = 0$$

$$(p^2 - 3p + 2)Y_{ab(D)} - p(R'_a + C'_b + T'_D) + (2 - p)Y_{ce(D)} + 2G' = 0$$
$$(p^2 - 3p + 2)Y_{ab(D)} + (2 - p)Y_{ce(D)} = p(R'_a + C'_b + T'_D) - 2G'$$

Let $N = p^2 - 3p + 2$, $M = 2 - p$, $Z_{ab(D)} = p(R'_a + C'_b + T'_D) - 2G'$

$$NY_{ab(D)} + MY_{ce(D)} = Z_{ab(D)} \qquad (1)$$

$$\frac{\partial SSE}{\partial Y_{ce(D)}} = 0$$

$$0 + 0 + 2Y_{ce(D)} - 0 - 0 - \frac{2\left(R'_c + Y_{ce(D)}\right)}{p} - 0 - 0 - \frac{2\left(C'_e + Y_{ce(D)}\right)}{p} - 0$$
$$- \frac{2\left(T'_D + Y_{ab(D)} + Y_{ce(D)}\right)}{p} + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(D)}\right)}{p^2} = 0$$

$$2Y_{ce(D)} - \frac{2\left(R'_c + Y_{ce(D)}\right)}{p} - \frac{2\left(C'_e + Y_{ce(D)}\right)}{p} - \frac{2\left(T'_D + Y_{ab(D)} + Y_{ce(D)}\right)}{p} + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(D)}\right)}{p^2} = 0$$

$$\frac{2p^2Y_{ce(D)} - 2p\left(R'_c + Y_{ce(D)}\right) - 2p\left(C'_e + Y_{ce(D)}\right) - 2p\left(T'_D + Y_{ab(D)} + Y_{ce(D)}\right) + 4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(D)}\right)}{p^2} = 0$$

$$\frac{2\left(p^2Y_{ce(D)} - p\left(R'_c + Y_{ce(D)}\right) - p\left(C'_e + Y_{ce(D)}\right) - p\left(T'_D + Y_{ab(D)} + Y_{ce(D)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(D)}\right)\right)}{p^2} = 0$$

$$p^2Y_{ce(D)} - p\left(R'_c + Y_{ce(D)}\right) - p\left(C'_e + Y_{ce(D)}\right) - p\left(T'_D + Y_{ab(D)} + Y_{ce(D)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(D)}\right) = 0$$

$$p^2Y_{ce(D)} - pR'_c - pY_{ce(D)} - pC'_e - pY_{ce(D)} - pT'_D - pY_{ab(D)} - pY_{ce(D)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + 2Y_{ab(D)} + 2Y_{ce(D)} = 0$$

$$(p^2 - 3p + 2)Y_{ce(D)} - p(R'_c + C'_e + T'_D) + (2 - p)Y_{ab(D)} + 2G' = 0$$

$$(p^2-3p+2)Y_{ce(D)}+(2-p)Y_{ab(D)}=p(R_c'+C_e'+T_D')-2G'$$

Let $N=p^2-3p+2,\quad M=2-p,\ \ Z_{ce(D)}=p(R_c'+C_e'+T_D')-2G'$

$$NY_{ce(D)}+MY_{ab(D)}=Z_{ce(D)} \qquad (2)$$

Multiply (1) by M and (2) by N and then subtract it

$$MNY_{ab(D)}+M^2Y_{ce(D)}=\ MZ_{ab(D)}$$
$$\pm MNY_{ab(D)}\pm N^2Y_{ce(D)}=\pm NZ_{ce(D)}$$
$$(M^2-N^2)Y_{ce(D)}=MZ_{ab(D)}-NZ_{ce(D)}$$

$$Y_{ce(D)}=\frac{MZ_{ab(D)}-NZ_{ce(D)}}{M^2-N^2}$$

Multiply (1) by N and (2) by M and then subtract it

$$MNY_{ce(D)}+N^2Y_{ab(D)}=\ NZ_{ab(D)}$$
$$\pm MNY_{ce(D)}\pm M^2Y_{ab(D)}=\pm MZ_{ce(D)}$$
$$(N^2-M^2)Y_{ab(D)}=NZ_{ab(D)}-MZ_{ce(D)}$$

$$Y_{ab(D)}=\frac{NZ_{ab(D)}-MZ_{ce(D)}}{N^2-M^2}$$

(d) Missing in different rows columns and treatments

Rows	**Columns**						
	1	**2**	…	***j***	…	***p***	**Total**
1	$Y_{11(A)}$	$Y_{21(B)}$	…	$Y_{1j(D)}$	…	$Y_{1p(Z)}$	R_1
2	$Y_{12(E)}$	$Y_{22(D)}$	…	$Y_{2j(M)}$	…	$Y_{2p(X)}$	R_2
⋮	⋮	⋮	$Y_{ab(D)}$	⋮		⋮	$R_a' + Y_{ab(D)}$
i	$Y_{i1(F)}$	$Y_{i2(G)}$	…	$Y_{ij(k)}$	…	$Y_{ip(O)}$	R_i
⋮	⋮	⋮		⋮	$Y_{ce(E)}$	⋮	$R_c' + Y_{ce(E)}$
p	$Y_{p1(Z)}$	$Y_{p2(N)}$	…	$Y_{pj(L)}$	…	$Y_{pp(Y)}$	R_p
Total	C_1	C_2	$C_b' + Y_{ab(D)}$	C_j	$C_e' + Y_{ce(E)}$	C_p	$G' + Y_{ab(D)} + Y_{ce(E)}$

Effected Total:

$$R_a=R_a'+Y_{ab(D)}$$
$$R_c=R_c'+Y_{ce(E)}$$
$$C_b=C_b'+Y_{ab(D)}$$
$$C_e=C_e'+Y_{ce(E)}$$
$$T_D=T_D'+Y_{ab(D)}$$
$$T_E=T_E'+Y_{ce(E)}$$
$$G=G'+Y_{ab(D)}+Y_{ce(E)}$$
$$C.F=\frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k)}'+Y_{ab(D)}+Y_{ce(E)}\right)^2}{p^2}$$
$$SSE=TSS-SSR-SSC-SST$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 + Y_{ab(D)}^2 + Y_{ce(E)}^2 - C.F - \frac{1}{p}\left[\sum_{i=1}^{p} R_i^2 + \left(R_a' + Y_{ab(D)}\right)^2 + \left(R_c' + Y_{ce(E)}\right)^2\right]$$
$$+ C.F - \frac{1}{p}\left[\sum_{j=1}^{p} C_j^2 + \left(C_b' + Y_{ab(D)}\right)^2 + \left(C_e' + Y_{ce(E)}\right)^2\right] + C.F$$
$$- \frac{1}{p}\left[\sum_{k=1}^{p} T_k^2 + \left(T_D' + Y_{ab(D)}\right)^2 + \left(T_E' + Y_{ce(E)}\right)^2\right] + C.F$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}^2 + Y_{ab(D)}^2 + Y_{ce(E)}^2 - \frac{1}{p}\left[\sum_{i=1}^{p} R_i^2 + \left(R_a' + Y_{ab(D)}\right)^2 + \left(R_c' + Y_{ce(E)}\right)^2\right]$$
$$- \frac{1}{p}\left[\sum_{j=1}^{p} C_j^2 + \left(C_b' + Y_{ab(D)}\right)^2 + \left(C_e' + Y_{ce(E)}\right)^2\right] - \frac{1}{p}\left[\sum_{k=1}^{p} T_k^2 + \left(T_D' + Y_{ab(D)}\right)^2 + \left(T_E' + Y_{ce(E)}\right)^2\right]$$
$$+ \frac{2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}' + Y_{ab(D)} + Y_{ce(E)}\right)^2}{p^2}$$

$$\frac{\partial SSE}{\partial Y_{ab(D)}} = 0$$

$$0 + 2Y_{ab(D)} + 0 - 0 - \frac{2\left(R_a' + Y_{ab(D)}\right)}{p} - 0 - 0 - \frac{2\left(C_b' + Y_{ab(D)}\right)}{p} - 0 - \frac{2\left(T_D' + Y_{ab(D)}\right)}{p} - 0$$
$$+ \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}' + Y_{ab(D)} + Y_{ce(E)}\right)}{p^2} = 0$$

$$2Y_{ab(D)} - \frac{2\left(R_a' + Y_{ab(D)}\right)}{p} - \frac{2\left(C_b' + Y_{ab(D)}\right)}{p} - \frac{2\left(T_D' + Y_{ab(D)}\right)}{p} + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}' + Y_{ab(D)} + Y_{ce(E)}\right)}{p^2} = 0$$

$$\frac{2p^2Y_{ab(D)} - 2p\left(R_a' + Y_{ab(D)}\right) - 2p\left(C_b' + Y_{ab(D)}\right) - 2p\left(T_D' + Y_{ab(D)}\right) + 4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}' + Y_{ab(D)} + Y_{ce(E)}\right)}{p^2} = 0$$

$$\frac{2\left(p^2Y_{ab(D)} - p\left(R_a' + Y_{ab(D)}\right) - p\left(C_b' + Y_{ab(D)}\right) - p\left(T_D' + Y_{ab(D)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}' + Y_{ab(D)} + Y_{ce(E)}\right)\right)}{p^2} = 0$$

$$p^2Y_{ab(D)} - p\left(R_a' + Y_{ab(D)}\right) - p\left(C_b' + Y_{ab(D)}\right) - p\left(T_D' + Y_{ab(D)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}' + Y_{ab(D)} + Y_{ce(E)}\right) = 0$$

$$p^2Y_{ab(D)} - pR_a' - pY_{ab(D)} - pC_b' - pY_{ab(D)} - pT_D' - pY_{ab(D)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}' + 2Y_{ab(D)} + 2Y_{ce(E)} = 0$$

$$(p^2 - 3p + 2)Y_{ab(D)} - p(R_a' + C_b' + T_D') + 2Y_{ce(E)} + 2G' = 0$$
$$(p^2 - 3p + 2)Y_{ab(D)} + 2Y_{ce(E)} = p(R_a' + C_b' + T_D') - 2G'$$

Let $F = p^2 - 3p + 2$, $\quad Z_{ab(D)} = p(R_a' + C_b' + T_D') - 2G'$

$$FY_{ab(D)} + 2Y_{ce(E)} = Z_{ab(D)} \qquad (1)$$

$$\frac{\partial SSE}{\partial Y_{ce(E)}} = 0$$

$$0 + 2Y_{ce(E)} + 0 - 0 - \frac{2\left(R_c' + Y_{ce(E)}\right)}{p} - 0 - 0 - \frac{2\left(C_e' + Y_{ce(E)}\right)}{p} - 0 - \frac{2\left(T_E' + Y_{ce(E)}\right)}{p} - 0$$
$$+ \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}' + Y_{ab(D)} + Y_{ce(E)}\right)}{p^2} = 0$$

$$2Y_{ce(E)} - \frac{2\left(R_c' + Y_{ce(E)}\right)}{p} - \frac{2\left(C_e' + Y_{ce(E)}\right)}{p} - \frac{2\left(T_E' + Y_{ce(E)}\right)}{p} + \frac{4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k)}' + Y_{ab(D)} + Y_{ce(E)}\right)}{p^2} = 0$$

$$\frac{2p^2Y_{ce(E)} - 2p\left(R'_c + Y_{ce(E)}\right) - 2p\left(C'_e + Y_{ce(E)}\right) - 2p\left(T'_E + Y_{ce(E)}\right) + 4\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(E)}\right)}{p^2} = 0$$

$$\frac{2\left(p^2Y_{ce(E)} - p\left(R'_c + Y_{ce(E)}\right) - p\left(C'_e + Y_{ce(E)}\right) - p\left(T'_E + Y_{ce(E)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(E)}\right)\right)}{p^2} = 0$$

$$p^2Y_{ce(E)} - p\left(R'_c + Y_{ce(E)}\right) - p\left(C'_e + Y_{ce(E)}\right) - p\left(T'_E + Y_{ce(E)}\right) + 2\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + Y_{ab(D)} + Y_{ce(E)}\right) = 0$$

$$p^2Y_{ce(E)} - pR'_c - pY_{ce(E)} - pC'_e - pY_{ce(E)} - pT'_E - pY_{ce(E)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p} Y'_{ij(k)} + 2Y_{ab(D)} + 2Y_{ce(E)} = 0$$

$$(p^2 - 3p + 2)Y_{ce(E)} - p(R'_c + C'_e + T'_E) + 2Y_{ab(D)} + 2G' = 0$$

$$(p^2 - 3p + 2)Y_{ce(E)} + 2Y_{ab(D)} = p(R'_c + C'_e + T'_E) - 2G'$$

Let $F = p^2 - 3p + 2$, $\quad Z_{ce(E)} = p(R'_c + C'_e + T'_E) - 2G'$

$$FY_{ce(E)} + 2Y_{ab(D)} = Z_{ce(E)} \qquad (2)$$

Multiply (1) by 2 and (2) by F and then subtract it

$$2FY_{ab(D)} + 4Y_{ce(E)} = 2Z_{ab(D)}$$

$$\pm 2FY_{ab(D)} \pm F^2Y_{ce(E)} = \pm FZ_{ce(E)}$$

$$(4 - F^2)Y_{ce(E)} = 2Z_{ab(D)} - FZ_{ce(E)}$$

$$Y_{ce(E)} = \frac{2Z_{ab(D)} - FZ_{ce(E)}}{4 - F^2}$$

Multiply (1) by F and (2) by 2 and then subtract it

$$2FY_{ce(E)} + F^2Y_{ab(D)} = FZ_{ab(D)}$$

$$\pm 2FY_{ce(E)} \pm 4Y_{ab(D)} = \pm 2Z_{ce(E)}$$

$$(F^2 - 4)Y_{ab(D)} = NZ_{ab(D)} - 2Z_{ce(E)}$$

$$Y_{ab(D)} = \frac{NZ_{ab(D)} - 2Z_{ce(E)}}{F^2 - 4}$$

Efficiency of LSD relative to CRD

RE (LS, CR): the relative efficiency of the Latin square design compared to a completely randomized design. Did accounting for row/column sources of variability increase the precision in estimating the treatment means?

$$RE(LS, CR) = \frac{MSE_{CR}}{MSE_{LS}} = \frac{MSR + MSC + (p-1)MSE}{(p+1)MSE}$$

Where p is equal to no. of treatments.

Efficiency of LSD relative to RCBD

RE (LS, RCB): the relative efficiency of the Latin square design compared to a Randomized complete block design.

RE using columns as Blocks in RCBD

$$RE(LS, RCB) = \frac{MSE_{RCB}}{MSE_{LSD}} = \frac{MSR + (p-1)MSE}{pMSE}$$

Where p is equal to no. of treatments.

RE using rows as Blocks in RCBD

$$RE(LS, RCB) = \frac{MSE_{RCB}}{MSE_{LSD}} = \frac{MSC + (p-1)MSE}{pMSE}$$

An estimated relative efficiency greater than 1 indicates that LSD is more efficient than RCBD whereas less than 1 indicates that RCBD is more efficient than LSD.

Greaco Latin Square Design

A Greaco-Latin square consists of two latin squares (one using the letters A, B, C, … the other using greek letters a, b, c, …) such that when the two latin square are supper imposed on each other the letters of one square appear once and only once with the letters of the other square. The two Latin squares are called mutually orthogonal.

Experimental Layout

a 7 x 7 Greaco-Latin Square

Aa	Be	Cb	Df	Ec	Fg	Gd
Bb	Cf	Dc	Eg	Fd	Ga	Ae
Cc	Dg	Ed	Fa	Ge	Ab	Bf
Dd	Ea	Fe	Gb	Af	Bc	Cg
Ee	Fb	Gf	Ac	Bg	Cd	Da
Ff	Gc	Ag	Bd	Ca	De	Eb
Gg	Ad	Ba	Ce	Db	Ef	Fc

Example:

A researcher is interested in determining the effect of two factors. The percentage of *Lysine* in the diet and the percentage of *Protein* in the diet have on *Milk Production* in cows.
For this reason it is decided to use a Greaco-Latin square design to experimentally determine the two effects of the two factors (*Lysine* and *Protein*).
Seven levels of each factor is selected

- 0.0(A), 0.1(B), 0.2(C), 0.3(D), 0.4(E), 0.5(F), and 0.6(G)% for *Lysine* and
- 2(a), 4(b), 6(c), 8(d), 10(e), 12(f) and 14(g)% for *Protein*.
- Seven animals (cows) are selected at random for the experiment which is to be carried out over seven three-month periods.

A Greaco-Latin Square is the used to assign the 7 X 7 combinations of levels of the two factors (*Lysine* and *Protein*) to a period and a cow. The data is tabulated on below:

		Period						
		1	2	3	4	5	6	7
Cows	1	304 (Aα)	436 (Bε)	350 (Cβ)	504 (Dϕ)	417 (Eχ)	519 (Fγ)	432 (Gδ)
	2	381 (Bβ)	505 (Cϕ)	425 (Dχ)	564 (Eγ)	494 (Fδ)	350 (Gα)	413 (Aε)
	3	432 (Cχ)	566 (Dγ)	479 (Eδ)	357 (Fα)	461 (Gε)	340 (Aβ)	502 (Bϕ)
	4	442 (Dδ)	372 (Eα)	536 (Fε)	366 (Gβ)	495 (Aϕ)	425 (Bχ)	507 (Cγ)
	5	496 (Eε)	449 (Fβ)	493 (Gϕ)	345 (Aχ)	509 (Bγ)	481 (Cδ)	380 (Dα)
	6	534 (Fϕ)	421 (Gχ)	452 (Aγ)	427 (Bδ)	346 (Cα)	478 (Dε)	397 (Eβ)
	7	543 (Gγ)	386 (Aδ)	435 (Bα)	485 (Cε)	406 (Dβ)	554 (Eϕ)	410 (Fχ)

Statistical Model and Analysis

The linear statistical model for Greaco LSD is

$$Y_{ij} = \mu + \tau_i + \beta_j + \gamma_k + \eta_m + e_{ij(k,m)} \qquad \begin{cases} i = 1,2,\dots,p \\ j = 1,2,\dots,p \\ k = 1,2,\dots,p \\ m = 1,2,\dots,p \end{cases}$$

Where

μ True mean effect

τ_i ith row effect

β_j jth column effect

γ_k kth Greek letter effect

η_m mth Latin letter effect

Formulation of Hypotheses

$$H_0: \tau_i = 0$$
$$H_0': \beta_j = 0$$
$$H_0'': \gamma_k = 0$$
$$H_0''': \eta_m = 0$$
$$H_1: \tau_i \neq 0$$
$$H_1': \beta_j \neq 0$$
$$H_1'': \gamma_k \neq 0$$
$$H_1''': \eta_m \neq 0$$

Level of significance

$$\alpha = 0.05, 0.01, 0.10, 0.001$$

Test Statistic

$$F_1 = \frac{s_r^2}{s_e^2}, \quad F_2 = \frac{s_C^2}{s_e^2}, \quad F_3 = \frac{s_G^2}{s_e^2}, \quad F_4 = \frac{s_L^2}{s_e^2}$$

S.O.V	df	SS	MS	F
Rows	$p-1$	SSR	$s_r^2=\frac{SSR}{p-1}$	$F_1=\frac{s_r^2}{s_e^2}$
Columns	$p-1$	SSC	$s_c^2=\frac{SSC}{p-1}$	$F_2=\frac{s_c^2}{s_e^2}$
Greek Letters	$p-1$	SSG	$s_G^2=\frac{SSG}{p-1}$	$F_3=\frac{s_G^2}{s_e^2}$
Latin Letters	$p-1$	SSL	$s_L^2=\frac{SSL}{p-1}$	$F_4=\frac{s_L^2}{s_e^2}$
Error	$(p-1)(p-3)$	SSE	$s_e^2=\frac{SSE}{(p-1)(p-3)}$	
Total	p^2-1	TSS		

$$C.F=\frac{Y_{....}}{p^2},\quad SSR=\frac{1}{p}\sum R_i^2-C.F,\quad SSC=\frac{1}{p}\sum C_j^2-C.F,\quad SSG=\frac{1}{p}\sum G_k^2-C.F$$

$$SSL=\frac{1}{p}\sum L_m^2-C.F,\quad SSE=TSS-SSR-SSC-SSG-SSL$$

C.R:

$F_1\geq F_{\alpha((p-1),(p-1)(p-3))}$

$F_2\geq F_{\alpha((p-1),(p-1)(p-3))}$

$F_3\geq F_{\alpha((p-1),(p-1)(p-3))}$

$F_4\geq F_{\alpha((p-1),(p-1)(p-3))}$

Conclusion:

If F_{cal} falls in critical region then we reject the null hypothesis.

Estimation of Model Parameters

The least square estimates of $\hat{\mu},\hat{\tau}_i,\hat{\beta}_j,\hat{\gamma}_k\ and\ \hat{\eta}_m$ are as:

$$S=\sum_{i=1}^{p}\sum_{j=1}^{p}\left(Y_{ij(k,m)}-\hat{\mu}-\hat{\tau}_i-\hat{\beta}_j-\hat{\gamma}_k-\hat{\eta}_m\right)^2$$

$$\frac{\partial S}{\partial\hat{\mu}}=2\sum_{i=1}^{p}\sum_{j=1}^{p}\left(Y_{ij(k,m)}-\hat{\mu}-\hat{\tau}_i-\hat{\beta}_j-\hat{\gamma}_k-\hat{\eta}_m\right)(-1)=0$$

$$-2\sum_{i=1}^{p}\sum_{j=1}^{p}\left(Y_{ij(k,m)}-\hat{\mu}-\hat{\tau}_i-\hat{\beta}_j-\hat{\gamma}_k-\hat{\eta}_m\right)=0$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k,m)}-p^2\hat{\mu}-p\sum_{i=1}^{p}\hat{\tau}_i-p\sum_{j=1}^{p}\hat{\beta}_j-p\sum_{k=1}^{p}\hat{\gamma}_k-p\sum_{m=1}^{p}\hat{\eta}_m=0$$

For unique solution Put $\sum_{i=1}^{p}\hat{\tau}_i=0,\sum_{j=1}^{p}\hat{\beta}_j=0,\sum_{k=1}^{p}\hat{\gamma}_k=0,\ \ \sum_{m=1}^{p}\hat{\eta}_m=0$

$$\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k,m)}-p^2\hat{\mu}=0$$

$$p^2\hat{\mu} = \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k,m)}$$

$$\hat{\mu} = \frac{\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k,m)}}{p^2} = \bar{\bar{Y}}$$

$$S = \sum_{j=1}^{p}\left(Y_{1j(k,m)} - \hat{\mu} - \hat{\tau}_1 - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_m\right)^2 + \sum_{j=1}^{p}\left(Y_{2j(k,m)} - \hat{\mu} - \hat{\tau}_2 - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_m\right)^2$$

$$+ \cdots + \sum_{j=1}^{p}\left(Y_{pj(k,m)} - \hat{\mu} - \hat{\tau}_p - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_m\right)^2$$

Differentiate w.r.t $\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_p$

$$\frac{\partial S}{\partial \hat{\tau}_1} = 2\sum_{j=1}^{p}\left(Y_{1j(k,m)} - \hat{\mu} - \hat{\tau}_1 - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_m\right)(-1) = 0$$

$$-2\sum_{j=1}^{p}\left(Y_{1j(k,m)} - \hat{\mu} - \hat{\tau}_1 - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_m\right) = 0$$

$$\sum_{j=1}^{p} Y_{1j(k,m)} - p\hat{\mu} - p\hat{\tau}_1 - \sum_{j=1}^{p}\hat{\beta}_j - \sum_{k=1}^{p}\hat{\gamma}_k - \sum_{m=1}^{p}\hat{\eta}_m = 0$$

For unique solution put $\sum_{j=1}^{p}\hat{\beta}_j = 0, \sum_{k=1}^{p}\hat{\gamma}_k = 0,\ \sum_{m=1}^{p}\hat{\eta}_m = 0$

$$\sum_{j=1}^{p} Y_{1j(k,m)} - p\hat{\mu} - p\hat{\tau}_1 = 0$$

$$p\hat{\tau}_1 = \sum_{j=1}^{p} Y_{1j(k,m)} - p\hat{\mu}$$

$$\hat{\tau}_1 = \frac{\sum_{j=1}^{p} Y_{1j(k,m)}}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\tau}_1 = \frac{R_1}{p} - \bar{\bar{Y}}$$

$$\frac{\partial S}{\partial \hat{\tau}_2} = 2\sum_{j=1}^{p}\left(Y_{2j(k,m)} - \hat{\mu} - \hat{\tau}_2 - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_m\right)(-1) = 0$$

$$-2\sum_{j=1}^{p}\left(Y_{2j(k,m)} - \hat{\mu} - \hat{\tau}_2 - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_m\right) = 0$$

$$\sum_{j=1}^{p} Y_{2j(k,m)} - p\hat{\mu} - p\hat{\tau}_1 - \sum_{j=1}^{p}\hat{\beta}_j - \sum_{k=1}^{p}\hat{\gamma}_k - \sum_{m=1}^{p}\hat{\eta}_m = 0$$

For unique solution put $\sum_{j=1}^{p}\hat{\beta}_j = 0, \sum_{k=1}^{p}\hat{\gamma}_k = 0,\ \sum_{m=1}^{p}\hat{\eta}_m = 0$

$$\sum_{j=1}^{p} Y_{2j(k,m)} - p\hat{\mu} - p\hat{\tau}_2 = 0$$

$$p\hat{\tau}_2 = \sum_{j=1}^{p} Y_{2j(k,m)} - p\hat{\mu}$$

$$\hat{\tau}_2 = \frac{\sum_{j=1}^{p} Y_{2j(k,m)}}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\tau}_2 = \frac{R_2}{p} - \bar{\bar{Y}}$$

$$\frac{\partial S}{\partial \hat{\tau}_p} = 2\sum_{j=1}^{p}\left(Y_{pj(k,m)} - \hat{\mu} - \hat{\tau}_p - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_m\right)(-1) = 0$$

$$-2\sum_{j=1}^{p}\left(Y_{pj(k,m)} - \hat{\mu} - \hat{\tau}_p - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_m\right) = 0$$

$$\sum_{j=1}^{p} Y_{pj(k,m)} - p\hat{\mu} - p\hat{\tau}_p - \sum_{j=1}^{p}\hat{\beta}_j - \sum_{k=1}^{p}\hat{\gamma}_k - \sum_{m=1}^{p}\hat{\eta}_m = 0$$

For unique solution put $\sum_{j=1}^{p}\hat{\beta}_j = 0, \sum_{k=1}^{p}\hat{\gamma}_k = 0,\ \sum_{m=1}^{p}\hat{\eta}_m = 0$

$$\sum_{j=1}^{p} Y_{pj(k,m)} - p\hat{\mu} - p\hat{\tau}_p = 0$$

$$p\hat{\tau}_p = \sum_{j=1}^{p} Y_{pj(k,m)} - p\hat{\mu}$$

$$\hat{\tau}_p = \frac{\sum_{j=1}^{p} Y_{pj(k,m)}}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\tau}_p = \frac{R_p}{p} - \bar{\bar{Y}}$$

$$S = \sum_{i=1}^{p}\left(Y_{i1(k,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_1 - \hat{\gamma}_k - \hat{\eta}_m\right)^2 + \sum_{i=1}^{p}\left(Y_{i2(k,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_2 - \hat{\gamma}_k - \hat{\eta}_m\right)^2 + \cdots$$

$$+ \sum_{i=1}^{p}\left(Y_{ip(k,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_p - \hat{\gamma}_k - \hat{\eta}_m\right)^2$$

Differentiate w.r.t $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_p$

$$\frac{\partial S}{\partial \hat{\beta}_1} = 2\sum_{i=1}^{p}\left(Y_{i1(k,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_1 - \hat{\gamma}_k - \hat{\eta}_m\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\left(Y_{i1(k,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_1 - \hat{\gamma}_k - \hat{\eta}_m\right) = 0$$

$$\sum_{i=1}^{p} Y_{i1(k,m)} - p\hat{\mu} - \sum_{i=1}^{p}\hat{\tau}_i - p\hat{\beta}_1 - \sum_{k=1}^{p}\hat{\gamma}_k - \sum_{m=1}^{p}\hat{\eta}_m = 0$$

For unique solution put $\sum_{i=1}^{p}\hat{\tau}_i = 0, \sum_{k=1}^{p}\hat{\gamma}_k = 0,\ \sum_{m=1}^{p}\hat{\eta}_m = 0$

$$\sum_{i=1}^{p} Y_{i1(k,m)} - p\hat{\mu} - p\hat{\beta}_1 = 0$$

$$p\hat{\beta}_1 = \sum_{i=1}^{p} Y_{i1(k,m)} - p\hat{\mu}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{p} Y_{i1(k,m)}}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\beta}_1 = \frac{C_1}{p} - \bar{\bar{Y}}$$

$$\frac{\partial S}{\partial \hat{\beta}_2} = 2\sum_{i=1}^{p}\left(Y_{i2(k,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_2 - \hat{\gamma}_k - \hat{\eta}_m\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\left(Y_{i2(k,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_2 - \hat{\gamma}_k - \hat{\eta}_m\right) = 0$$

$$\sum_{i=1}^{p} Y_{i2(k,m)} - p\hat{\mu} - \sum_{i=1}^{p}\hat{\tau}_i - p\hat{\beta}_2 - \sum_{k=1}^{p}\hat{\gamma}_k - \sum_{m=1}^{p}\hat{\eta}_m = 0$$

For unique solution put $\sum_{i=1}^{p}\hat{\tau}_i = 0, \sum_{k=1}^{p}\hat{\gamma}_k = 0,\ \sum_{m=1}^{p}\hat{\eta}_m = 0$

$$\sum_{i=1}^{p} Y_{i2(k,m)} - p\hat{\mu} - p\hat{\beta}_2 = 0$$

$$p\hat{\beta}_2 = \sum_{i=1}^{p} Y_{i2(k,m)} - p\hat{\mu}$$

$$\hat{\beta}_2 = \frac{\sum_{i=1}^{p} Y_{i2(k,m)}}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\beta}_2 = \frac{C_2}{p} - \bar{\bar{Y}}$$

$$\frac{\partial S}{\partial \hat{\beta}_p} = 2\sum_{i=1}^{p}\left(Y_{ip(k,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_p - \hat{\gamma}_k - \hat{\eta}_m\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\left(Y_{ip(k,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_p - \hat{\gamma}_k - \hat{\eta}_m\right) = 0$$

$$\sum_{i=1}^{p} Y_{ip(k,m)} - p\hat{\mu} - \sum_{i=1}^{p}\hat{\tau}_i - p\hat{\beta}_p - \sum_{k=1}^{p}\hat{\gamma}_k - \sum_{m=1}^{p}\hat{\eta}_m = 0$$

For unique solution put $\sum_{i=1}^{p}\hat{\tau}_i = 0, \sum_{k=1}^{p}\hat{\gamma}_k = 0,\ \sum_{m=1}^{p}\hat{\eta}_m = 0$

$$\sum_{i=1}^{p} Y_{ip(k,m)} - p\hat{\mu} - p\hat{\beta}_p = 0$$

$$p\hat{\beta}_p = \sum_{i=1}^{p} Y_{ip(k,m)} - p\hat{\mu}$$

$$\hat{\beta}_p = \frac{\sum_{i=1}^{p} Y_{ip(k,m)}}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\beta}_p = \frac{C_p}{p} - \bar{\bar{Y}}$$

$$S = \sum_{i=1}^{p}\left(Y_{ij(1,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_1 - \hat{\eta}_m\right)^2 + \sum_{i=1}^{p}\left(Y_{ij(2,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_2 - \hat{\eta}_m\right)^2 + \cdots$$

$$+ \sum_{i=1}^{p}\left(Y_{ij(p,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_p - \hat{\eta}_m\right)^2$$

Differentiate w.r.t $\hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_p$

$$\frac{\partial S}{\partial \hat{\gamma}_1} = 2\sum_{i=1}^{p}\left(Y_{ij(1,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_1 - \hat{\eta}_m\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\left(Y_{ij(1,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_1 - \hat{\eta}_m\right) = 0$$

$$\sum_{i=1}^{p} Y_{ij(1,m)} - p\hat{\mu} - \sum_{i=1}^{p}\hat{\tau}_i - \sum_{j=1}^{p}\hat{\beta}_j - p\hat{\gamma}_1 - \sum_{m=1}^{p}\hat{\eta}_m = 0$$

For unique solution put $\sum_{i=1}^{p}\hat{\tau}_i = 0, \sum_{j=1}^{p}\hat{\beta}_j = 0,\ \sum_{m=1}^{p}\hat{\eta}_m = 0$

$$\sum_{i=1}^{p} Y_{ij(1,m)} - p\hat{\mu} - p\hat{\gamma}_1 = 0$$

$$p\hat{\gamma}_1 = \sum_{i=1}^{p} Y_{ij(1,m)} - p\hat{\mu}$$

$$\hat{\gamma}_1 = \frac{\sum_{i=1}^{p} Y_{ij(1,m)}}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\gamma}_1 = \frac{G_1}{p} - \bar{\bar{Y}}$$

$$\frac{\partial S}{\partial \hat{\gamma}_2} = 2\sum_{i=1}^{p}\left(Y_{ij(2,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_2 - \hat{\eta}_m\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\left(Y_{ij(2,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_2 - \hat{\eta}_m\right) = 0$$

$$\sum_{i=1}^{p} Y_{ij(2,m)} - p\hat{\mu} - \sum_{i=1}^{p}\hat{\tau}_i - \sum_{j=1}^{p}\hat{\beta}_j - p\hat{\gamma}_2 - \sum_{m=1}^{p}\hat{\eta}_m = 0$$

For unique solution put $\sum_{i=1}^{p}\hat{\tau}_i = 0, \sum_{j=1}^{p}\hat{\beta}_j = 0,\ \sum_{m=1}^{p}\hat{\eta}_m = 0$

$$\sum_{i=1}^{p} Y_{ij(2,m)} - p\hat{\mu} - p\hat{\gamma}_2 = 0$$

$$p\hat{\gamma}_2 = \sum_{i=1}^{p} Y_{ij(2,m)} - p\hat{\mu}$$

$$\hat{\gamma}_2 = \frac{\sum_{i=1}^{p} Y_{ij(2,m)}}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\gamma}_2 = \frac{G_2}{p} - \bar{\bar{Y}}$$

$$\frac{\partial S}{\partial \hat{\gamma}_p} = 2\sum_{i=1}^{p}\left(Y_{ij(p,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_p - \hat{\eta}_m\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\left(Y_{ij(p,m)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_p - \hat{\eta}_m\right) = 0$$

$$\sum_{i=1}^{p} Y_{ij(p,m)} - p\hat{\mu} - \sum_{i=1}^{p}\hat{\tau}_i - \sum_{j=1}^{p}\hat{\beta}_j - p\hat{\gamma}_p - \sum_{m=1}^{p}\hat{\eta}_m = 0$$

For unique solution put $\sum_{i=1}^{p}\hat{\tau}_i = 0, \sum_{j=1}^{p}\hat{\beta}_j = 0,\ \sum_{m=1}^{p}\hat{\eta}_m = 0$

$$\sum_{i=1}^{p} Y_{ij(p,m)} - p\hat{\mu} - p\hat{\gamma}_p = 0$$

$$p\hat{\gamma}_p = \sum_{i=1}^{p} Y_{ij(p,m)} - p\hat{\mu}$$

$$\hat{\gamma}_p = \frac{\sum_{i=1}^{p} Y_{ij(p,m)}}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\gamma}_p = \frac{G_p}{p} - \bar{\bar{Y}}$$

$$S = \sum_{i=1}^{p}\left(Y_{ij(k,1)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_1\right)^2 + \sum_{i=1}^{p}\left(Y_{ij(k,2)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_2\right)^2 + \cdots$$

$$+ \sum_{i=1}^{p}\left(Y_{ij(k,p)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_p\right)^2$$

Differentiate w.r.t $\hat{\eta}_1, \hat{\eta}_2, \ldots, \hat{\eta}_p$

$$\frac{\partial S}{\partial \hat{\eta}_1} = 2\sum_{i=1}^{p}\left(Y_{ij(k,1)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_1\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\left(Y_{ij(k,1)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_1\right) = 0$$

$$\sum_{i=1}^{p} Y_{ij(k,1)} - p\hat{\mu} - \sum_{i=1}^{p}\hat{\tau}_i - \sum_{j=1}^{p}\hat{\beta}_j - \sum_{k=1}^{p}\gamma_k - p\hat{\eta}_1 = 0$$

For unique solution put $\sum_{i=1}^{p}\hat{\tau}_i = 0, \sum_{j=1}^{p}\hat{\beta}_j = 0,\ \sum_{k=1}^{p}\hat{\gamma}_k = 0$

$$\sum_{i=1}^{p} Y_{ij(k,1)} - p\hat{\mu} - p\hat{\eta}_1 = 0$$

$$p\hat{\eta}_1 = \sum_{i=1}^{p} Y_{ij(k,1)} - p\hat{\mu}$$

$$\hat{\eta}_1 = \frac{\sum_{i=1}^{p} Y_{ij(k,1)}}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\eta}_1 = \frac{L_1}{p} - \bar{\bar{Y}}$$

$$\frac{\partial S}{\partial \hat{\eta}_2} = 2\sum_{i=1}^{p}\left(Y_{ij(k,2)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_2\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\left(Y_{ij(k,2)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_2\right) = 0$$

$$\sum_{i=1}^{p} Y_{ij(k,2)} - p\hat{\mu} - \sum_{i=1}^{p}\hat{\tau}_i - \sum_{j=1}^{p}\hat{\beta}_j - \sum_{k=1}^{p}\gamma_k - p\hat{\eta}_2 = 0$$

For unique solution put $\sum_{i=1}^{p}\hat{\tau}_i = 0, \sum_{j=1}^{p}\hat{\beta}_j = 0,\ \sum_{k=1}^{p}\hat{\gamma}_k = 0$

$$\sum_{i=1}^{p} Y_{ij(k,2)} - p\hat{\mu} - p\hat{\eta}_2 = 0$$

$$p\hat{\eta}_2 = \sum_{i=1}^{p} Y_{ij(k,2)} - p\hat{\mu}$$

$$\hat{\eta}_2 = \frac{\sum_{i=1}^{p} Y_{ij(k,2)}}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\eta}_2 = \frac{L_2}{p} - \bar{\bar{Y}}$$

$$\frac{\partial S}{\partial \hat{\eta}_p} = 2\sum_{i=1}^{p}\left(Y_{ij(k,p)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_p\right)(-1) = 0$$

$$-2\sum_{i=1}^{p}\left(Y_{ij(k,p)} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - \hat{\gamma}_k - \hat{\eta}_p\right) = 0$$

$$\sum_{i=1}^{p} Y_{ij(k,p)} - p\hat{\mu} - \sum_{i=1}^{p}\hat{\tau}_i - \sum_{j=1}^{p}\hat{\beta}_j - \sum_{k=1}^{p}\gamma_k - p\hat{\eta}_p = 0$$

For unique solution put $\sum_{i=1}^{p}\hat{\tau}_i = 0, \sum_{j=1}^{p}\hat{\beta}_j = 0,\ \sum_{k=1}^{p}\hat{\gamma}_k = 0$

$$\sum_{i=1}^{p} Y_{ij(k,p)} - p\hat{\mu} - p\hat{\eta}_p = 0$$

$$p\hat{\eta}_p = \sum_{i=1}^{p} Y_{ij(k,p)} - p\hat{\mu}$$

$$\hat{\eta}_p = \frac{\sum_{i=1}^{p} Y_{ij(k,p)}}{p} - \frac{p\hat{\mu}}{p}$$

$$\hat{\eta}_p = \frac{L_p}{p} - \bar{\bar{Y}}$$

Expected Mean Square Error

Fixed Effect Model

Assumptions:

1. $e_{ij(k,m)} \sim iidN(0, \sigma^2)$
2. $E\left(e_{ij(k,m)} e_{gh(l,z)}\right) = 0$
3. $\sum_{i=1}^{p} \tau_i = 0$
4. $\sum_{j=1}^{p} \beta_j = 0$
5. $\sum_{k=1}^{p} \gamma_k = 0$
6. $\sum_{m=1}^{p} \eta_m = 0$

$$E(SSE) = E(TSS) - E(SSR) - E(SSC) - E(SSG) - E(SSL)$$

$$TSS = \sum_{i=1}^{p} \sum_{j=1}^{p} Y_{ij(k,m)}^2 - C.F$$

$$C.F = \frac{\left(\sum_{i=1}^{p} \sum_{j=1}^{p} Y_{ij(k,m)}\right)^2}{p^2} = \frac{\left(\sum_{i=1}^{p} \sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + \eta_m + e_{ij(k,m)}\right)\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p} \tau_i + p\sum_{j=1}^{p} \beta_j + p\sum_{k=1}^{p} \gamma_k + p\sum_{m=1}^{p} \eta_m + \sum_{i=1}^{p} \sum_{j=1}^{p} e_{ij(k,m)}\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + \sum_{i=1}^{p} \sum_{j=1}^{p} e_{ij(k,m)}\right)^2}{p^2}$$

$$= \frac{p^4\mu^2 + \sum_{i=1}^{p} \sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i \neq g} \sum_{j \neq h} e_{ij(k,m)} e_{gh(l,z)} + 2p^2\mu \sum_{i=1}^{p} \sum_{j=1}^{p} e_{ij(k,m)}}{p^2}$$

Apply expectation on both sides

$E(C.F)$

$$= \frac{p^4\mu^2 + \sum_{i=1}^{p} \sum_{j=1}^{p} E(e_{ij(k,m)}^2) + \sum_{i \neq g} \sum_{j \neq h} E(e_{ij(k,m)} e_{gh(l,z)}) + 2p^2\mu \sum_{i=1}^{p} \sum_{j=1}^{p} E(e_{ij(k,m)})}{p^2}$$

$$= \frac{p^4\mu^2 + p^2\sigma^2 + 0 + 0}{p^2} = \frac{p^4\mu^2 + p^2\sigma^2}{p^2}$$

$$= p^2\mu^2 + \sigma^2$$

$$\sum_{i=1}^{p} \sum_{j=1}^{p} Y_{ij(k,m)}^2 = \sum_{i=1}^{p} \sum_{j=1}^{p} \left(\mu + \tau_i + \beta_j + \gamma_k + \eta_m + e_{ij(k,m)}\right)^2$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p}\big(\mu^2+\tau_i^2+\beta_j^2+\gamma_k^2+\eta_m^2+e_{ij(k,m)}^2+2\mu\tau_i+2\mu\beta_j+2\mu\gamma_k+2\mu\eta_m+2\mu e_{ij(k,m)}$$
$$+2\tau_i\beta_j+2\tau_i\gamma_k+2\tau_i\eta_m+2\tau_i e_{ij(k,m)}+2\beta_j\gamma_k+2\beta_j\eta_m+2\beta_j e_{ij(k,m)}$$
$$+2\gamma_k\eta_m+2\gamma_k e_{ij(k,m)}+2\eta_m e_{ij(k,m)}\big)$$

$$= p^2\mu^2+p\sum_{i=1}^{p}\tau_i^2+p\sum_{j=1}^{p}\beta_j^2+p\sum_{k=1}^{p}\gamma_k^2+p\sum_{m=1}^{p}\eta_m^2+\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k,m)}^2+2p\mu\sum_{i=1}^{p}\tau_i$$
$$+2p\mu\sum_{j=1}^{p}\beta_j+2p\mu\sum_{k=1}^{p}\gamma_k+2p\mu\sum_{m=1}^{p}\eta_m+2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k,m)}$$
$$+2\sum_{i=1}^{p}\tau_i\sum_{j=1}^{p}\beta_j+2\sum_{i=1}^{p}\tau_i\sum_{k=1}^{p}\gamma_k+2\sum_{i=1}^{p}\tau_i\sum_{m=1}^{p}\eta_m+2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k,m)}$$
$$+2\sum_{j=1}^{p}\beta_j\sum_{k=1}^{p}\gamma_k+2\sum_{j=1}^{p}\beta_j\sum_{m=1}^{p}\eta_m+2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\, e_{ij(k,m)}+2\sum_{k=1}^{p}\gamma_k\sum_{m=1}^{p}\eta_m$$
$$+2\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k,m)}+2\sum_{i=1}^{p}\sum_{j=1}^{p}\eta_m e_{ij(k,m)}$$

Apply expectation on both sides

$$= p^2\mu^2+p\sum_{i=1}^{p}\tau_i^2+p\sum_{j=1}^{p}\beta_j^2+p\sum_{k=1}^{p}\gamma_k^2+p\sum_{m=1}^{p}\eta_m^2+\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k,m)}^2)+2p\mu\sum_{i=1}^{p}\tau_i$$
$$+2p\mu\sum_{j=1}^{p}\beta_j+2p\mu\sum_{k=1}^{p}\gamma_k+2p\mu\sum_{m=1}^{p}\eta_m+2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k,m)})$$
$$+2\sum_{i=1}^{p}\tau_i\sum_{j=1}^{p}\beta_j+2\sum_{i=1}^{p}\tau_i\sum_{k=1}^{p}\gamma_k+2\sum_{i=1}^{p}\tau_i\sum_{m=1}^{p}\eta_m+2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i E(e_{ij(k,m)})$$
$$+2\sum_{j=1}^{p}\beta_j\sum_{k=1}^{p}\gamma_k+2\sum_{j=1}^{p}\beta_j\sum_{m=1}^{p}\eta_m+2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j\, E(\,e_{ij(k,m)})$$
$$+2\sum_{k=1}^{p}\gamma_k\sum_{m=1}^{p}\eta_m+2\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k E(e_{ij(k,m)})+2\sum_{i=1}^{p}\sum_{j=1}^{p}\eta_m E(e_{ij(k,m)})$$

$$= p^2\mu^2+p\sum_{i=1}^{p}\tau_i^2+p\sum_{j=1}^{p}\beta_j^2+p\sum_{k=1}^{p}\gamma_k^2+p\sum_{m=1}^{p}\eta_m^2+p^2\sigma^2+0+0+0+0+0+0+0$$
$$+0+0+0+0+0+0+0+0$$

$$= p^2\mu^2+p\sum_{i=1}^{p}\tau_i^2+p\sum_{j=1}^{p}\beta_j^2+p\sum_{k=1}^{p}\gamma_k^2+p\sum_{m=1}^{p}\eta_m^2+p^2\sigma^2$$

$$E(TSS)=E\left(\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k,m)}^2\right)-E(C.F)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum_{m=1}^{p}\eta_m^2 + p^2\sigma^2 - p^2\mu^2 - \sigma^2$$

$$E(TSS) = p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum_{m=1}^{p}\eta_m^2 + (p^2-1)\sigma^2$$

$$SSR = \frac{\sum_{i=1}^{p} R_i^2}{p} - C.F$$

$$\frac{\sum_{i=1}^{p} R_i^2}{p} = \frac{\sum_{i=1}^{p}\left(\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + \eta_m + e_{ij(k,m)}\right)\right)^2}{p}$$

$$= \frac{\sum_{i=1}^{p}\left(p\mu + p\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{m=1}^{p}\eta_m + \sum_{j=1}^{p} e_{ij(k,m)}\right)^2}{p}$$

$$= \frac{\sum_{i=1}^{p}\left(p\mu + p\tau_i + \sum_{j=1}^{p} e_{ij(k,m)}\right)^2}{p}$$

$$= \frac{\sum_{i=1}^{p}\left(p^2\mu^2 + p^2\tau_i^2 + \sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{j\neq h} e_{ij(k,m)}e_{ih(l,z)} + 2p^2\mu\tau_i + 2p\mu\sum_{j=1}^{p} e_{ij(k,m)} + 2p\tau_i\sum_{j=1}^{p} e_{ij(k,m)}\right)}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}e_{gh(l,z)} + 2p^2\mu\sum_{i=1}^{p}\tau_i \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k,m)} \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}e_{gh(l,z)} + 0 \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k,m)} \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}e_{gh(l,z)} + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)} \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k,m)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k,m)}e_{gh(l,z)}) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}) \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i E(e_{ij(k,m)}) \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p^2\sigma^2 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p^2\sigma^2}{p} = p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sigma^2$$

$$E(SSR) = E\left[\frac{\sum_{i=1}^{p} R_i^2}{p}\right] - E(C.F)$$

$$= p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sigma^2 - p^2\mu^2 - \sigma^2$$

$$= p\sum_{i=1}^{p}\tau_i^2 + (p-1)\sigma^2$$

$$SSC = \frac{\sum_{j=1}^{p} C_j^2}{p} - C.F$$

$$\frac{\sum_{j=1}^{p} C_j^2}{p} = \frac{\sum_{j=1}^{p}\left(\sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + \eta_m + e_{ij(k,m)}\right)\right)^2}{p}$$

$$= \frac{\sum_{j=1}^{p}\left(p\mu + \sum_{i=1}^{p}\tau_i + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{m=1}^{p}\eta_m + \sum_{i=1}^{p} e_{ij(k,m)}\right)^2}{p}$$

$$= \frac{\sum_{j=1}^{p}\left(p\mu + p\beta_j + \sum_{i=1}^{p} e_{ij(k,m)}\right)^2}{p}$$

$$= \frac{\sum_{j=1}^{p}\left(p^2\mu^2 + p^2\beta_j^2 + \sum_{i=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g} e_{ij(k,m)}e_{gj(l,z)} + 2p^2\mu\beta_j + 2p\mu\sum_{i=1}^{p} e_{ij(k,m)} + 2p\beta_j\sum_{i=1}^{p} e_{ij(k,m)}\right)}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}e_{gh(l,z)} + 2p^2\mu\sum_{j=1}^{p}\beta_j \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k,m)} \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}e_{gh(l,z)} + 0 \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k,m)} \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}e_{gh(l,z)} + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)} \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k,m)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k,m)}e_{gh(l,z)}) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}) \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j E(e_{ij(k,m)}) \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + p^2\sigma^2 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{j=1}^{p}\beta_j^2 + p^2\sigma^2}{p} = p^2\mu^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sigma^2$$

$$E(SSC) = E\left[\frac{\sum_{j=1}^{p} C_j^2}{p}\right] - E(C.F)$$

$$= p^2\mu^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sigma^2 - p^2\mu^2 - \sigma^2$$

$$= p\sum_{j=1}^{p}\beta_j^2 + (p-1)\sigma^2$$

$$SSG = \frac{\sum_{k=1}^{p} G_k^2}{p} - C.F$$

$$\frac{\sum_{k=1}^{p} G_k^2}{p} = \frac{\sum_{k=1}^{p}\left(\sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + \eta_m + e_{ij(k,m)}\right)\right)^2}{p}$$

$$= \frac{\sum_{k=1}^{p}\left(p\mu + \sum_{i=1}^{p}\tau_i + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{m=1}^{p}\eta_m + \sum_{i=1}^{p} e_{ij(k,m)}\right)^2}{p}$$

$$= \frac{\sum_{k=1}^{p}\left(p\mu + p\gamma_k + \sum_{i=1}^{p} e_{ij(k,m)}\right)^2}{p}$$

$$= \frac{\sum_{k=1}^{p}\left(p^2\mu^2 + p^2\gamma_k^2 + \sum_{i=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g} e_{ij(k,m)}e_{gj(l,z)} + 2p^2\mu\gamma_k + 2p\mu\sum_{i=1}^{p} e_{ij(k,m)} + 2p\gamma_k\sum_{i=1}^{p} e_{ij(k,m)}\right)}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}e_{gh(l,z)} + 2p^2\mu\sum_{k=1}^{p}\gamma_k \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k,m)} \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}e_{gh(l,z)} + 0 \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k,m)} \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}e_{gh(l,z)} + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)} \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k,m)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k,m)}e_{gh(l,z)}) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}) \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k E(e_{ij(k,m)}) \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{k=1}^{p}\gamma_k^2 + p^2\sigma^2}{p} = p^2\mu^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sigma^2$$

$$E(SSG) = E\left[\frac{\sum_{k=1}^{p} G_k^2}{p}\right] - E(C.F)$$

$$= p^2\mu^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sigma^2 - p^2\mu^2 - \sigma^2$$

$$= p\sum_{k=1}^{p}\gamma_k^2 + (p-1)\sigma^2$$

$$SSL = \frac{\sum_{m=1}^{p} L_m^2}{p} - C.F$$

$$\frac{\sum_{m=1}^{p} L_m^2}{p} = \frac{\sum_{m=1}^{p}\left(\sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + \eta_m + e_{ij(k,m)}\right)\right)^2}{p}$$

$$= \frac{\sum_{m=1}^{p}\left(p\mu + \sum_{i=1}^{p}\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{k=1}^{p}\gamma_k + p\eta_m + \sum_{i=1}^{p} e_{ij(k,m)}\right)^2}{p}$$

$$= \frac{\sum_{m=1}^{p}\left(p\mu + p\eta_m + \sum_{i=1}^{p} e_{ij(k,m)}\right)^2}{p}$$

$$= \frac{\sum_{m=1}^{p}\left(p^2\mu^2 + p^2\eta_m^2 + \sum_{i=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g} e_{ij(k,m)}e_{gh(l,m)} + 2p^2\mu\eta_m + 2p\mu\sum_{i=1}^{p} e_{ij(k,m)} + 2p\eta_m\sum_{i=1}^{p} e_{ij(k,m)}\right)}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{m=1}^{p}\eta_m^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}e_{gh(l,z)} + 2p^2\mu\sum_{m=1}^{p}\eta_m \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\eta_m e_{ij(k,m)} \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{m=1}^{p}\eta_m^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}e_{gh(l,z)} + 0 \\ +2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\eta_m e_{ij(k,m)} \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{m=1}^{p}\eta_m^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}e_{gh(l,z)} + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)} \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\eta_m e_{ij(k,m)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{m=1}^{p}\eta_m^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k,m)}e_{gh(l,z)}) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}) \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\eta_m E(e_{ij(k,m)}) \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{m=1}^{p}\eta_m^2 + p^2\sigma^2 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sum_{m=1}^{p}\eta_m^2 + p^2\sigma^2}{p} = p^2\mu^2 + p\sum_{m=1}^{p}\eta_m^2 + p\sigma^2$$

$$E(SSL) = E\left[\frac{\sum_{m=1}^{p} L_m^2}{p}\right] - E(C.F)$$

$$= p^2\mu^2 + p\sum_{m=1}^{p}\eta_m^2 + p\sigma^2 - p^2\mu^2 - \sigma^2$$

$$= p\sum_{m=1}^{p}\eta_m^2 + (p-1)\sigma^2$$

$$E(SSE) = E(TSS) - E(SSR) - E(SSC) - E(SSG) - E(SSL)$$

$$= p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum_{m=1}^{p}\eta_m^2 + (p^2-1)\sigma^2 - p\sum_{i=1}^{p}\tau_i^2 - (p-1)\sigma^2$$

$$- p\sum_{j=1}^{p}\beta_j^2 - (p-1)\sigma^2 - p\sum_{k=1}^{p}\gamma_k^2 - (p-1)\sigma^2 - p\sum_{m=1}^{p}\eta_m^2 - (p-1)\sigma^2$$

$$= (p^2 - 1 - p + 1 - p + 1 - p + 1 - p + 1)\sigma^2$$

$$= (p^2 - 4p + 3)\sigma^2 = (p^2 - 3p - p + 3)\sigma^2$$

$$= (p(p-3) - 1(p-3))\sigma^2 = (p-1)(p-3)\sigma^2$$

$$E(MSE) = \frac{E(SSE)}{(p-1)(p-3)} = \frac{(p-1)(p-3)\sigma^2}{(p-1)(p-3)} = \sigma^2$$

$$E(MSR) = \frac{E(SSR)}{p-1} = \frac{p\sum_{i=1}^{p}\tau_i^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{i=1}^{p}\tau_i^2$$

$$E(MSC) = \frac{E(SSC)}{p-1} = \frac{p\sum_{j=1}^{p}\beta_j^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{j=1}^{p}\beta_j^2$$

$$E(MSG) = \frac{E(SSG)}{p-1} = \frac{p\sum_{k=1}^{p}G_k^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{k=1}^{p}G_k^2$$

$$E(MSL) = \frac{E(SSL)}{p-1} = \frac{p\sum_{m=1}^{p}\eta_m^2 + (p-1)\sigma^2}{p-1} = \sigma^2 + \frac{p}{p-1}\sum_{m=1}^{p}\eta_m^2$$

Random Effect Model

Assumptions:

1. $e_{ij(k,m)} \sim iidN(0, \sigma^2)$
2. $E\left(e_{ij(k,m)}e_{gh(l,z)}\right) = 0$
3. $E\left(\tau_i e_{ij(k,m)}\right) = 0$
4. $E\left(\beta_j e_{ij(k,m)}\right) = 0$
5. $E\left(\gamma_k e_{ij(k,m)}\right) = 0$
6. $E\left(\eta_m e_{ij(k,m)}\right) = 0$
7. $\tau_i \sim iidN(0, \sigma_\tau^2)$
8. $\beta_j \sim iidN(0, \sigma_\beta^2)$
9. $\gamma_k \sim iidN(0, \sigma_\gamma^2)$
10. $\eta_m \sim iidN(0, \sigma_\eta^2)$
11. $E\left(\tau_i\tau_j\right) = 0$
12. $E\left(\beta_i\beta_j\right) = 0$
13. $E(\gamma_k\gamma_l) = 0$
14. $E(\eta_m\eta_n) = 0$
15. $E\left(\tau_i\beta_j\right) = 0$
16. $E(\tau_i\gamma_k) = 0$
17. $E(\tau_i\eta_m) = 0$
18. $E\left(\beta_j\gamma_k\right) = 0$
19. $E\left(\beta_j\eta_m\right) = 0$
20. $E(\gamma_k\eta_m) = 0$

$$E(SSE) = E(TSS) - E(SSR) - E(SSC) - E(SSG) - E(SSL)$$

$$TSS = \sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k,m)}^2 - C.F$$

$$C.F = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij(k,m)}\right)^2}{p^2} = \frac{\left(\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + \eta_m + e_{ij(k,m)}\right)\right)^2}{p^2}$$

$$= \frac{\left(p^2\mu + p\sum_{i=1}^{p}\tau_i + p\sum_{j=1}^{p}\beta_j + p\sum_{k=1}^{p}\gamma_k + p\sum_{m=1}^{p}\eta_m + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}\right)^2}{p^2}$$

$$= \frac{\begin{array}{c} p^4\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + p^2\sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + p^2\sum_{k=1}^{p}\gamma_k^2 + \sum\sum_{k\neq l}\gamma_k\gamma_l + p^2\sum_{m=1}^{p}\eta_m^2 \\ \sum\sum_{m\neq n}\eta_m\eta_n + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k,m)}e_{gh(l,n)} + 2p^3\mu\sum_{i=1}^{p}\tau_i + 2p^3\mu\sum_{j=1}^{p}\beta_j + 2p^3\mu\sum_{k=1}^{p}\gamma_k \\ +2p^3\mu\sum_{m=1}^{p}\eta_m + 2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k,m)} + 2p^2\sum_{i=1}^{p}\tau_i\sum_{j=1}^{p}\beta_j + 2p^2\sum_{i=1}^{p}\tau_i\sum_{k=1}^{p}\gamma_k + 2p^2\sum_{i=1}^{p}\tau_i\sum_{m=1}^{p}\eta_m \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k,m)} + 2p^2\sum_{j=1}^{p}\beta_j\sum_{k=1}^{p}\gamma_k + 2p^2\sum_{j=1}^{p}\beta_j\sum_{m=1}^{p}\eta_m + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k,m)} \\ +2p^2\sum_{k=1}^{p}\gamma_k\sum_{m=1}^{p}\eta_m + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k,m)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\eta_m e_{ij(k,m)} \end{array}}{p^2}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^4\mu^2 + p^2\sum_{i=1}^{p}E(\tau_i^2) + \sum\sum_{i\neq j}E(\tau_i\tau_j) + p^2\sum_{j=1}^{p}E(\beta_j^2) + \sum\sum_{i\neq j}E(\beta_i\beta_j) + p^2\sum_{k=1}^{p}E(\gamma_k^2) + \sum\sum_{k\neq l}E(\gamma_k\gamma_l) \\ +p^2\sum_{m=1}^{p}E(\eta_m^2) + \sum\sum_{m\neq n}E(\eta_m\eta_n) + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k,m)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k,m)}e_{gh(l,n)}) + 2p^3\mu\sum_{i=1}^{p}E(\tau_i) \\ +2p^3\mu\sum_{j=1}^{p}E(\beta_j) + 2p^3\mu\sum_{k=1}^{p}E(\gamma_k) + 2p^3\mu\sum_{m=1}^{p}E(\eta_m) + 2p^2\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k,m)}) + 2p^2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i\beta_j) \\ +2p^2\sum_{i=1}^{p}\sum_{k=1}^{p}E(\tau_i\gamma_k) + 2p^2\sum_{i=1}^{p}\sum_{m=1}^{p}E(\tau_i\eta_m) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i e_{ij(k,m)}) + 2p^2\sum_{j=1}^{p}\sum_{k=1}^{p}E(\beta_j\gamma_k) \\ +2p^2\sum_{j=1}^{p}\sum_{m=1}^{p}E(\eta_m\beta_j) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j e_{ij(k,m)}) + 2p^2\sum_{k=1}^{p}\sum_{m=1}^{p}E(\gamma_k\eta_m) \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\gamma_k e_{ij(k,m)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\eta_m e_{ij(k,m)}) \end{array}}{p^2}$$

$$= \frac{p^4\mu^2 + p^3\sigma_\tau^2 + 0 + p^3\sigma_\beta^2 + 0 + p^3\sigma_\gamma^2 + 0 + p^3\sigma_\eta^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p^2}$$

$$= \frac{p^4\mu^2 + p^3\sigma_\tau^2 + p^3\sigma_\beta^2 + p^3\sigma_\gamma^2 + p^3\sigma_\eta^2 + p^2\sigma^2}{p^2}$$

$$E(C.F) = p^2\mu^2 + p\sigma_\tau^2 + p\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma_\eta^2 + \sigma^2$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p}Y_{ij(k,m)}^2 = \sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + \eta_m + e_{ij(k,m)}\right)^2$$

$$\begin{aligned} = \sum_{i=1}^{p}\sum_{j=1}^{p}\big(&\mu^2 + \tau_i^2 + \beta_j^2 + \gamma_k^2 + \eta_m^2 + e_{ij(k,m)}^2 + 2\mu\tau_i + 2\mu\beta_j + 2\mu\gamma_k + 2\mu\eta_m + 2\mu e_{ij(k,m)} \\ &+ 2\tau_i\beta_j + 2\tau_i\gamma_k + 2\tau_i\eta_m + 2\tau_i e_{ij(k,m)} + 2\beta_j\gamma_k + 2\beta_j\eta_m + 2\beta_j e_{ij(k,m)} \\ &+ 2\gamma_k\eta_m + 2\gamma_k e_{ij(k,m)} + 2\eta_m e_{ij(k,m)}\big) \end{aligned}$$

$$\begin{aligned} = p^2\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum_{m=1}^{p}\eta_m^2 + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k,m)}^2 + 2p\mu\sum_{i=1}^{p}\tau_i \\ + 2p\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{m=1}^{p}\eta_m + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k,m)} \\ + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2\sum_{i=1}^{p}\sum_{m=1}^{p}\tau_i\eta_m + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k,m)} \\ + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\beta_j\gamma_k + 2\sum_{j=1}^{p}\sum_{m=1}^{p}\beta_j\eta_m + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k,m)} + 2\sum_{k=1}^{p}\sum_{m=1}^{p}\gamma_k\eta_m \\ + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_k e_{ij(k,m)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\eta_m e_{ij(k,m)} \end{aligned}$$

Apply expectation on both sides

$$= p^2\mu^2 + p\sum_{i=1}^{p} E(\tau_i^2) + p\sum_{j=1}^{p} E(\beta_j^2) + p\sum_{k=1}^{p} E(\gamma_k^2) + p\sum_{m=1}^{p} E(\eta_m^2) + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}^2)$$
$$+ 2p\mu\sum_{i=1}^{p} E(\tau_i) + 2p\mu\sum_{j=1}^{p} E(\beta_j) + 2p\mu\sum_{k=1}^{p} E(\gamma_k) + 2p\mu\sum_{m=1}^{p} E(\eta_m)$$
$$+ 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i\beta_j) + 2\sum_{i=1}^{p}\sum_{k=1}^{p} E(\tau_i\gamma_k)$$
$$+ 2\sum_{i=1}^{p}\sum_{m=1}^{p} E(\tau_i\eta_m) + 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i e_{ij(k,m)}) + 2\sum_{j=1}^{p}\sum_{k=1}^{p} E(\beta_j\gamma_k)$$
$$+ 2\sum_{j=1}^{p}\sum_{m=1}^{p} E(\beta_j\eta_m) + 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\beta_j\, e_{ij(k,m)}) + 2\sum_{k=1}^{p}\sum_{m=1}^{p} E(\gamma_k\eta_m)$$
$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\gamma_k e_{ij(k,m)}) + 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\eta_m e_{ij(k,m)})$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p^2\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma_\eta^2 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+ 0 + 0 + 0 + 0 + 0$$
$$= p^2\mu^2 + p^2\sigma_\tau^2 + p^2\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma_\eta^2 + p^2\sigma^2$$

$$E(TSS) = E\left[\sum_{i=1}^{p}\sum_{j=1}^{p} Y_{ij}^2\right] - E(C.F)$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p^2\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma_\eta^2 + p^2\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\beta^2 - p\sigma_\gamma^2 - p\sigma_\eta^2 - \sigma^2$$
$$= p(p-1)\sigma_\tau^2 + p(p-1)\sigma_\beta^2 + p(p-1)\sigma_\gamma^2 + p(p-1)\sigma_\eta^2 + (p^2-1)\sigma^2$$

$$SSR = \frac{\sum_{i=1}^{p} R_i^2}{p} - C.F$$

$$\frac{\sum_{i=1}^{p} R_i^2}{p} = \frac{\sum_{i=1}^{p}\left(\sum_{j=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + \eta_m + e_{ij(k,m)}\right)\right)^2}{p}$$

$$= \frac{\sum_{i=1}^{p}\left(p\mu + p\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{m=1}^{p}\eta_m + \sum_{j=1}^{p} e_{ij(k,m)}\right)^2}{p}$$

$$= \frac{\sum_{i=1}^{p}\left(\begin{array}{c} p^2\mu^2 + p^2\tau_i^2 + \sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + \sum_{k=1}^{p}\gamma_k^2 + \sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{m=1}^{p}\eta_m^2 \\ + \sum\sum_{m\neq n}\eta_m\eta_n + \sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{j\neq h} e_{ij(k,m)}e_{gh(l,z)} + 2p^2\mu\tau_i + 2p\mu\sum_{j=1}^{p}\beta_j \\ + 2p\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{m=1}^{p}\eta_m + 2p\mu\sum_{j=1}^{p} e_{ij(k,m)} + 2p\tau_i\sum_{j=1}^{p}\beta_j + 2p\tau_i\sum_{k=1}^{p}\gamma_k \\ + 2p\tau_i\sum_{m=1}^{p}\eta_m + 2p\tau_i\sum_{j=1}^{p} e_{ij(k,m)} + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\beta_j\gamma_k + 2\sum_{j=1}^{p}\sum_{m=1}^{p}\beta_j\eta_m \\ + 2\sum_{j=1}^{p}\beta_j e_{ij(k,m)} + 2\sum_{k=1}^{p}\sum_{m=1}^{p}\gamma_k\eta_m + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k,m)} + 2\sum_{j=1}^{p}\sum_{m=1}^{p}\eta_m e_{ij(k,m)} \end{array}\right)}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p}\tau_i^2 + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + p\sum_{k=1}^{p}\gamma_k^2 + p\sum\sum_{k\neq l}\gamma_k\gamma_l + p\sum_{m=1}^{p}\eta_m^2 \\ +p\sum\sum_{m\neq n}\eta_m\eta_n + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k,m)}e_{gh(l,z)} + 2p^2\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\sum_{j=1}^{p}\beta_j \\ +2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p^2\mu\sum_{m=1}^{p}\eta_m + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k,m)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2p\sum_{k=1}^{p}\sum_{i=1}^{p}\tau_i\gamma_k \\ +2p\sum_{m=1}^{p}\sum_{i=1}^{p}\tau_i\eta_m + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k,m)} + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\beta_j\gamma_k + 2\sum_{j=1}^{p}\sum_{m=1}^{p}\beta_j\eta_m \\ +2\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k,m)} + 2\sum_{k=1}^{p}\sum_{m=1}^{p}\gamma_k\eta_m + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k,m)} + 2\sum_{j=1}^{p}\sum_{m=1}^{p}\eta_m e_{ij(k,m)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sum_{i=1}^{p}E(\tau_i^2) + p\sum_{j=1}^{p}E(\beta_j^2) + p\sum\sum_{i\neq j}E(\beta_i\beta_j) + p\sum_{k=1}^{p}E(\gamma_k^2) + p\sum\sum_{k\neq l}E(\gamma_k\gamma_l) + p\sum_{m=1}^{p}E(\eta_m^2) \\ +p\sum\sum_{m\neq n}E(\eta_m\eta_n) + \sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k,m)}^2) + \sum_{i\neq g}\sum_{j\neq h}E(e_{ij(k,m)}e_{gh(l,z)}) + 2p^2\mu\sum_{i=1}^{p}E(\tau_i) + 2p^2\mu\sum_{j=1}^{p}E(\beta_j) \\ +2p^2\mu\sum_{k=1}^{p}E(\gamma_k) + 2p^2\mu\sum_{m=1}^{p}E(\eta_m) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}E(e_{ij(k,m)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i\beta_j) + \\ 2p\sum_{k=1}^{p}\sum_{i=1}^{p}E(\tau_i\gamma_k) + 2p\sum_{m=1}^{p}\sum_{i=1}^{p}E(\tau_i\eta_m) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}E(\tau_i e_{ij(k,m)}) + 2\sum_{j=1}^{p}\sum_{k=1}^{p}E(\beta_j\gamma_k) \\ +2\sum_{j=1}^{p}\sum_{m=1}^{p}E(\beta_j\eta_m) + 2\sum_{i=1}^{p}\sum_{j=1}^{p}E(\beta_j e_{ij(k,m)}) + 2\sum_{k=1}^{p}\sum_{m=1}^{p}E(\gamma_k\eta_m) \\ +2\sum_{j=1}^{p}\sum_{k=1}^{p}E(\gamma_k e_{ij(k,m)}) + 2\sum_{j=1}^{p}\sum_{m=1}^{p}E(\eta_m e_{ij(k,m)}) \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^3\sigma_\tau^2 + p^2\sigma_\beta^2 + 0 + p^2\sigma_\gamma^2 + 0 + p^2\sigma_\eta^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 \\ +0 + 0 + 0 + 0 + 0 + 0 + 0 \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^3\sigma_\tau^2 + p^2\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma_\eta^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma_\eta^2 + p\sigma^2$$

$$E(SSR) = E\left[\frac{\sum_{i=1}^{p}R_i^2}{p}\right] - E(C.F)$$

$$= p^2\mu^2 + p^2\sigma_\tau^2 + p\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma_\eta^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\beta^2 - p\sigma_\gamma^2 - p\sigma_\eta^2 - \sigma^2$$

$$= p(p-1)\sigma_\tau^2 + (p-1)\sigma^2$$

$$SSC = \frac{\sum_{j=1}^{p}C_j^2}{p} - C.F$$

$$\frac{\sum_{j=1}^{p}C_j^2}{p} = \frac{\sum_{j=1}^{p}\left(\sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + \eta_m + e_{ij(k,m)}\right)\right)^2}{p}$$

$$= \frac{\sum_{j=1}^{p}\left(p\mu + \sum_{i=1}^{p}\tau_i + p\beta_j + \sum_{k=1}^{p}\gamma_k + \sum_{m=1}^{p}\eta_m + \sum_{i=1}^{p}e_{ij(k,m)}\right)^2}{p}$$

$$= \frac{\sum_{j=1}^{p}\left(\begin{array}{c} p^2\mu^2 + \sum_{i=1}^{p}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + p^2\beta_j^2 + \sum_{k=1}^{p}\gamma_k^2 + \sum\sum_{k\neq l}\gamma_k\gamma_l + \sum_{m=1}^{p}\eta_m^2 \\ +\sum\sum_{m\neq n}\eta_m\eta_n + \sum_{j=1}^{p}e_{ij(k,m)}^2 + \sum_{j\neq h}e_{ij(k,m)}e_{gh(l,z)} + 2p\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\beta_j \\ +2p\mu\sum_{k=1}^{p}\gamma_k + 2p\mu\sum_{m=1}^{p}\eta_m + 2p\mu\sum_{j=1}^{p}e_{ij(k,m)} + 2p\sum_{i=1}^{p}\tau_i\beta_j + 2\sum_{k=1}^{p}\sum_{i=1}^{p}\tau_i\gamma_k \\ +2\sum_{m=1}^{p}\sum_{i=1}^{p}\tau_i\eta_m + 2\sum_{i=1}^{p}\tau_i e_{ij(k,m)} + 2p\sum_{k=1}^{p}\beta_j\gamma_k + 2p\sum_{m=1}^{p}\beta_j\eta_m \\ +2p\sum_{i=1}^{p}\beta_j e_{ij(k,m)} + 2\sum_{k=1}^{p}\sum_{m=1}^{p}\gamma_k\eta_m + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k,m)} + 2\sum_{j=1}^{p}\sum_{m=1}^{p}\eta_m e_{ij(k,m)} \end{array}\right)}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum\sum_{i\neq j}\tau_i\tau_j + p^2\sum_{j=1}^{p}\beta_j^2 + p\sum_{k=1}^{p}\gamma_k^2 + p\sum\sum_{k\neq l}\gamma_k\gamma_l + p\sum_{m=1}^{p}\eta_m^2 \\ +p\sum\sum_{m\neq n}\eta_m\eta_n + \sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h}e_{ij(k,m)}e_{gh(l,z)} + 2p^2\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\sum_{j=1}^{p}\beta_j \\ +2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p^2\mu\sum_{m=1}^{p}\eta_m + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p}e_{ij(k,m)} + 2p\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2\sum_{k=1}^{p}\sum_{i=1}^{p}\tau_i\gamma_k \\ +2\sum_{m=1}^{p}\sum_{i=1}^{p}\tau_i\eta_m + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i e_{ij(k,m)} + 2p\sum_{k=1}^{p}\sum_{j=1}^{p}\beta_j\gamma_k + 2p\sum_{m=1}^{p}\sum_{j=1}^{p}\beta_j\eta_m \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p}\beta_j e_{ij(k,m)} + 2\sum_{k=1}^{p}\sum_{m=1}^{p}\gamma_k\eta_m + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k,m)} + 2\sum_{j=1}^{p}\sum_{m=1}^{p}\eta_m e_{ij(k,m)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p\sum_{i=1}^{p} E(\tau_i^2) + p\sum\sum_{i\neq j} E(\tau_i\tau_j) + p^2\sum_{j=1}^{p} E(\beta_j^2) + p\sum_{k=1}^{p} E(\gamma_k^2) + p\sum\sum_{k\neq l} E(\gamma_k\gamma_l) + p\sum_{m=1}^{p} E(\eta_m^2) \\ +p\sum\sum_{m\neq n} E(\eta_m\eta_n) + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k,m)}e_{gh(l,z)}) + 2p^2\mu\sum_{i=1}^{p} E(\tau_i) + 2p^2\mu\sum_{j=1}^{p} E(\beta_j) \\ +2p^2\mu\sum_{k=1}^{p} E(\gamma_k) + 2p^2\mu\sum_{m=1}^{p} E(\eta_m) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}) + 2p\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i\beta_j) + 2\sum_{k=1}^{p}\sum_{i=1}^{p} E(\tau_i\gamma_k) \\ +2\sum_{m=1}^{p}\sum_{i=1}^{p} E(\tau_i\eta_m) + 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i e_{ij(k,m)}) + 2p\sum_{k=1}^{p}\sum_{j=1}^{p} E(\beta_j\gamma_k) + 2p\sum_{m=1}^{p}\sum_{j=1}^{p} E(\beta_j\eta_m) \\ +2p\sum_{i=1}^{p}\sum_{j=1}^{p} E(\beta_j e_{ij(k,m)}) + 2\sum_{k=1}^{p}\sum_{m=1}^{p} E(\gamma_k\eta_m) + 2\sum_{j=1}^{p}\sum_{k=1}^{p} E(\gamma_k e_{ij(k,m)}) + 2\sum_{j=1}^{p}\sum_{m=1}^{p} E(\eta_m e_{ij(k,m)}) \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + 0 + p^3\sigma_\beta^2 + p^2\sigma_\gamma^2 + 0 + p^2\sigma_\eta^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + p^3\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^2\sigma_\eta^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p^2\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma_\eta^2 + p\sigma^2$$

$$E(SSC) = E\left[\frac{\sum_{j=1}^{p} C_j^2}{p}\right] - E(C.F)$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p^2\sigma_\beta^2 + p\sigma_\gamma^2 + p\sigma_\eta^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\beta^2 - p\sigma_\gamma^2 - p\sigma_\eta^2 - \sigma^2$$

$$= p(p-1)\sigma_\beta^2 + (p-1)\sigma^2$$

$$SSG = \frac{\sum_{k=1}^{p} G_k^2}{p} - C.F$$

$$\frac{\sum_{k=1}^{p} G_k^2}{p} = \frac{\sum_{k=1}^{p}\left(\sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + \eta_m + e_{ij(k,m)}\right)\right)^2}{p}$$

$$= \frac{\sum_{k=1}^{p}\left(p\mu + \sum_{i=1}^{p}\tau_i + \sum_{j=1}^{p}\beta_j + p\gamma_k + \sum_{m=1}^{p}\eta_m + \sum_{i=1}^{p} e_{ij(k,m)}\right)^2}{p}$$

$$= \frac{\sum_{k=1}^{p}\left(\begin{array}{c} p^2\mu^2 + \sum_{i=1}^{p}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + \sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + p^2\gamma_k^2 \\ + \sum_{m=1}^{p}\eta_m^2 + \sum\sum_{m\neq n}\eta_m\eta_n + \sum_{i=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g} e_{ij(k,m)}\, e_{gj(l,n)} + 2p\mu\sum_{i=1}^{p}\tau_i \\ +2p\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\gamma_k + 2p\mu\sum_{m=1}^{p}\eta_m + 2p\mu\sum_{i=1}^{p} e_{ij(k,m)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j \\ +2p\gamma_k\sum_{i=1}^{p}\tau_i + 2\sum_{i=1}^{p}\sum_{m=1}^{p}\tau_i\eta_m + 2\sum_{i=1}^{p}\tau_i e_{ij(k,m)} + 2p\gamma_k\sum_{j=1}^{p}\beta_j \\ +2\sum_{j=1}^{p}\sum_{m=1}^{p}\eta_m\beta_j + 2\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}\beta_j + 2p\gamma_k\sum_{m=1}^{p}\eta_m + 2p\gamma_k\sum_{i=1}^{p} e_{ij(k,m)} \\ +2\sum_{i=1}^{p}\sum_{m=1}^{p}\eta_m e_{ij(k,m)} \end{array}\right)}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum\sum_{i\neq j}\tau_i\tau_j + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + p^2\sum_{k=1}^{p}\gamma_k^2 \\ +p\sum_{m=1}^{p}\eta_m^2 + p\sum\sum_{m\neq n}\eta_m\eta_n + \sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}\, e_{gh(l,n)} \\ +2p^2\mu\sum_{i=1}^{p}\tau_i + 2p^2\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p^2\mu\sum_{m=1}^{p}\eta_m + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)} \\ +2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j + 2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k\tau_i + 2\sum_{i=1}^{p}\sum_{m=1}^{p}\tau_i\eta_m + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i e_{ij(k,m)} \\ +2p\sum_{j=1}^{p}\sum_{k=1}^{p}\gamma_k\beta_j + 2\sum_{j=1}^{p}\sum_{m=1}^{p}\eta_m\beta_j + 2\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}\beta_j + 2p\sum_{m=1}^{p}\sum_{k=1}^{p}\gamma_k\eta_m \\ +2p\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k,m)} + 2\sum_{i=1}^{p}\sum_{m=1}^{p}\eta_m e_{ij(k,m)} \end{array}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{gathered} p^3\mu^2 + p\sum_{i=1}^{p} E(\tau_i^2) + p\sum\sum_{i\neq j} E(\tau_i\tau_j) + p\sum_{j=1}^{p} E(\beta_j^2) + p\sum\sum_{i\neq j} E(\beta_i\beta_j) + p^2\sum_{k=1}^{p} E(\gamma_k^2) \\ +p\sum_{m=1}^{p} E(\eta_m^2) + p\sum\sum_{m\neq n} E(\eta_m\eta_n) + \sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k,m)}\, e_{gh(l,n)}) \\ +2p^2\mu\sum_{i=1}^{p} E(\tau_i) + 2p^2\mu\sum_{j=1}^{p} E(\beta_j) + 2p^2\mu\sum_{k=1}^{p} E(\gamma_k) + 2p^2\mu\sum_{m=1}^{p} E(\eta_m) + 2p\mu\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}) \\ +2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i\beta_j) + 2p\sum_{i=1}^{p}\sum_{k=1}^{p} E(\gamma_k\tau_i) + 2\sum_{i=1}^{p}\sum_{m=1}^{p} E(\tau_i\eta_m) + 2\sum_{i=1}^{p}\sum_{k=1}^{p} E(\tau_i e_{ij(k,m)}) \\ +2p\sum_{j=1}^{p}\sum_{k=1}^{p} E(\gamma_k\beta_j) + 2\sum_{j=1}^{p}\sum_{m=1}^{p} E(\eta_m\,\beta_j) + 2\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}\,\beta_j) + 2p\sum_{m=1}^{p}\sum_{k=1}^{p} E(\gamma_k\eta_m) \\ +2p\sum_{i=1}^{p}\sum_{k=1}^{p} E(\gamma_k e_{ij(k,m)}) + 2\sum_{i=1}^{p}\sum_{m=1}^{p} E(\eta_m\, e_{ij(k,m)}) \end{gathered}}{p}$$

$$= \frac{\begin{gathered} p^3\mu^2 + p^2\sigma_\tau^2 + 0 + p^2\sigma_\beta^2 + 0 + p^3\sigma_\gamma^2 + p^2\sigma_\eta^2 + 0 + p^2\sigma^2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 \\ +0 + 0 + 0 + 0 + 0 + 0 \end{gathered}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + p^2\sigma_\beta^2 + p^3\sigma_\gamma^2 + p^2\sigma_\eta^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p\sigma_\beta^2 + p^2\sigma_\gamma^2 + p\sigma_\eta^2 + p\sigma^2$$

$$E(SSG) = E\left[\frac{\sum_{k=1}^{p} G_k^2}{p}\right] - E(C.F)$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p\sigma_\beta^2 + p^2\sigma_\gamma^2 + p\sigma_\eta^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\beta^2 - p\sigma_\gamma^2 - p\sigma_\eta^2 - \sigma^2$$

$$= p(p-1)\sigma_\gamma^2 + (p-1)\sigma^2$$

$$SSL = \frac{\sum_{m=1}^{p} L_m^2}{p} - C.F$$

$$\frac{\sum_{m=1}^{p} L_m^2}{p} = \frac{\sum_{m=1}^{p}\left(\sum_{i=1}^{p}\left(\mu + \tau_i + \beta_j + \gamma_k + \eta_m + e_{ij(k,m)}\right)\right)^2}{p}$$

$$= \frac{\sum_{m=1}^{p}\left(p\mu + \sum_{i=1}^{p}\tau_i + \sum_{j=1}^{p}\beta_j + \sum_{k=1}^{p}\gamma_k + p\eta_m + \sum_{i=1}^{p} e_{ij(k,m)}\right)^2}{p}$$

$$= \frac{\sum_{m=1}^{p}\left(\begin{gathered} p^2\mu^2 + \sum_{i=1}^{p}\tau_i^2 + \sum\sum_{i\neq j}\tau_i\tau_j + \sum_{j=1}^{p}\beta_j^2 + \sum\sum_{i\neq j}\beta_i\beta_j + \sum_{k=1}^{p}\gamma_k^2 \\ + \sum\sum_{k\neq l}\gamma_k\gamma_l + p^2\eta_m^2 + \sum_{i=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g} e_{ij(k,m)}\, e_{gj(l,n)} + 2p\mu\sum_{i=1}^{p}\tau_i \\ +2p\mu\sum_{j=1}^{p}\beta_j + 2p\mu\sum_{k=1}^{p}\gamma_k + 2p^2\mu\eta_m + 2p\mu\sum_{i=1}^{p} e_{ij(k,m)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j \\ +2\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2p\eta_m\sum_{i=1}^{p}\tau_i + 2\sum_{i=1}^{p}\tau_i e_{ij(k,m)} + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\gamma_k\,\beta_j \\ +2p\eta_m\sum_{j=1}^{p}\beta_j + 2\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}\,\beta_j + 2p\eta_m\sum_{k=1}^{p}\gamma_k + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k,m)} \\ +2p\eta_m\sum_{i=1}^{p} e_{ij(k,m)} \end{gathered}\right)}{p}$$

$$= \frac{\begin{gathered} p^3\mu^2 + p\sum_{i=1}^{p}\tau_i^2 + p\sum\sum_{i\neq j}\tau_i\tau_j + p\sum_{j=1}^{p}\beta_j^2 + p\sum\sum_{i\neq j}\beta_i\beta_j + p\sum_{k=1}^{p}\gamma_k^2 \\ +p\sum\sum_{k\neq l}\gamma_k\gamma_l + p^2\sum_{m=1}^{p}\eta_m^2 + \sum_{i=1}^{p}\sum_{m=1}^{p} e_{ij(k,m)}^2 + \sum_{i\neq g}\sum_{j\neq h} e_{ij(k,m)}\, e_{gj(l,n)} + 2p^2\mu\sum_{i=1}^{p}\tau_i \\ +2p^2\mu\sum_{j=1}^{p}\beta_j + 2p^2\mu\sum_{k=1}^{p}\gamma_k + 2p^2\mu\sum_{m=1}^{p}\eta_m + 2p\mu\sum_{i=1}^{p}\sum_{m=1}^{p} e_{ij(k,m)} + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\tau_i\beta_j \\ +2\sum_{i=1}^{p}\sum_{k=1}^{p}\tau_i\gamma_k + 2p\sum_{i=1}^{p}\sum_{m=1}^{p}\tau_i\,\eta_m + 2\sum_{i=1}^{p}\sum_{m=1}^{p}\tau_i e_{ij(k,m)} + 2\sum_{j=1}^{p}\sum_{k=1}^{p}\gamma_k\,\beta_j \\ +2p\sum_{j=1}^{p}\sum_{m=1}^{p}\eta_m\beta_j + 2\sum_{i=1}^{p}\sum_{j=1}^{p} e_{ij(k,m)}\,\beta_j + 2p\sum_{k=1}^{p}\sum_{m=1}^{p}\eta_m\gamma_k + 2\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_k e_{ij(k,m)} \\ +2p\sum_{i=1}^{p}\sum_{m=1}^{p}\eta_m e_{ij(k,m)} \end{gathered}}{p}$$

Apply expectation on both sides

$$= \frac{\begin{array}{c} p^3\mu^2 + p\sum_{i=1}^{p} E(\tau_i^2) + p\sum\sum_{i\neq j} E(\tau_i\tau_j) + p\sum_{j=1}^{p} E(\beta_j^2) + p\sum\sum_{i\neq j} E(\beta_i\beta_j) + p\sum_{k=1}^{p} E(\gamma_k^2) \\ +p\sum\sum_{k\neq l} E(\gamma_k\gamma_l) + p^2\sum_{m=1}^{p} E(\eta_m^2) + \sum_{i=1}^{p}\sum_{m=1}^{p} E(e_{ij(k,m)}^2) + \sum_{i\neq g}\sum_{j\neq h} E(e_{ij(k,m)}\, e_{gj(l,n)}) \\ +2p^2\mu\sum_{i=1}^{p} E(\tau_i) + 2p^2\mu\sum_{j=1}^{p} E(\beta_j) + 2p^2\mu\sum_{k=1}^{p} E(\gamma_k) + 2p^2\mu\sum_{m=1}^{p} E(\eta_m) \\ +2\sum_{i=1}^{p}\sum_{j=1}^{p} E(\tau_i\beta_j) + 2p\mu\sum_{i=1}^{p}\sum_{m=1}^{p} E(e_{ij(k,m)}) + 2\sum_{i=1}^{p}\sum_{k=1}^{p} E(\tau_i\gamma_k) + 2p\sum_{i=1}^{p}\sum_{m=1}^{p} E(\tau_i\,\eta_m) \\ +2\sum_{i=1}^{p}\sum_{m=1}^{p} E(\tau_i e_{ij(k,m)}) + 2\sum_{j=1}^{p}\sum_{k=1}^{p} E(\gamma_k\,\beta_j) + 2p\sum_{j=1}^{p}\sum_{m=1}^{p} E(\eta_m\beta_j) \\ +2\sum_{i=1}^{p}\sum_{j=1}^{p} E(e_{ij(k,m)}\,\beta_j) + 2p\sum_{k=1}^{p}\sum_{m=1}^{p} E(\eta_m\gamma_k) + 2\sum_{i=1}^{p}\sum_{k=1}^{p} E(\gamma_k e_{ij(k,m)}) \\ +2p\sum_{i=1}^{p}\sum_{m=1}^{p} E(\eta_m e_{ij(k,m)}) \end{array}}{p}$$

$$= \frac{\begin{array}{c} p^3\mu^2 + p^2\sigma_\tau^2 + 0 + p^2\sigma_\beta^2 + 0 + p^2\sigma_\gamma^2 + 0 + p^3\sigma_\eta^2 + p^2\sigma^2 + 0+0+0+0+0+0+0+0+0 \\ +0+0+0+0+0+0+0 \end{array}}{p}$$

$$= \frac{p^3\mu^2 + p^2\sigma_\tau^2 + p^2\sigma_\beta^2 + p^2\sigma_\gamma^2 + p^3\sigma_\eta^2 + p^2\sigma^2}{p}$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p\sigma_\beta^2 + p\sigma_\gamma^2 + p^2\sigma_\eta^2 + p\sigma^2$$

$$E(SSL) = E\left[\frac{\sum_{m=1}^{p} L_m^2}{p}\right] - E[C.F]$$

$$= p^2\mu^2 + p\sigma_\tau^2 + p\sigma_\beta^2 + p\sigma_\gamma^2 + p^2\sigma_\eta^2 + p\sigma^2 - p^2\mu^2 - p\sigma_\tau^2 - p\sigma_\beta^2 - p\sigma_\gamma^2 - p\sigma_\eta^2 - \sigma^2$$

$$= p(p-1)\sigma_\eta^2 + (p-1)\sigma^2$$

$$E(SSE) = E(TSS) - E(SSR) - E(SSC) - E(SSG) - E(SSL)$$

$$= p(p-1)\sigma_\tau^2 + p(p-1)\sigma_\beta^2 + p(p-1)\sigma_\gamma^2 + p(p-1)\sigma_\eta^2 + (p^2-1)\sigma^2 - p(p-1)\sigma_\tau^2 - (p-1)\sigma^2$$
$$-p(p-1)\sigma_\beta^2 - (p-1)\sigma^2 - p(p-1)\sigma_\gamma^2 - (p-1)\sigma^2 - p(p-1)\sigma_\eta^2 - (p-1)\sigma^2$$

$$= (p^2 - 1 - p + 1 - p + 1 - p + 1 - p + 1)\sigma^2$$

$$= (p^2 - 4p + 3)\sigma^2 = (p^2 - 3p - p + 3)\sigma^2$$

$$= (p(p-3) - 1(p-3))\sigma^2$$

$$= (p-3)(p-1)\sigma^2$$

$$E(MSE) = \frac{E(SSE)}{(p-1)(p-3)} = \frac{(p-3)(p-1)\sigma^2}{(p-1)(p-3)} = \sigma^2$$

$$E(MSR) = \frac{E(SSR)}{p-1} = \frac{p(p-1)\sigma_\tau^2 + (p-1)\sigma^2}{p-1} = p\sigma_\tau^2 + \sigma^2$$

$$E(MSC) = \frac{E(SSC)}{p-1} = \frac{p(p-1)\sigma_\beta^2 + (p-1)\sigma^2}{p-1} = p\sigma_\beta^2 + \sigma^2$$

$$E(MSG) = \frac{E(SSG)}{p-1} = \frac{p(p-1)\sigma_\gamma^2 + (p-1)\sigma^2}{p-1} = p\sigma_\gamma^2 + \sigma^2$$

$$E(MSL) = \frac{E(SSL)}{p-1} = \frac{p(p-1)\sigma_\eta^2 + (p-1)\sigma^2}{p-1} = p\sigma_\eta^2 + \sigma^2$$