Runge-Kutta Methods

RK Method of Order 2 :

We begin with Rungu-kutta method of order 2.

The RK methods have the general form:

$$y_{n+1} = y_n + hF(x_n, y_n ; h) + O(h^2) \quad ; \quad n \geq 0, \quad y_0 = Y_0 \qquad (1).$$

The quantity $F(x_n, y_n ; h)$ can be thought of as some kind of "average slope" of the solution on the interval $[x_n, x_{n+1}]$. But its construction is based on making (1) act like a Taylor's method.

For methods of order 2, we generally have

$$F(x, y ; h) = \gamma_1 f(x, y) + \gamma_2 f\big(x + \alpha h, y + \beta h f(x, y)\big) \qquad (2).$$

and determine the constants $\{\alpha, \beta, \gamma_1, \gamma_2\}$.

This method is of order $O(h^3)$, just as Taylor's method of order 2.

Runge-Kutta methods have the high-order local Truncation error of the Taylor methods but eliminate the need to compute and Evaluate the derivatives of $f(t,y)$.

Before presenting the ideas, we need to consider Taylor's Theorem in two variables.

<u>THEOREM</u>: Suppose that $f(t,y)$ and all its partial derivatives of order less than or equal to $n+1$ are continuous on

$$D = \{(t,y) \mid a \le t \le b;\ c \le y \le d\}$$

and let $(t_0, y_0) \in D$. For every $(t,y) \in D$, there exists c between t & t_0 and μ between y & y_0 with

$$f(t,y) = P_n(t,y) + R_n(t,y)$$

where

$$P_n(t,y) = f(t_0,y_0) + \left[(t-t_0)\frac{\partial f}{\partial t}(t_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)\right]$$

$$+ \left[\frac{(t-t_0)^2}{2!}\frac{\partial^2 f}{\partial t^2}(t_0,y_0) + (t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t\,\partial y}(t_0,y_0)\right.$$

$$\left. + \frac{(y-y_0)^2}{2!}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right] + (\text{P.T.O})$$

$$+\left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t-t_0)^{n-j}(y-y_0)^{j}\frac{\partial^n f(t_0,y_0)}{\partial t^{n-j}\partial y^j}\right] \longrightarrow (3)$$

and

$$R_n(t,y)=\frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(t-t_0)^{n+1}(y-y_0)^{j}\frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(t_0,y_0) \longrightarrow (4)$$

The function $P_n(t,y)$ is called the n^{th} Taylor's polynomial in two variables for the function f about (t_0, y_0) and $R_n(t,y)$ is the remainder term associated with $P_n(t,y)$.

Taylor's Method of order 2:

We know that Taylor's method of order 2 is

$$y_{n+1} = y_n + h\left[T^{(2)}(t,y) + O(h^2)\right].$$

or

$$y_{n+1} = y_n + h\left[f(t_n,y_n) + \frac{h}{2}f'(t_n,y_n)\right] + O(h^3). \longrightarrow (5)$$

As $$\frac{df}{dt} = f'(t,y) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)\cdot y'(t)$$

$$= \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)\, f(t,y).$$

$\because y' = f(t,y).$

Thus (5) can be written as:

$$y_{n+1} = y_n + h\left[f(t_n, y_n) + \frac{h}{2}\frac{\partial f}{\partial t}(t_n, y_n) + \frac{h}{2}\frac{\partial f}{\partial y}(t_n, y_n) \cdot f(t_n, y_n)\right] + O(h^3) \quad (6)$$

We can expand $\gamma_2 f(t+\alpha h, y+\beta h f(t,y))$ in equation (2) using Taylor's polynomial of degree one about (t_n, y_n) as:

$$\gamma_2 f(t_n+\alpha h, y_n+\beta h f(t_n, y_n)) = \gamma_2 f(t_n, y_n) + \gamma_2 \alpha h \frac{\partial f}{\partial t}(t_n, y_n) + \gamma_2 \beta h f(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n) + O(h^2) \quad (7)$$

Using (7) in equation (2) ~~gives~~ and then in (1) gives

$$y_{n+1} = y_n + h\left[\gamma_1 f(t_n, y_n) + \gamma_2 f(t_n, y_n) + \gamma_2 \alpha h \frac{\partial f}{\partial t}(t_n, y_n) + \gamma_2 \beta h f(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n) + O(h^2)\right]$$

Or

$$y_{n+1} = y_n + h(\gamma_1+\gamma_2) f(t_n, y_n) + \gamma_2 \alpha h^2 \frac{\partial f}{\partial t}(t_n, y_n) + \gamma_2 \beta h^2 \frac{\partial f}{\partial y}(t_n, y_n) \cdot f(t_n, y_n) + O(h^3) \longrightarrow (8)$$

Comparison of (6) & (8) yields:

$$\left.\begin{aligned} \gamma_1 + \gamma_2 &= 1. \\ \gamma_2 \alpha &= \frac{1}{2} \\ \gamma_2 \beta &= \frac{1}{2} \end{aligned}\right\} \longrightarrow (9)$$

Since there are only three equations in four unknowns, the system (9) have infinite solutions. We are permitted to choose one of the coefficients. There are several special choices that have been studied in literature.

We mention two choices:

<u>Case-(i)</u>: $\gamma_1 = \frac{1}{2} \Rightarrow \gamma_2 = \frac{1}{2}$

$$\Rightarrow \alpha = 1$$

$$\& \quad \beta = 1.$$

Thus we have: (Using (1) & (2)):

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f\left(t_n + h_n, y_n + h f(t_n, y_n)\right)\right]$$

Which is called RK method of order 2. (Same as Heun's Method).

CASE-(ii): $\gamma_1 = 0 \Rightarrow \gamma_2 = 1, \ \alpha = \frac{1}{2}, \ \beta = \frac{1}{2}$.

Thus we can write (using (1) & (2)):

$$y_{n+1} = y_n + h f\left(t_n + \frac{h}{2}, \ y_n + \frac{h}{2} f(t_n, y_n)\right)$$

Which is called the Modified Euler Cauchy method.

Using the comparison of Taylor's method of order 4 with the order 4 expansion of (2), we can derive RK method of order 4

Runge-Kutta Method Order 4:

The method has local Truncation error $O(h^4)$, provided the solution $y(t)$ has five continuous derivatives.

$$K_1 = hf(t_k, y_k)$$

$$K_2 = hf\left(t_k + \frac{h}{2}, \ y_k + \frac{1}{2} K_1\right)$$

$$K_3 = hf\left(t_k + \frac{h}{2}, \ y_k + \frac{1}{2} K_2\right)$$

$$K_4 = hf(t_k + h, \ y_k + K_3)$$

$$y_{k+1} = y_k + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

$k = 0, 1, 2, \ldots, n-1$.

Example: Use RK4 with $h = 0.2$, $N = 10$ & $t_k = 0.2k$ to obtain approximations to the solution of the initial value problem

$$y' = y - t^2 + 1 \;,\; 0 \le t \le 2 \;,\; y(0) = 0.5 .$$

Sol: As $y_0 = 0.5$, $t_0 = 0$. $h = \frac{2-0}{10} = 0.2$.

$t_k = 0 + 0.2k$. ; $k = 0, 1, 2, 3, \ldots, 9$.

$k = 0$:

$$y_1 = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$K_1 = 0.2\, f(0, 0.5) = 0.2\left[0.5 - 0 + 1\right] = 0.3 .$$

$$= h\, f(t_0, y_0) = h\left[y_0 - t_0^2 + 1\right]$$

$$K_2 = h\, f\left(t_0 + \frac{h}{2}, y_0 + \frac{1}{2}K_1\right)$$

$$= h\left[y_0 + \frac{1}{2}K_1 - \left(t_0 + \frac{h}{2}\right)^2 + 1\right]$$

$$= (0.2)\left[0.5 + \frac{1}{2}(0.3) - \left(0 + \frac{0.2}{2}\right)^2 + 1\right]$$

$$K_2 = 0.328 .$$

$$K_3 = h\, f\left(t_0 + \frac{h}{2}, y_0 + \frac{1}{2}K_2\right)$$

$$= 0.2\left[y_0 + \frac{1}{2}K_2 - \left(t_0 + \frac{h}{2}\right)^2 + 1\right]$$

$$= 0.2\left[0.5 + \frac{1}{2}(0.328) - \left(0 + \frac{0.2}{2}\right)^2 + 1\right]$$

$$K_3 = 0.3308 .$$

$$K_4 = hF(t_0+h,\ y_0+K_3)$$

$$= h\left[y_0 + K_3 - (t_0+h)^2 + 1\right]$$

$$= (0.2)\left[0.5 + 0.3308 - (0+0.2)^2 + 1\right]$$

$$K_4 = 0.35816$$

$$y_1 = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$= 0.5 + \frac{1}{6}\left[0.3 + 2(0.328) + 2(0.3308) + 0.35816\right]$$

$$y_1 = 0.8292933$$

$\underline{\underline{k=1}}$: $\quad y_2 = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

& So on.

EXERCISES: Approximate the solutions to each of the following IVP's and compare the results to the actual value

(i) $y' = te^{3t} - 2y \; ; \; 0 \le t \le 1 \, , \; y(0) = 0 \, , \; h = 0.5 .$

$$y_{exact}(t) = \frac{1}{3} t e^{3t} - \frac{1}{25} e^{3t} + \frac{1}{25} e^{-2t} .$$

(ii) $y' = \cos 2t + \sin 3t \, , \quad 0 \le t \le 1 \, , \; y(0) = 1 \, , \; h = 0.25 ,$

$$y_{exact}(t) = \frac{1}{2} \sin 2t - \frac{1}{3} \cos 3t + \frac{4}{3} .$$

(iii) $y' = \dfrac{1+t}{1+y} \; ; \quad 1 \le t \le 2 \, , \; y(1) = 2 \, , \; h = 0.5 .$

$$y_{exact}(t) = \sqrt{t^2 + 2t + 6} - 1 .$$

Using the following Methods.

(a) Taylor's Method of Order 3.

(b) Modified Euler Method.

(c) Heun's Method.

(d) RK method of Order 4.

Systems of Differential Equations:

Consider the m^{th} order system of first order initial value problems:

$$
\left.\begin{aligned}
\frac{du_1}{dt} &= f_1(t, u_1, u_2, \cdots, u_m) \\
\frac{du_2}{dt} &= f_2(t, u_1, u_2, \cdots, u_m) \\
&\vdots \\
\frac{du_m}{dt} &= f_m(t, u_1, u_2, \cdots, u_m)
\end{aligned}\right\} \longrightarrow (1).
$$

for all $a \le t \le b$, with initial conditions:

$$u_1(a) = \alpha_1, \; u_2(a) = \alpha_2, \; \cdots, \; u_m(a) = \alpha_m \longrightarrow (2).$$

The objective is to find m functions

$$u_1(t), u_2(t), \cdots, u_m(t)$$

That satisfy each of the diff. equation together with initial condition (2).

REMARK: See Lipschitz-Conditions & Existence & Uniquess theorem in Book.

Methods to solve systems of IVPs (1-2) are generalizations of the methods for a single IVP. for example:

Euler Method for System (1-2):

$$t_{k+1} = t_0 + kh, \quad h = \frac{b-a}{N}, \qquad a \le t \le b.$$

$$u_i(a) = \alpha_i.$$

$$u_i^{k+1} = u_i^k + h f(t_k, u_i^k)$$

$$\forall\ i = 1, 2, \dots, m.$$
$$\&\ k = 0, 1, \dots, N-1.$$

OR

$$\begin{aligned}
u_1^{k+1} &= u_1^k + h f(t_k, u_1^k) \\
u_2^{k+1} &= u_2^k + h f(t_k, u_2^k) \\
&\ \vdots \\
u_m^{k+1} &= u_m^k + h f(t_k, u_m^k).
\end{aligned}$$

RK Method of Order 4 for System:

Let an integer $N>0$ be chosen and set $h=\frac{b-a}{N}$.

Partition the interval $[a,b]$ into N subintervals with the mesh points

$$t_j = a + jh \quad ; \; j=0,1,2,\dots,N.$$

Use the notation u_{ij} ; for each $j=0,1,\dots,N$ and $i=1,2,\dots,m$ to denote approximation to $u_i(t_j)$

That is, u_{ij} approximates the ith solution $u_i(t)$ of the system (1-2) at the jth mesh point t_j.

For initial conditions set

$$u_{1,0}=\alpha_1 \;,\; u_{2,0}=\alpha_2 \;,\dots,\; u_{m,0}=\alpha_m .$$

Suppose that the values $u_{1,j}, u_{2,j}, \dots, u_{m,j}$ have been computed. We obtain $u_{1,j+1}, u_{2,j+1}, \dots, u_{m,j+1}$ by first calculating:

$$K_{1,i} = h f_i\left(t_j, w_{1,j}, u_{2,j}, \dots, u_{m,j}\right) ; \; i=1,2,\dots,m$$

$$K_{2,i} = h f_i\left(t_j+\frac{h}{2}, w_{1,j}+\frac{1}{2}K_{1,1}, w_{2,j}+\frac{1}{2}K_{1,2}, \dots, w_{m,j}+\frac{1}{2}K_{1,m}\right)$$

$$K_{3,i} = hf_i\left(t_j + \frac{h}{2}, u_{1,j} + \frac{1}{2}K_{2,1}, u_{2,j} + \frac{1}{2}K_{2,2}, \cdots, u_{m,j} + \frac{1}{2}K_{2,m}\right)$$

$$K_{4,i} = hf_i\left(t_j + h, u_{1,j} + K_{3,1}, u_{2,j} + K_{3,2}, \cdots, u_{m,j} + K_{3,m}\right).$$

for each $i = 1, 2, \cdots, m$,

and THEN

$$u_{i,j+1} = u_{i,j} + \frac{1}{6}\left[K_{1,i} + 2K_{2,i} + 2K_{3,i} + K_{4,i}\right]$$

$$\forall\ i = 1, 2, \cdots, m.$$

<u>EXERCISES</u>: Use RK4 method for systems to approximate the solutions of the following systems

(i) $u_1' = 3u_1 + 2u_2 - (2t^2+1)e^{2t}$; $u_1(0) = 1$

$u_2' = 4u_1 + u_2 + (t^2 + 2t - 4)e^{2t}$; $u_2(0) = 1$

$0 \le t \le 1$, $h = 0.2$.

(ii) $u_1' = -4u_1 - 2u_2 + \cos t + 4\sin t$; $u_1(0) = 0$.

$u_2' = 3u_1 + u_2 - 3\sin t$; $u_2(0) = -1$,

$0 \le t \le 2$, $h = 0.1$

Higher-Order Differential Equations:

A general mth order initial-value problem has the form:

$$y^{(m)}(t) = f(t, y, y', \ldots, y^{(m-1)}) \quad ; \quad a \le t \le b \qquad \rightarrow (3)$$

with Initial-value conditions:

$$y(a) = \alpha_1 \ , \ y'(a) = \alpha_2 \ , \ \ldots \ , \ y^{(m-1)}(a) = \alpha_m \ . \qquad \rightarrow (4)$$

The IVP of order m (3-4) can be converted into a system of first order Initial-value problems (like equations (1-2) on page (10)):

Let $u_1(t) = y(t)$, $u_2(t) = y'(t)$, ..., $u_m(t) = y^{(m-1)}(t)$.

This produces the first-order system:

$$u_1(t) = y(t).$$

$$\frac{du_1}{dt} = \frac{dy}{dt} = u_2$$

$$\frac{du_2}{dt} = \frac{dy'}{dt} = \frac{d^2y}{dt^2} = u_3$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\frac{du_{m-1}}{dt} = \frac{dy^{(m-2)}}{dt} = \frac{d^{(m-2)}y}{dt^{m-2}} = u_m .$$

Using above expressions, we can write

$$\frac{du_m}{dt} = \frac{dy^{(m-1)}}{dt} = y^{(m)} = f(t, y, y', ---, y^{(m-1)})$$

or

$$\frac{du_m}{dt} = f(t, u_1, u_2, ---, u_m)$$

with initial Conditions :

$$u_1(a) = y(a) = \alpha_1 , \ u_2(a) = y'(a) = \alpha_2 , ----, u_m(a) = y^{(m-1)}(a) = \alpha_m .$$

Example Transform the second order Initial-Value problem

$$y'' - 2y' + 2y = e^{2t}\sin t \quad ; \ 0 \le t \le 1$$

with $y(0) = -0.4$, $y'(0) = -0.6$

Into a system of first order Initial-Value problems and use RK4 method with $h = 0.1$ to approximate the solution.

Sol: Let $u_1(t) = y(t)$, $u_2(t) = y'(t)$.

This transforms the second-order equation into system

$$u_1'(t) = u_2(t)$$
$$u_2'(t) = e^{2t}\sin t - 2u_1 + 2u_2 .$$

with IC's $u_1(0) = -0.4$, $u_2(0) = -0.6$.

Now we need to solve the above system using RK4. (Do yourself).

EXERCISES: Transform the following (IVP's) ODEs into system of first order IVP's

(i) $y''' + 2y'' - y' - 2y = e^t$; $0 \le t \le 3$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 0$ with $h = 0.2$.

(ii) $t^3 y''' - t^2 y'' + 3ty' - 4y = 5t^3 \ln t + 9t^3$

$1 \le t \le 2$, $y(1) = 0$, $y'(1) = 1$, $y''(1) = 3$.

with $h = 0.1$.

Using RK4, find the app. solution.