22 GAUSS'S LAW

? This child acquires an electric charge by touching the charged metal sphere. The charged hairs on the child's head repel and stand out. If the child stands *inside* a large, charged metal sphere, will her hair stand on end?

LEARNING GOALS

By studying this chapter, you will learn:

- How you can determine the amount of charge within a closed surface by examining the electric field on the surface.
- What is meant by electric flux, and how to calculate it.
- How Gauss's law relates the electric flux through a closed surface to the charge enclosed by the surface.
- How to use Gauss's law to calculate the electric field due to a symmetric charge distribution.
- Where the charge is located on a charged conductor.

Often, there are both an easy way and a hard way to do a job; the easy way may involve nothing more than using the right tools. In physics, an important tool for simplifying problems is the *symmetry properties* of systems. Many physical systems have symmetry; for example, a cylindrical body doesn't look any different after you've rotated it around its axis, and a charged metal sphere looks just the same after you've turned it about any axis through its center.

Gauss's law is part of the key to using symmetry considerations to simplify electric-field calculations. For example, the field of a straight-line or plane-sheet charge distribution, which we derived in Section 21.5 using some fairly strenuous integrations, can be obtained in a few lines with the help of Gauss's law. But Gauss's law is more than just a way to make certain calculations easier. Indeed, it is a fundamental statement about the relationship between electric charges and electric fields. Among other things, Gauss's law can help us understand how electric charge distributes itself over conducting bodies.

Here's what Gauss's law is all about. Given any general distribution of charge, we surround it with an imaginary surface that encloses the charge. Then we look at the electric field at various points on this imaginary surface. Gauss's law is a relationship between the field at *all* the points on the surface and the total charge enclosed within the surface. This may sound like a rather indirect way of expressing things, but it turns out to be a tremendously useful relationship. Above and beyond its use as a calculational tool, Gauss's law can help us gain deeper insights into electric fields. We will make use of these insights repeatedly in the next several chapters as we pursue our study of electromagnetism.

22.1 Charge and Electric Flux

In Chapter 21 we asked the question, "Given a charge distribution, what is the electric field produced by that distribution at a point P?" We saw that the answer could be found by representing the distribution as an assembly of point charges,

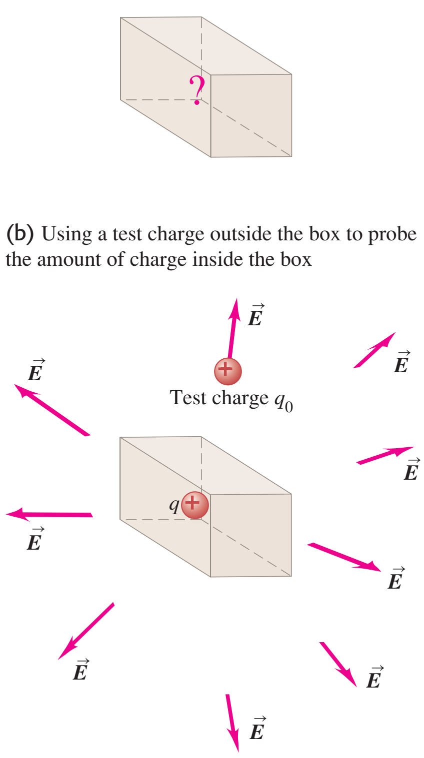
The discussion of Gauss's law in this section is based on and inspired by the innovative ideas of Ruth W. Chabay and Bruce A. Sherwood in *Electric and Magnetic Interactions* (John Wiley & Sons, 1994).

MasteringPHYSICS
ActivPhysics 11.7: Electric Flux

22.1 How can you measure the charge inside a box without opening it?

(a) A box containing an unknown amount of charge

(b) Using a test charge outside the box to probe the amount of charge inside the box



each of which produces an electric field $\vec{E}$ given by Eq. (21.7). The total field at P is then the vector sum of the fields due to all the point charges.

But there is an alternative relationship between charge distributions and electric fields. To discover this relationship, let's stand the question of Chapter 21 on its head and ask, "If the electric field pattern is known in a given region, what can we determine about the charge distribution in that region?"

Here's an example. Consider the box shown in Fig. 22.1a, which may or may not contain electric charge. We'll imagine that the box is made of a material that has no effect on any electric fields; it's of the same breed as the massless rope and the frictionless incline. Better still, let the box represent an *imaginary* surface that may or may not enclose some charge. We'll refer to the box as a **closed surface** because it completely encloses a volume. How can you determine how much (if any) electric charge lies within the box?

Knowing that a charge distribution produces an electric field and that an electric field exerts a force on a test charge, you move a test charge q_0 around the vicinity of the box. By measuring the force $\vec{F}$ experienced by the test charge at different positions, you make a three-dimensional map of the electric field $\vec{E} = \vec{F}/q_0$ outside the box. In the case shown in Fig. 22.1b, the map turns out to be the same as that of the electric field produced by a positive point charge (Fig. 21.28a). From the details of the map, you can find the exact value of the point charge inside the box.

To determine the contents of the box, we actually need to measure $\vec{E}$ only on the *surface* of the box. In Fig. 22.2a there is a single *positive* point charge inside the box, and in Fig. 22.2b there are two such charges. The field patterns on the surfaces of the boxes are different in detail, but in each case the electric field points *out* of the box. Figures 22.2c and 22.2d show cases with one and two *negative* point charges, respectively, inside the box. Again, the details of $\vec{E}$ are different for the two cases, but the electric field points *into* each box.

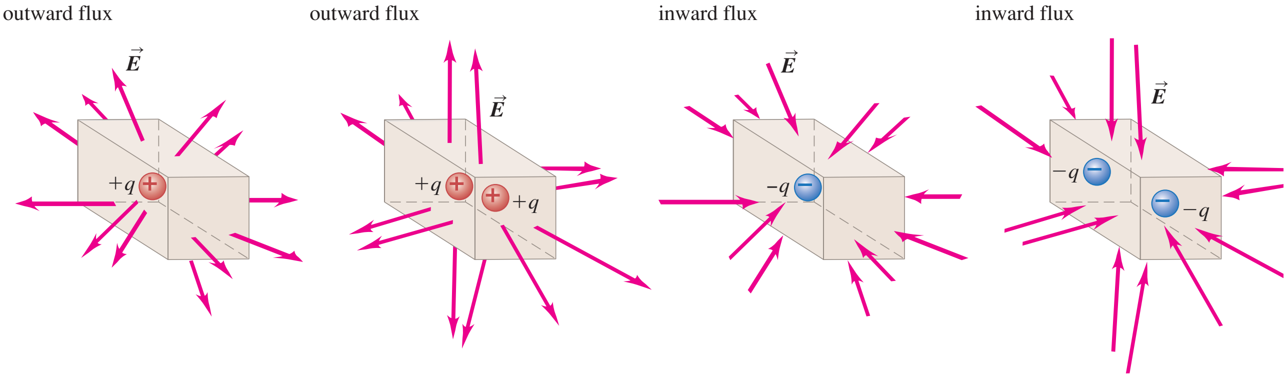
Electric Flux and Enclosed Charge

In Section 21.4 we mentioned the analogy between electric-field vectors and the velocity vectors of a fluid in motion. This analogy can be helpful, even though an electric field does not actually "flow." Using this analogy, in Figs. 22.2a and 22.2b, in which the electric field vectors point out of the surface, we say that there is an *outward* **electric flux.** (The word "flux" comes from a Latin word meaning "flow.") In Figs. 22.2c and 22.2d the $\vec{E}$ vectors point into the surface, and the electric flux is *inward.*

Figure 22.2 suggests a simple relationship: Positive charge inside the box goes with an outward electric flux through the box's surface, and negative charge inside goes with an inward electric flux. What happens if there is *zero* charge

22.2 The electric field on the surface of boxes containing **(a)** a single positive point charge, **(b)** two positive point charges, **(c)** a single negative point charge, or **(d)** two negative point charges.

(a) Positive charge inside box, outward flux
(b) Positive charges inside box, outward flux
(c) Negative charge inside box, inward flux
(d) Negative charges inside box, inward flux



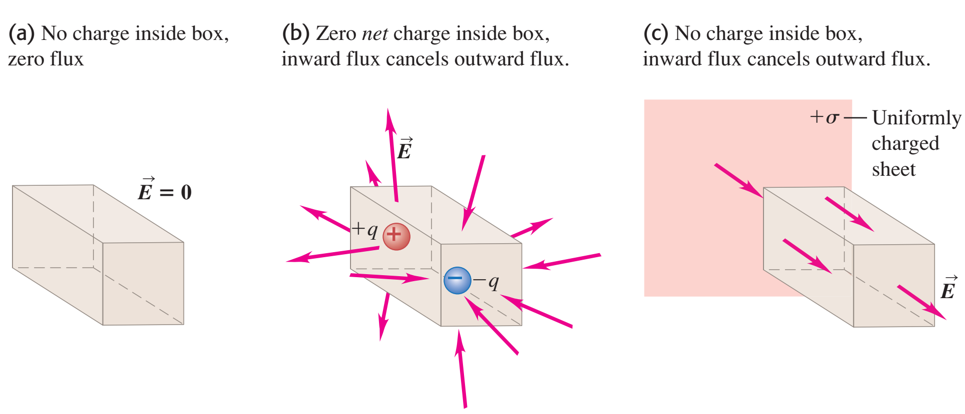
inside the box? In Fig. 22.3a the box is empty and $\vec{E} = \mathbf{0}$ everywhere, so there is no electric flux into or out of the box. In Fig. 22.3b, one positive and one negative point charge of equal magnitude are enclosed within the box, so the *net* charge inside the box is zero. There is an electric field, but it "flows into" the box on half of its surface and "flows out of" the box on the other half. Hence there is no *net* electric flux into or out of the box.

The box is again empty in Fig. 22.3c. However, there is charge present *outside* the box; the box has been placed with one end parallel to a uniformly charged infinite sheet, which produces a uniform electric field perpendicular to the sheet (as we learned in Example 21.11 of Section 21.5). On one end of the box, $\vec{E}$ points into the box; on the opposite end, $\vec{E}$ points out of the box; and on the sides, $\vec{E}$ is parallel to the surface and so points neither into nor out of the box. As in Fig. 22.3b, the inward electric flux on one part of the box exactly compensates for the outward electric flux on the other part. So in all of the cases shown in Fig. 22.3, there is no *net* electric flux through the surface of the box, and no *net* charge is enclosed in the box.

Figures 22.2 and 22.3 demonstrate a connection between the *sign* (positive, negative, or zero) of the *net* charge enclosed by a closed surface and the sense (outward, inward, or none) of the net electric flux through the surface. There is also a connection between the *magnitude* of the net charge inside the closed surface and the *strength* of the net "flow" of $\vec{E}$ over the surface. In both Figs. 22.4a and 22.4b there is a single point charge inside the box, but in Fig. 22.4b the magnitude of the charge is twice as great, and so $\vec{E}$ is everywhere twice as great in magnitude as in Fig. 22.4a. If we keep in mind the fluid-flow analogy, this means that the net outward electric flux is also twice as great in Fig. 22.4b as in Fig. 22.4a. This suggests that the net electric flux through the surface of the box is *directly proportional* to the magnitude of the net charge enclosed by the box.

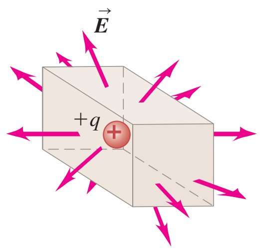
This conclusion is independent of the size of the box. In Fig. 22.4c the point charge $+q$ is enclosed by a box with twice the linear dimensions of the box in Fig. 22.4a. The magnitude of the electric field of a point charge decreases with distance according to $1/r^2$, so the average magnitude of $\vec{E}$ on each face of the large box in Fig. 22.4c is just $\frac{1}{4}$ of the average magnitude on the corresponding face in Fig. 22.4a. But each face of the large box has exactly four times the area of the corresponding face of the small box. Hence the outward electric flux is the *same* for the two boxes if we *define* electric flux as follows: For each face of the box, take the product of the average perpendicular component of $\vec{E}$ and the area of that face; then add up the results from all faces of the box. With this definition the net electric flux due to a single point charge inside the box is independent of the size of the box and depends only on the net charge inside the box.

22.3 Three cases in which there is zero *net* charge inside a box and no net electric flux through the surface of the box. (a) An empty box with $\vec{E} = \mathbf{0}$. (b) A box containing one positive and one equal-magnitude negative point charge. (c) An empty box immersed in a uniform electric field.

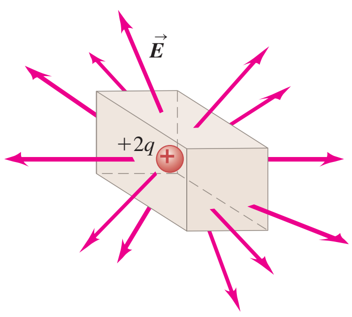


22.4 (a) A box enclosing a positive point charge $+q$. (b) Doubling the charge causes the magnitude of $\vec{E}$ to double, and it doubles the electric flux through the surface. (c) If the charge stays the same but the dimensions of the box are doubled, the flux stays the same. The magnitude of $\vec{E}$ on the surface decreases by a factor of $\frac{1}{4}$, but the area through which $\vec{E}$ "flows" increases by a factor of 4.

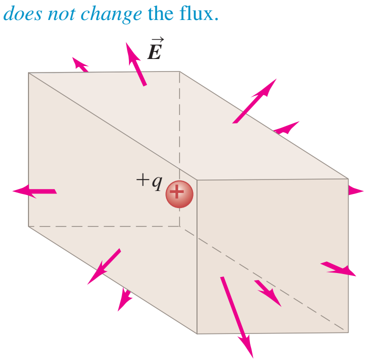
(a) A box containing a charge



(b) Doubling the enclosed charge doubles the flux.



(c) Doubling the box dimensions *does not change* the flux.



To summarize, for the special cases of a closed surface in the shape of a rectangular box and charge distributions made up of point charges or infinite charged sheets, we have found:

1. Whether there is a net outward or inward electric flux through a closed surface depends on the sign of the enclosed charge.
2. Charges *outside* the surface do not give a net electric flux through the surface.
3. The net electric flux is directly proportional to the net amount of charge enclosed within the surface but is otherwise independent of the size of the closed surface.

These observations are a qualitative statement of *Gauss's law.*

Do these observations hold true for other kinds of charge distributions and for closed surfaces of arbitrary shape? The answer to these questions will prove to be yes. But to explain why this is so, we need a precise mathematical statement of what we mean by electric flux. We develop this in the next section.

Test Your Understanding of Section 22.1 If all of the dimensions of the box in Fig. 22.2a are increased by a factor of 3, what effect will this change have on the electric flux through the box? (i) The flux will be $3^2 = 9$ times greater; (ii) the flux will be 3 times greater; (iii) the flux will be unchanged; (iv) the flux will be $\frac{1}{3}$ as great; (v) the flux will be $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ as great; (vi) not enough information is given to decide. ❙

22.2 Calculating Electric Flux

In the preceding section we introduced the concept of *electric flux*. We used this to give a rough qualitative statement of Gauss's law: The net electric flux through a closed surface is directly proportional to the net charge inside that surface. To be able to make full use of this law, we need to know how to *calculate* electric flux. To do this, let's again make use of the analogy between an electric field $\vec{E}$ and the field of velocity vectors $\vec{v}$ in a flowing fluid. (Again, keep in mind that this is only an analogy; an electric field is *not* a flow.)

Flux: Fluid-Flow Analogy

Figure 22.5 shows a fluid flowing steadily from left to right. Let's examine the volume flow rate dV/dt (in, say, cubic meters per second) through the wire rectangle with area A. When the area is perpendicular to the flow velocity $\vec{v}$ (Fig. 22.5a) and the flow velocity is the same at all points in the fluid, the volume flow rate dV/dt is the area A multiplied by the flow speed v:

$$\frac{dV}{dt} = vA$$

When the rectangle is tilted at an angle ϕ (Fig. 22.5b) so that its face is not perpendicular to $\vec{v}$, the area that counts is the silhouette area that we see when we look in the direction of $\vec{v}$. This area, which is outlined in red and labeled $A_\perp$ in Fig. 22.5b, is the *projection* of the area A onto a surface perpendicular to $\vec{v}$. Two sides of the projected rectangle have the same length as the original one, but the other two are foreshortened by a factor of $\cos\phi$, so the projected area $A_\perp$ is equal to $A\cos\phi$. Then the volume flow rate through A is

$$\frac{dV}{dt} = vA\cos\phi$$

If $\phi = 90°$, $dV/dt = 0$; the wire rectangle is edge-on to the flow, and no fluid passes through the rectangle.

Also, $v\cos\phi$ is the component of the vector $\vec{v}$ perpendicular to the plane of the area A. Calling this component $v_\perp$, we can rewrite the volume flow rate as

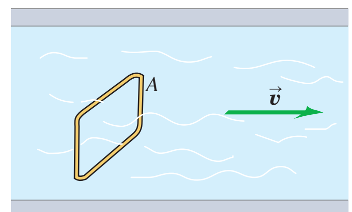
$$\frac{dV}{dt} = v_\perp A$$

We can express the volume flow rate more compactly by using the concept of *vector area* $\vec{A}$, a vector quantity with magnitude A and a direction perpendicular to the plane of the area we are describing. The vector area $\vec{A}$ describes both the size of an area and its orientation in space. In terms of $\vec{A}$, we can write the volume flow rate of fluid through the rectangle in Fig. 22.5b as a scalar (dot) product:

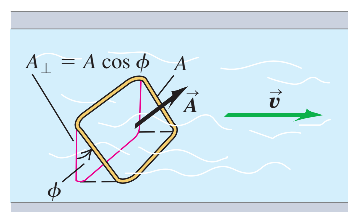
$$\frac{dV}{dt} = \vec{v}\cdot\vec{A}$$

22.5 The volume flow rate of fluid through the wire rectangle (a) is vA when the area of the rectangle is perpendicular to $\vec{v}$ and (b) is $vA\cos\phi$ when the rectangle is tilted at an angle ϕ.

(a) A wire rectangle in a fluid



(b) The wire rectangle tilted by an angle ϕ



Flux of a Uniform Electric Field

Using the analogy between electric field and fluid flow, we now define electric flux in the same way as we have just defined the volume flow rate of a fluid; we simply replace the fluid velocity $\vec{v}$ by the electric field $\vec{E}$. The symbol that we use for electric flux is Φ_E (the capital Greek letter phi; the subscript E is a reminder that this is *electric* flux). Consider first a flat area A perpendicular to a uniform electric field $\vec{E}$ (Fig. 22.6a). We define the electric flux through this area to be the product of the field magnitude E and the area A:

$$\Phi_E = EA$$

Roughly speaking, we can picture Φ_E in terms of the field lines passing through A. Increasing the area means that more lines of $\vec{E}$ pass through the area, increasing the flux; a stronger field means more closely spaced lines of $\vec{E}$ and therefore more lines per unit area, so again the flux increases.

If the area A is flat but not perpendicular to the field $\vec{E}$, then fewer field lines pass through it. In this case the area that counts is the silhouette area that we see when looking in the direction of $\vec{E}$. This is the area $A_\perp$ in Fig. 22.6b and is equal to $A\cos\phi$ (compare to Fig. 22.5b). We generalize our definition of electric flux for a uniform electric field to

$$\Phi_E = EA\cos\phi \qquad \text{(electric flux for uniform } \vec{E}\text{, flat surface)} \qquad (22.1)$$

Since $E\cos\phi$ is the component of $\vec{E}$ perpendicular to the area, we can rewrite Eq. (22.1) as

$$\Phi_E = E_\perp A \qquad \text{(electric flux for uniform } \vec{E}\text{, flat surface)} \qquad (22.2)$$

In terms of the vector area $\vec{A}$ perpendicular to the area, we can write the electric flux as the scalar product of $\vec{E}$ and $\vec{A}$:

$$\Phi_E = \vec{E}\cdot\vec{A} \qquad \text{(electric flux for uniform } \vec{E}\text{, flat surface)} \qquad (22.3)$$

Equations (22.1), (22.2), and (22.3) express the electric flux for a *flat* surface and a *uniform* electric field in different but equivalent ways. The SI unit for electric flux is $1\ \text{N}\cdot\text{m}^2/\text{C}$. Note that if the area is edge-on to the field, $\vec{E}$ and $\vec{A}$ are perpendicular and the flux is zero (Fig. 22.6c).

We can represent the direction of a vector area $\vec{A}$ by using a *unit vector* $\hat{n}$ perpendicular to the area; $\hat{n}$ stands for "normal." Then

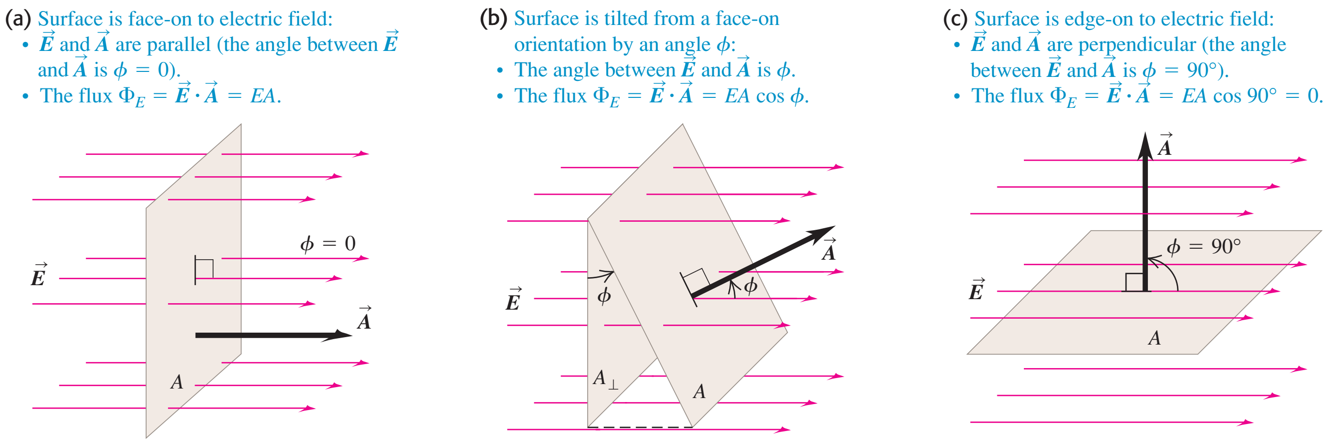
$$\vec{A} = A\hat{n} \qquad (22.4)$$

A surface has two sides, so there are two possible directions for $\hat{n}$ and $\vec{A}$. We must always specify which direction we choose. In Section 22.1 we related the charge inside a *closed* surface to the electric flux through the surface. With a closed surface we will always choose the direction of $\hat{n}$ to be *outward,* and we

Application Flux Through a Basking Shark's Mouth

Unlike aggressive carnivorous sharks such as great whites, a basking shark feeds passively on plankton in the water that passes through the shark's gills as it swims. To survive on these tiny organisms requires a huge flux of water through a basking shark's immense mouth, which can be up to a meter across. The water flux—the product of the shark's speed through the water and the area of its mouth—can be up to $0.5\ \text{m}^3/\text{s}$ (500 liters per second, or almost 5×10^5 gallons per hour). In a similar way, the flux of electric field through a surface depends on the magnitude of the field and the area of the surface (as well as the relative orientation of the field and surface).

22.6 A flat surface in a uniform electric field. The electric flux Φ_E through the surface equals the scalar product of the electric field $\vec{E}$ and the area vector $\vec{A}$.



will speak of the flux *out of* a closed surface. Thus what we called "outward electric flux" in Section 22.1 corresponds to a *positive* value of Φ_E, and what we called "inward electric flux" corresponds to a *negative* value of Φ_E.

Flux of a Nonuniform Electric Field

What happens if the electric field $\vec{E}$ isn't uniform but varies from point to point over the area A? Or what if A is part of a curved surface? Then we divide A into many small elements dA, each of which has a unit vector $\hat{n}$ perpendicular to it and a vector area $d\vec{A} = \hat{n}\, dA$. We calculate the electric flux through each element and integrate the results to obtain the total flux:

$$\Phi_E = \int E \cos\phi \, dA = \int E_\perp \, dA = \int \vec{E} \cdot d\vec{A} \quad \text{(general definition of electric flux)} \tag{22.5}$$

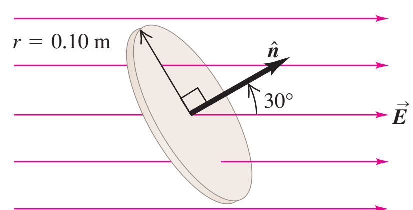
We call this integral the **surface integral** of the component $E_\perp$ over the area, or the surface integral of $\vec{E} \cdot d\vec{A}$. In specific problems, one form of the integral is sometimes more convenient than another. Example 22.3 at the end of this section illustrates the use of Eq. (22.5).

In Eq. (22.5) the electric flux $\int E_\perp \, dA$ is equal to the *average* value of the perpendicular component of the electric field, multiplied by the area of the surface. This is the same definition of electric flux that we were led to in Section 22.1, now expressed more mathematically. In the next section we will see the connection between the total electric flux through *any* closed surface, no matter what its shape, and the amount of charge enclosed within that surface.

Example 22.1 Electric flux through a disk

A disk of radius 0.10 m is oriented with its normal unit vector $\hat{n}$ at 30° to a uniform electric field $\vec{E}$ of magnitude 2.0×10^3 N/C (Fig. 22.7). (Since this isn't a closed surface, it has no "inside" or "outside." That's why we have to specify the direction of $\hat{n}$ in the figure.) (a) What is the electric flux through the disk? (b) What is the flux through the disk if it is turned so that $\hat{n}$ is perpendicular to $\vec{E}$? (c) What is the flux through the disk if $\hat{n}$ is parallel to $\vec{E}$?

22.7 The electric flux Φ_E through a disk depends on the angle between its normal $\hat{n}$ and the electric field $\vec{E}$.



SOLUTION

IDENTIFY and SET UP: This problem is about a flat surface in a uniform electric field, so we can apply the ideas of this section. We calculate the electric flux using Eq. (22.1).

EXECUTE: (a) The area is $A = \pi(0.10\text{ m})^2 = 0.0314\text{ m}^2$ and the angle between $\vec{E}$ and $\vec{A} = A\hat{n}$ is $\phi = 30°$, so from Eq. (22.1),

$$\Phi_E = EA\cos\phi = (2.0 \times 10^3\text{ N/C})(0.0314\text{ m}^2)(\cos 30°) = 54\text{ N}\cdot\text{m}^2/\text{C}$$

(b) The normal to the disk is now perpendicular to $\vec{E}$, so $\phi = 90°$, $\cos\phi = 0$, and $\Phi_E = 0$.

(c) The normal to the disk is parallel to $\vec{E}$, so $\phi = 0$ and $\cos\phi = 1$:

$$\Phi_E = EA\cos\phi = (2.0 \times 10^3\text{ N/C})(0.0314\text{ m}^2)(1) = 63\text{ N}\cdot\text{m}^2/\text{C}$$

EVALUATE: As a check on our results, note that our answer to part (b) is smaller than that to part (a), which is in turn smaller than that to part (c). Is all this as it should be?

Example 22.2 Electric flux through a cube

An imaginary cubical surface of side L is in a region of uniform electric field $\vec{E}$. Find the electric flux through each face of the cube and the total flux through the cube when (a) it is oriented with two of its faces perpendicular to $\vec{E}$ (Fig. 22.8a) and (b) the cube is turned by an angle θ about a vertical axis (Fig. 22.8b).

SOLUTION

IDENTIFY and SET UP: Since $\vec{E}$ is uniform and each of the six faces of the cube is flat, we find the flux Φ_{Ei} through each face using Eqs. (22.3) and (22.4). The total flux through the cube is the sum of the six individual fluxes.

EXECUTE: (a) Figure 22.8a shows the unit vectors $\hat{n}_1$ through $\hat{n}_6$ for each face; each unit vector points *outward* from the cube's closed surface. The angle between $\vec{E}$ and $\hat{n}_1$ is 180°, the angle between $\vec{E}$ and $\hat{n}_2$ is 0°, and the angle between $\vec{E}$ and each of the other four unit vectors is 90°. Each face of the cube has area L^2, so the fluxes through the faces are

$$\Phi_{E1} = \vec{E}\cdot\hat{n}_1 A = EL^2\cos 180° = -EL^2$$
$$\Phi_{E2} = \vec{E}\cdot\hat{n}_2 A = EL^2\cos 0° = +EL^2$$
$$\Phi_{E3} = \Phi_{E4} = \Phi_{E5} = \Phi_{E6} = EL^2\cos 90° = 0$$

The flux is negative on face 1, where $\vec{E}$ is directed into the cube, and positive on face 2, where $\vec{E}$ is directed out of the cube. The total flux through the cube is

$$\Phi_E = \Phi_{E1} + \Phi_{E2} + \Phi_{E3} + \Phi_{E4} + \Phi_{E5} + \Phi_{E6} = -EL^2 + EL^2 + 0 + 0 + 0 + 0 = 0$$

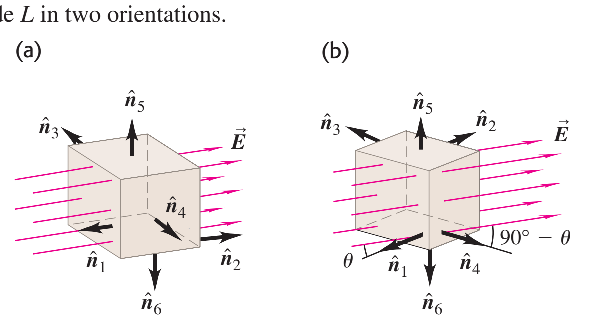
(b) The field $\vec{E}$ is directed into faces 1 and 3, so the fluxes through them are negative; $\vec{E}$ is directed out of faces 2 and 4, so the fluxes through them are positive. We find

$$\Phi_{E1} = \vec{E}\cdot\hat{n}_1 A = EL^2\cos(180° - \theta) = -EL^2\cos\theta$$
$$\Phi_{E2} = \vec{E}\cdot\hat{n}_2 A = +EL^2\cos\theta$$
$$\Phi_{E3} = \vec{E}\cdot\hat{n}_3 A = EL^2\cos(90° + \theta) = -EL^2\sin\theta$$
$$\Phi_{E4} = \vec{E}\cdot\hat{n}_4 A = EL^2\cos(90° - \theta) = +EL^2\sin\theta$$
$$\Phi_{E5} = \Phi_{E6} = EL^2\cos 90° = 0$$

The total flux $\Phi_E = \Phi_{E1} + \Phi_{E2} + \Phi_{E3} + \Phi_{E4} + \Phi_{E5} + \Phi_{E6}$ through the surface of the cube is again zero.

EVALUATE: We came to the same conclusion in our discussion of Fig. 22.3c: There is zero net flux of a uniform electric field through a closed surface that contains no electric charge.

22.8 Electric flux of a uniform field $\vec{E}$ through a cubical box of side L in two orientations.



Example 22.3 Electric flux through a sphere

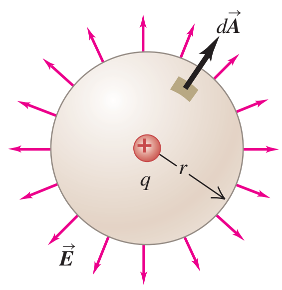
A point charge $q = +3.0\ \mu\text{C}$ is surrounded by an imaginary sphere of radius $r = 0.20$ m centered on the charge (Fig. 22.9). Find the resulting electric flux through the sphere.

SOLUTION

IDENTIFY and SET UP: The surface is not flat and the electric field is not uniform, so to calculate the electric flux (our target variable) we must use the general definition, Eq. (22.5). We use Eq. (22.5) to calculate the electric flux (our target variable). Because the sphere is centered on the point charge, at any point on the spherical surface, $\vec{E}$ is directed out of the sphere perpendicular to the surface. The positive direction for both $\hat{n}$ and $E_\perp$ is outward, so $E_\perp = E$ and the flux through a surface element dA is $\vec{E}\cdot d\vec{A} = E\,dA$. This greatly simplifies the integral in Eq. (22.5).

Continued

22.9 Electric flux through a sphere centered on a point charge.



EXECUTE: We must evaluate the integral of Eq. (22.5), $\Phi_E = \int E\,dA$. At any point on the sphere of radius r the electric field has the same magnitude $E = q/4\pi\epsilon_0 r^2$. Hence E can be taken outside the integral, which becomes $\Phi_E = E\int dA = EA$, where A is the area of the spherical surface: $A = 4\pi r^2$. Hence the total flux through the sphere is

$$\Phi_E = EA = \frac{q}{4\pi\epsilon_0 r^2}4\pi r^2 = \frac{q}{\epsilon_0}$$
$$= \frac{3.0\times10^{-6}\ \text{C}}{8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2} = 3.4\times10^5\ \text{N}\cdot\text{m}^2/\text{C}$$

EVALUATE: The radius r of the sphere cancels out of the result for Φ_E. We would have obtained the same flux with a sphere of radius 2.0 m or 200 m. We came to essentially the same conclusion in our discussion of Fig. 22.4 in Section 22.1, where we considered rectangular closed surfaces of two different sizes enclosing a point charge. There we found that the flux of $\vec{E}$ was independent of the size of the surface; the same result holds true for a spherical surface. Indeed, the flux through *any* surface enclosing a single point charge is independent of the shape or size of the surface, as we'll soon see.

Test Your Understanding of Section 22.2 Rank the following surfaces in order from most positive to most negative electric flux. (i) a flat rectangular surface with vector area $\vec{A} = (6.0\ \text{m}^2)\hat{\imath}$ in a uniform electric field $\vec{E} = (4.0\ \text{N/C})\hat{\jmath}$; (ii) a flat circular surface with vector area $\vec{A} = (3.0\ \text{m}^2)\hat{\jmath}$ in a uniform electric field $\vec{E} = (4.0\ \text{N/C})\hat{\imath} + (2.0\ \text{N/C})\hat{\jmath}$; (iii) a flat square surface with vector area $\vec{A} = (3.0\ \text{m}^2)\hat{\imath} + (7.0\ \text{m}^2)\hat{\jmath}$ in a uniform electric field $\vec{E} = (4.0\ \text{N/C})\hat{\imath} - (2.0\ \text{N/C})\hat{\jmath}$; (iv) a flat oval surface with vector area $\vec{A} = (3.0\ \text{m}^2)\hat{\imath} - (7.0\ \text{m}^2)\hat{\jmath}$ in a uniform electric field $\vec{E} = (4.0\ \text{N/C})\hat{\imath} - (2.0\ \text{N/C})\hat{\jmath}$.

22.3 Gauss's Law

Gauss's law is an alternative to Coulomb's law. While completely equivalent to Coulomb's law, Gauss's law provides a different way to express the relationship between electric charge and electric field. It was formulated by Carl Friedrich Gauss (1777–1855), one of the greatest mathematicians of all time (Fig. 22.10).

22.10 Carl Friedrich Gauss helped develop several branches of mathematics, including differential geometry, real analysis, and number theory. The "bell curve" of statistics is one of his inventions. Gauss also made state-of-the-art investigations of the earth's magnetism and calculated the orbit of the first asteroid to be discovered.

Point Charge Inside a Spherical Surface

Gauss's law states that the total electric flux through any closed surface (a surface enclosing a definite volume) is proportional to the total (net) electric charge inside the surface. In Section 22.1 we observed this relationship qualitatively for certain special cases; now we'll develop it more rigorously. We'll start with the field of a single positive point charge q. The field lines radiate out equally in all directions. We place this charge at the center of an imaginary spherical surface with radius R. The magnitude E of the electric field at every point on the surface is given by

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}$$

At each point on the surface, $\vec{E}$ is perpendicular to the surface, and its magnitude is the same at every point, just as in Example 22.3 (Section 22.2). The total electric flux is the product of the field magnitude E and the total area $A = 4\pi R^2$ of the sphere:

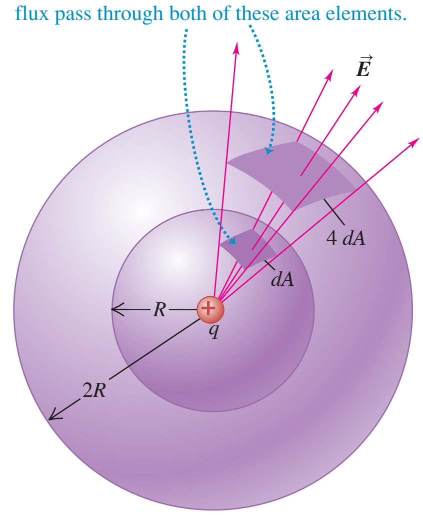
$$\Phi_E = EA = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}(4\pi R^2) = \frac{q}{\epsilon_0} \qquad (22.6)$$

The flux is independent of the radius R of the sphere. It depends only on the charge q enclosed by the sphere.

We can also interpret this result in terms of field lines. Figure 22.11 shows two spheres with radii R and $2R$ centered on the point charge q. Every field line that passes through the smaller sphere also passes through the larger sphere, so the total flux through each sphere is the same.

What is true of the entire sphere is also true of any portion of its surface. In Fig. 22.11 an area dA is outlined on the sphere of radius R and then projected onto the sphere of radius $2R$ by drawing lines from the center through points on the boundary of dA. The area projected on the larger sphere is clearly $4\,dA$. But since the electric field due to a point charge is inversely proportional to r^2, the field magnitude is $\frac{1}{4}$ as great on the sphere of radius $2R$ as on the sphere of radius R. Hence the electric flux is the same for both areas and is independent of the radius of the sphere.

22.11 Projection of an element of area dA of a sphere of radius R onto a concentric sphere of radius $2R$. The projection multiplies each linear dimension by 2, so the area element on the larger sphere is $4\,dA$.



Point Charge Inside a Nonspherical Surface

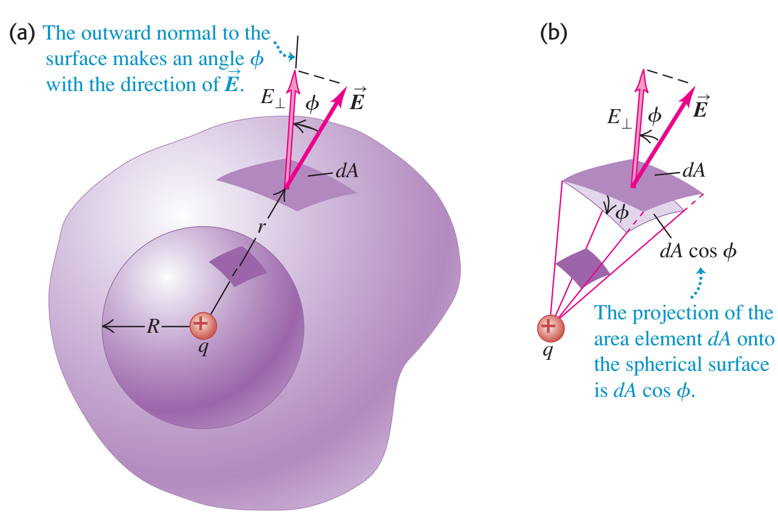
This projection technique shows us how to extend this discussion to nonspherical surfaces. Instead of a second sphere, let us surround the sphere of radius R by a surface of irregular shape, as in Fig. 22.12a. Consider a small element of area dA on the irregular surface; we note that this area is *larger* than the corresponding element on a spherical surface at the same distance from q. If a normal to dA makes an angle ϕ with a radial line from q, two sides of the area projected onto the spherical surface are foreshortened by a factor $\cos\phi$ (Fig. 22.12b). The other two sides are unchanged. Thus the electric flux through the spherical surface element is equal to the flux $E\,dA\cos\phi$ through the corresponding irregular surface element.

We can divide the entire irregular surface into elements dA, compute the electric flux $E\,dA\cos\phi$ for each, and sum the results by integrating, as in Eq. (22.5). Each of the area elements projects onto a corresponding spherical surface element. Thus the *total* electric flux through the irregular surface, given by any of the forms of Eq. (22.5), must be the same as the total flux through a sphere, which Eq. (22.6) shows is equal to q/ϵ_0. Thus, for the irregular surface,

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \qquad (22.7)$$

Equation (22.7) holds for a surface of *any* shape or size, provided only that it is a *closed* surface enclosing the charge q. The circle on the integral sign reminds us that the integral is always taken over a *closed* surface.

The area elements $d\vec{A}$ and the corresponding unit vectors $\hat{n}$ always point *out of* the volume enclosed by the surface. The electric flux is then positive in areas



22.12 Calculating the electric flux through a nonspherical surface.

where the electric field points out of the surface and negative where it points inward. Also, $E_\perp$ is positive at points where $\vec{E}$ points out of the surface and negative at points where $\vec{E}$ points into the surface.

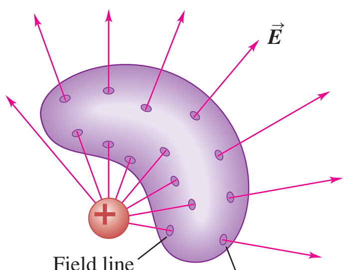
If the point charge in Fig. 22.12 is negative, the $\vec{E}$ field is directed radially *inward;* the angle ϕ is then greater than 90°, its cosine is negative, and the integral in Eq. (22.7) is negative. But since q is also negative, Eq. (22.7) still holds.

For a closed surface enclosing *no* charge,

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = 0$$

This is the mathematical statement that when a region contains no charge, any field lines caused by charges *outside* the region that enter on one side must leave again on the other side. (In Section 22.1 we came to the same conclusion by considering the special case of a rectangular box in a uniform field.) Figure 22.13 illustrates this point. *Electric field lines can begin or end inside a region of space only when there is charge in that region.*

22.13 A point charge *outside* a closed surface that encloses no charge. If an electric field line from the external charge enters the surface at one point, it must leave at another.



General Form of Gauss's Law

Now comes the final step in obtaining the general form of Gauss's law. Suppose the surface encloses not just one point charge q but several charges q_1, q_2, q_3, ... The total (resultant) electric field $\vec{E}$ at any point is the vector sum of the $\vec{E}$ fields of the individual charges. Let Q_{encl} be the *total* charge enclosed by the surface: $Q_{encl} = q_1 + q_2 + q_3 + \cdots$. Also let $\vec{E}$ be the *total* field at the position of the surface area element $d\vec{A}$, and let $E_\perp$ be its component perpendicular to the plane of that element (that is, parallel to $d\vec{A}$). Then we can write an equation like Eq. (22.7) for each charge and its corresponding field and add the results. When we do, we obtain the general statement of Gauss's law:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0} \qquad \text{(Gauss's law)} \qquad (22.8)$$

The total electric flux through a closed surface is equal to the total (net) electric charge inside the surface, divided by ϵ_0.

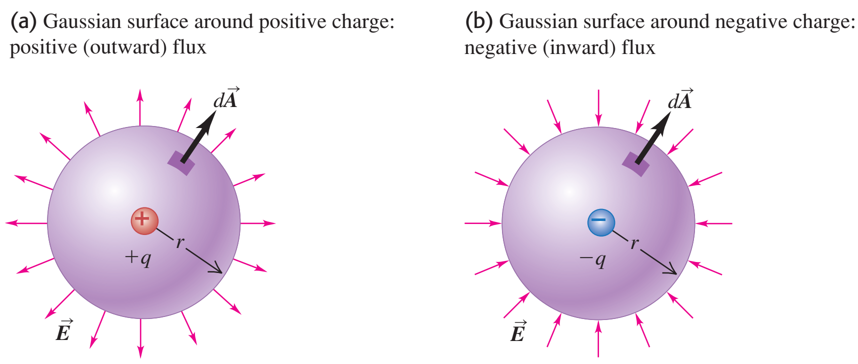
CAUTION Gaussian surfaces are imaginary Remember that the closed surface in Gauss's law is *imaginary;* there need not be any material object at the position of the surface. We often refer to a closed surface used in Gauss's law as a **Gaussian surface.**

Using the definition of Q_{encl} and the various ways to express electric flux given in Eq. (22.5), we can express Gauss's law in the following equivalent forms:

$$\Phi_E = \oint E\cos\phi \, dA = \oint E_\perp \, dA = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0} \qquad \text{(various forms of Gauss's law)} \qquad (22.9)$$

As in Eq. (22.5), the various forms of the integral all express the same thing, the total electric flux through the Gaussian surface, in different terms. One form is sometimes more convenient than another.

As an example, Fig. 22.14a shows a spherical Gaussian surface of radius r around a positive point charge $+q$. The electric field points out of the Gaussian surface, so at every point on the surface $\vec{E}$ is in the same direction as $d\vec{A}$, $\phi = 0$, and $E_\perp$ is equal to the field magnitude $E = q/4\pi\epsilon_0 r^2$. Since E is the same at all points



22.14 Spherical Gaussian surfaces around (a) a positive point charge and (b) a negative point charge.

on the surface, we can take it outside the integral in Eq. (22.9). Then the remaining integral is $\int dA = A = 4\pi r^2$, the area of the sphere. Hence Eq. (22.9) becomes

$$\Phi_E = \oint E_\perp \, dA = \oint \left(\frac{q}{4\pi\epsilon_0 r^2}\right) dA = \frac{q}{4\pi\epsilon_0 r^2} \oint dA = \frac{q}{4\pi\epsilon_0 r^2} 4\pi r^2 = \frac{q}{\epsilon_0}$$

The enclosed charge Q_{encl} is just the charge $+q$, so this agrees with Gauss's law. If the Gaussian surface encloses a *negative* point charge as in Fig. 22.14b, then $\vec{E}$ points *into* the surface at each point in the direction opposite $d\vec{A}$. Then $\phi = 180°$ and $E_\perp$ is equal to the negative of the field magnitude: $E_\perp = -E = -|-q|/4\pi\epsilon_0 r^2 = -q/4\pi\epsilon_0 r^2$. Equation (22.9) then becomes

$$\Phi_E = \oint E_\perp \, dA = \oint \left(\frac{-q}{4\pi\epsilon_0 r^2}\right) dA = \frac{-q}{4\pi\epsilon_0 r^2} \oint dA = \frac{-q}{4\pi\epsilon_0 r^2} 4\pi r^2 = \frac{-q}{\epsilon_0}$$

This again agrees with Gauss's law because the enclosed charge in Fig. 22.14b is $Q_{encl} = -q$.

In Eqs. (22.8) and (22.9), Q_{encl} is always the algebraic sum of all the positive and negative charges enclosed by the Gaussian surface, and $\vec{E}$ is the *total* field at each point on the surface. Also note that in general, this field is caused partly by charges inside the surface and partly by charges outside. But as Fig. 22.13 shows, the outside charges do *not* contribute to the total (net) flux through the surface. So Eqs. (22.8) and (22.9) are correct even when there are charges outside the surface that contribute to the electric field at the surface. When $Q_{encl} = 0$, the total flux through the Gaussian surface must be zero, even though some areas may have positive flux and others may have negative flux (see Fig. 22.3b).

Gauss's law is the definitive answer to the question we posed at the beginning of Section 22.1: "If the electric field pattern is known in a given region, what can we determine about the charge distribution in that region?" It provides a relationship between the electric field on a closed surface and the charge distribution within that surface. But in some cases we can use Gauss's law to answer the reverse question: "If the charge distribution is known, what can we determine about the electric field that the charge distribution produces?" Gauss's law may seem like an unappealing way to address this question, since it may look as though evaluating the integral in Eq. (22.8) is a hopeless task. Sometimes it is, but other times it is surprisingly easy. Here's an example in which *no* integration is involved at all; we'll work out several more examples in the next section.

Conceptual Example 22.4 Electric flux and enclosed charge

Figure 22.15 shows the field produced by two point charges $+q$ and $-q$ (an electric dipole). Find the electric flux through each of the closed surfaces A, B, C, and D.

SOLUTION

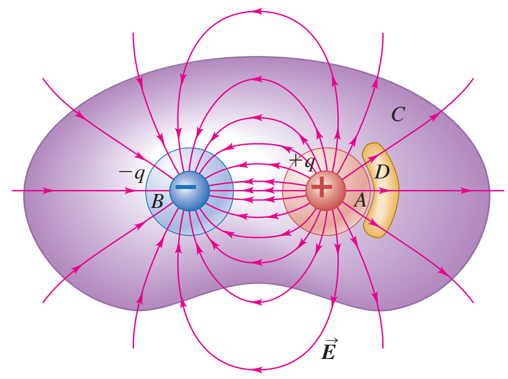
Gauss's law, Eq. (22.8), says that the total electric flux through a closed surface is equal to the total enclosed charge divided by ϵ_0. In

Continued

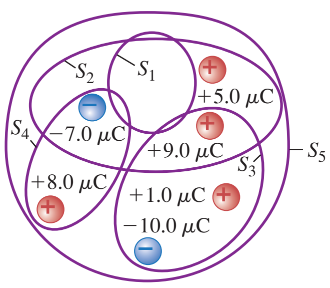
Fig. 22.15, surface A (shown in red) encloses the positive charge, so $Q_{encl} = +q$; surface B (in blue) encloses the negative charge, so $Q_{encl} = -q$; surface C (in purple) encloses *both* charges, so $Q_{encl} = +q + (-q) = 0$; and surface D (in yellow) encloses no charges, so $Q_{encl} = 0$. Hence, without having to do any integration, we have $\Phi_{EA} = +q/\epsilon_0$, $\Phi_{EB} = -q/\epsilon_0$, and $\Phi_{EC} = \Phi_{ED} = 0$. These results depend only on the charges enclosed within each Gaussian surface, not on the precise shapes of the surfaces.

We can draw similar conclusions by examining the electric field lines. All the field lines that cross surface A are directed out of the surface, so the flux through A must be positive. Similarly, the flux through B must be negative since all of the field lines that cross that surface point inward. For both surface C and surface D, there are as many field lines pointing into the surface as there are field lines pointing outward, so the flux through each of these surfaces is zero.

22.15 The net number of field lines leaving a closed surface is proportional to the total charge enclosed by that surface.



22.16 Five Gaussian surfaces and six point charges.



Test Your Understanding of Section 22.3 Figure 22.16 shows six point charges that all lie in the same plane. Five Gaussian surfaces—S_1, S_2, S_3, S_4, and S_5—each enclose part of this plane, and Fig. 22.16 shows the intersection of each surface with the plane. Rank these five surfaces in order of the electric flux through them, from most positive to most negative. ❙

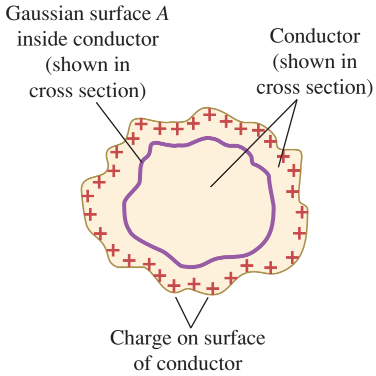
22.4 Applications of Gauss's Law

Gauss's law is valid for *any* distribution of charges and for *any* closed surface. Gauss's law can be used in two ways. If we know the charge distribution, and if it has enough symmetry to let us evaluate the integral in Gauss's law, we can find the field. Or if we know the field, we can use Gauss's law to find the charge distribution, such as charges on conducting surfaces.

In this section we present examples of both kinds of applications. As you study them, watch for the role played by the symmetry properties of each system. We will use Gauss's law to calculate the electric fields caused by several simple charge distributions; the results are collected in a table in the chapter summary.

In practical problems we often encounter situations in which we want to know the electric field caused by a charge distribution on a conductor. These calculations are aided by the following remarkable fact: *When excess charge is placed on a solid conductor and is at rest, it resides entirely on the surface, not in the interior of the material.* (By *excess* we mean charges other than the ions and free electrons that make up the neutral conductor.) Here's the proof. We know from Section 21.4 that in an electrostatic situation (with all charges at rest) the electric field $\vec{E}$ at every point in the interior of a conducting material is zero. If $\vec{E}$ were *not* zero, the excess charges would move. Suppose we construct a Gaussian surface inside the conductor, such as surface A in Fig. 22.17. Because $\vec{E} = \mathbf{0}$ everywhere on this surface, Gauss's law requires that the net charge inside the surface is zero. Now imagine shrinking the surface like a collapsing balloon until it encloses a region so small that we may consider it as a point P; then the charge at that point must be zero. We can do this anywhere inside the conductor, so *there can be no excess charge at any point within a solid conductor; any excess charge must reside on the conductor's surface.* (This result is for a *solid* conductor. In the next section we'll discuss what can happen if the conductor has cavities in its interior.) We will make use of this fact frequently in the examples that follow.

22.17 Under electrostatic conditions (charges not in motion), any excess charge on a solid conductor resides entirely on the conductor's surface.



Problem-Solving Strategy 22.1 Gauss's Law

IDENTIFY *the relevant concepts:* Gauss's law is most useful when the charge distribution has spherical, cylindrical, or planar symmetry. In these cases the symmetry determines the direction of $\vec{E}$. Then Gauss's law yields the magnitude of $\vec{E}$ if we are given the charge distribution, and vice versa. In either case, begin the analysis by asking the question: What is the symmetry?

SET UP *the problem* using the following steps:

1. List the known and unknown quantities and identify the target variable.
2. Select the appropriate closed, imaginary Gaussian surface. For spherical symmetry, use a concentric spherical surface. For cylindrical symmetry, use a coaxial cylindrical surface with flat ends perpendicular to the axis of symmetry (like a soup can). For planar symmetry, use a cylindrical surface (like a tuna can) with its flat ends parallel to the plane.

EXECUTE *the solution* as follows:

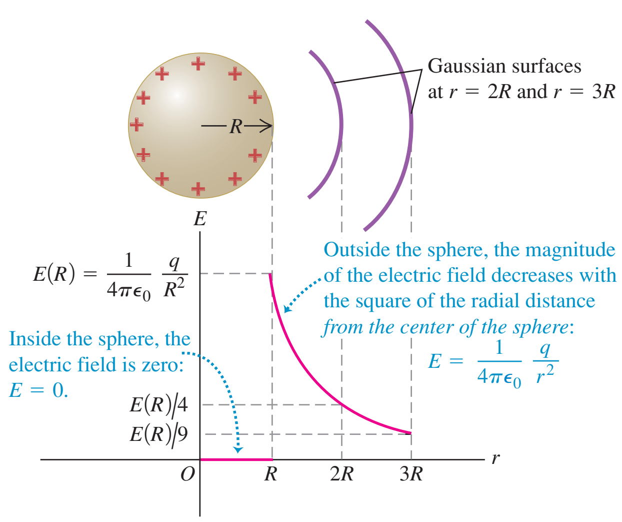
1. Determine the appropriate size and placement of your Gaussian surface. To evaluate the field magnitude at a particular point, the surface must include that point. It may help to place one end of a can-shaped surface within a conductor, where $\vec{E}$ and therefore Φ are zero, or to place its ends equidistant from a charged plane.
2. Evaluate the integral $\oint E_\perp\, dA$ in Eq. (22.9). In this equation $E_\perp$ is the perpendicular component of the *total* electric field at each point on the Gaussian surface. A well-chosen Gaussian surface should make integration trivial or unnecessary. If the surface comprises several separate surfaces, such as the sides and ends of a cylinder, the integral $\oint E_\perp\, dA$ over the entire closed surface is the sum of the integrals $\int E_\perp\, dA$ over the separate surfaces. Consider points 3–6 as you work.
3. If $\vec{E}$ is *perpendicular* (normal) at every point to a surface with area A, if it points *outward* from the interior of the surface, and if it has the same *magnitude* at every point on the surface, then $E_\perp = E =$ constant, and $\int E_\perp\, dA$ over that surface is equal to EA. (If $\vec{E}$ is inward, then $E_\perp = -E$ and $\int E_\perp\, dA = -EA$.) This should be the case for part or all of your Gaussian surface. If $\vec{E}$ is tangent to a surface at every point, then $E_\perp = 0$ and the integral over that surface is zero. This may be the case for parts of a cylindrical Gaussian surface. If $\vec{E} = \mathbf{0}$ at every point on a surface, the integral is zero.
4. Even when there is *no* charge within a Gaussian surface, the field at any given point on the surface is not necessarily zero. In that case, however, the total electric flux through the surface is always zero.
5. The flux integral $\oint E_\perp\, dA$ can be approximated as the difference between the numbers of electric lines of force leaving and entering the Gaussian surface. In this sense the flux gives the sign of the enclosed charge, but is only proportional to it; zero flux corresponds to zero enclosed charge.
6. Once you have evaluated $\oint E_\perp\, dA$, use Eq. (22.9) to solve for your target variable.

EVALUATE *your answer:* If your result is a *function* that describes how the magnitude of the electric field varies with position, ensure that it makes sense.

Example 22.5 Field of a charged conducting sphere

We place a total positive charge q on a solid conducting sphere with radius R (Fig. 22.18). Find $\vec{E}$ at any point inside or outside the sphere.

22.18 Calculating the electric field of a conducting sphere with positive charge q. Outside the sphere, the field is the same as if all of the charge were concentrated at the center of the sphere.



SOLUTION

IDENTIFY and SET UP: As we discussed earlier in this section, all of the charge must be on the surface of the sphere. The charge is free to move on the conductor, and there is no preferred position on the surface; the charge is therefore distributed *uniformly* over the surface, and the system is spherically symmetric. To exploit this symmetry, we take as our Gaussian surface a sphere of radius r centered on the conductor. We can calculate the field inside or outside the conductor by taking $r < R$ or $r > R$, respectively. In either case, the point at which we want to calculate $\vec{E}$ lies on the Gaussian surface.

EXECUTE: The spherical symmetry means that the direction of the electric field must be radial; that's because there is no preferred direction parallel to the surface, so $\vec{E}$ can have no component parallel to the surface. There is also no preferred orientation of the sphere, so the field magnitude E can depend only on the distance r from the center and must have the same value at all points on the Gaussian surface.

For $r > R$ the entire conductor is within the Gaussian surface, so the enclosed charge is q. The area of the Gaussian surface is $4\pi r^2$, and $\vec{E}$ is uniform over the surface and perpendicular to it at each point. The flux integral $\oint E_\perp\, dA$ is then just $E(4\pi r^2)$, and Eq. (22.8) gives

Continued

$$E(4\pi r^2) = \frac{q}{\epsilon_0} \quad \text{and}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \quad \text{(outside a charged conducting sphere)}$$

This expression is the same as that for a point charge; outside the charged sphere, its field is the same as though the entire charge were concentrated at its center. Just outside the surface of the sphere, where $r = R$,

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2} \quad \text{(at the surface of a charged conducting sphere)}$$

CAUTION **Flux can be positive or negative** Remember that we have chosen the charge q to be *positive*. If the charge is negative, the electric field is radially *inward* instead of radially outward, and the electric flux through the Gaussian surface is negative. The electric-field magnitudes outside and at the surface of the sphere are given by the same expressions as above, except that q denotes the *magnitude* (absolute value) of the charge.

For $r < R$ we again have $E(4\pi r^2) = Q_{encl}/\epsilon_0$. But now our Gaussian surface (which lies entirely within the conductor) encloses *no* charge, so $Q_{encl} = 0$. The electric field inside the conductor is therefore zero.

EVALUATE: We already knew that $\vec{E} = \mathbf{0}$ inside a solid conductor (whether spherical or not) when the charges are at rest. Figure 22.18 shows E as a function of the distance r from the center of the sphere. Note that in the limit as $R \rightarrow 0$, the sphere becomes a point charge; there is then only an "outside," and the field is everywhere given by $E = q/4\pi\epsilon_0 r^2$. Thus we have deduced Coulomb's law from Gauss's law. (In Section 22.3 we deduced Gauss's law from Coulomb's law; the two laws are equivalent.)

We can also use this method for a conducting spherical *shell* (a spherical conductor with a concentric spherical hole inside) if there is no charge inside the hole. We use a spherical Gaussian surface with radius r less than the radius of the hole. If there *were* a field inside the hole, it would have to be radial and spherically symmetric as before, so $E = Q_{encl}/4\pi\epsilon_0 r^2$. But now there is no enclosed charge, so $Q_{encl} = 0$ and $E = 0$ inside the hole.

Can you use this same technique to find the electric field in the region between a charged sphere and a concentric hollow conducting sphere that surrounds it?

Example 22.6 Field of a uniform line charge

Electric charge is distributed uniformly along an infinitely long, thin wire. The charge per unit length is λ (assumed positive). Find the electric field using Gauss's law.

SOLUTION

IDENTIFY and SET UP: We found in Example 21.10 (Section 21.5) that the field $\vec{E}$ of a uniformly charged, infinite wire is radially outward if λ is positive and radially inward if λ is negative, and that the field magnitude E depends only on the radial distance from the wire. This suggests that we use a *cylindrical* Gaussian surface, of radius r and arbitrary length l, coaxial with the wire and with its ends perpendicular to the wire (Fig. 22.19).

EXECUTE: The flux through the flat ends of our Gaussian surface is zero because the radial electric field is parallel to these ends, and so $\vec{E}\cdot\hat{n} = 0$. On the cylindrical part of our surface we have $\vec{E}\cdot\hat{n} = E_\perp = E$ everywhere. (If λ were negative, we would have $\vec{E}\cdot\hat{n} = E_\perp = -E$ everywhere.) The area of the cylindrical surface is $2\pi rl$, so the flux through it—and hence the total flux Φ_E through the Gaussian surface—is $EA = 2\pi rlE$. The total enclosed charge is $Q_{encl} = \lambda l$, and so from Gauss's law, Eq. (22.8),

$$\Phi_E = 2\pi rlE = \frac{\lambda l}{\epsilon_0} \quad \text{and}$$

$$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r} \quad \text{(field of an infinite line of charge)}$$

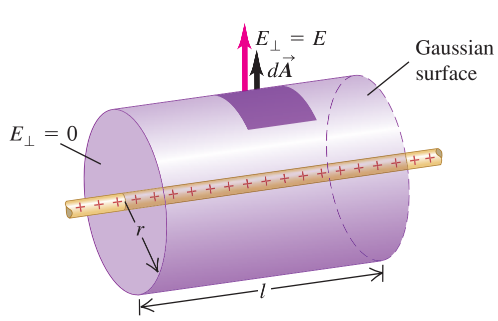
We found this same result in Example 21.10 with *much* more effort.

If λ is *negative*, $\vec{E}$ is directed radially inward, and in the above expression for E we must interpret λ as the absolute value of the charge per unit length.

EVALUATE: We saw in Example 21.10 that the *entire* charge on the wire contributes to the field at any point, and yet we consider only that part of the charge $Q_{encl} = \lambda l$ within the Gaussian surface when we apply Gauss's law. There's nothing inconsistent here; it takes the entire charge to give the field the properties that allow us to calculate Φ_E so easily, and Gauss's law always applies to the enclosed charge only. If the wire is short, the symmetry of the infinite wire is lost, and E is not uniform over a coaxial, cylindrical Gaussian surface. Gauss's law then *cannot* be used to find Φ_E; we must solve the problem the hard way, as in Example 21.10.

We can use the Gaussian surface in Fig. 22.19 to show that the field outside a long, uniformly charged cylinder is the same as though all the charge were concentrated on a line along its axis (see Problem 22.42). We can also calculate the electric field in the space between a charged cylinder and a coaxial hollow conducting cylinder surrounding it (see Problem 22.39).

22.19 A coaxial cylindrical Gaussian surface is used to find the electric field outside an infinitely long, charged wire.



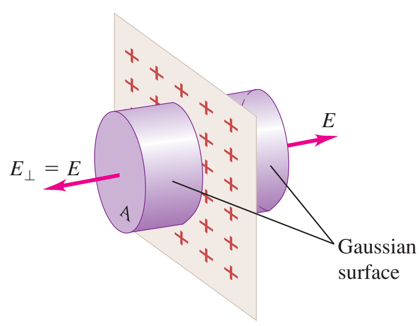
Example 22.7 Field of an infinite plane sheet of charge

Use Gauss's law to find the electric field caused by a thin, flat, infinite sheet with a uniform positive surface charge density σ.

SOLUTION

IDENTIFY and SET UP: In Example 21.11 (Section 21.5) we found that the field $\vec{E}$ of a uniformly charged infinite sheet is normal to the sheet, and that its magnitude is independent of the distance from the sheet. To take advantage of these symmetry properties, we use a cylindrical Gaussian surface with ends of area A and with its axis perpendicular to the sheet of charge (Fig. 22.20).

22.20 A cylindrical Gaussian surface is used to find the field of an infinite plane sheet of charge.



EXECUTE: The flux through the cylindrical part of our Gaussian surface is zero because $\vec{E}\cdot\hat{n} = 0$ everywhere. The flux through each flat end of the surface is $+EA$ because $\vec{E}\cdot\hat{n} = E_\perp = E$ everywhere, so the total flux through both ends—and hence the total flux Φ_E through the Gaussian surface—is $+2EA$. The total enclosed charge is $Q_{encl} = \sigma A$, and so from Gauss's law,

$$2EA = \frac{\sigma A}{\epsilon_0} \quad \text{and}$$

$$E = \frac{\sigma}{2\epsilon_0} \quad \text{(field of an infinite sheet of charge)}$$

In Example 21.11 we found this same result using a much more complex calculation.

If σ is negative, $\vec{E}$ is directed *toward* the sheet, the flux through the Gaussian surface in Fig. 22.20 is negative, and σ in the expression $E = \sigma/2\epsilon_0$ denotes the magnitude (absolute value) of the charge density.

EVALUATE: Again we see that, given favorable symmetry, we can deduce electric fields using Gauss's law much more easily than using Coulomb's law.

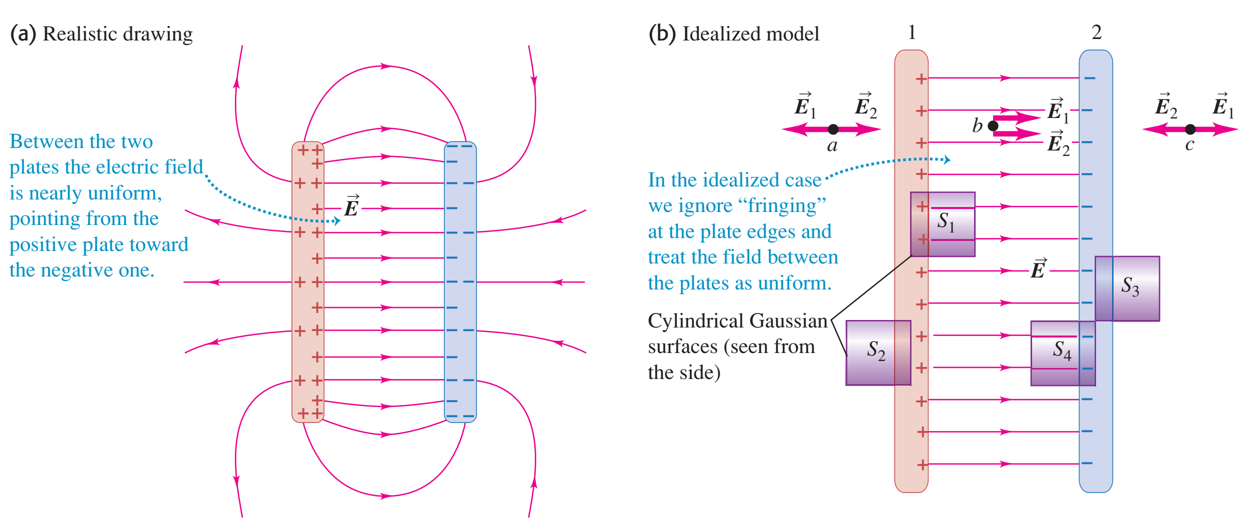
Example 22.8 Field between oppositely charged parallel conducting plates

Two large plane parallel conducting plates are given charges of equal magnitude and opposite sign; the surface charge densities are $+\sigma$ and $-\sigma$. Find the electric field in the region between the plates.

SOLUTION

IDENTIFY and SET UP: Figure 22.21a shows the field. Because opposite charges attract, most of the charge accumulates at the opposing faces of the plates. A small amount of charge resides on the *outer* surfaces of the plates, and there is some spreading or "fringing" of the field at the edges. But if the plates are very large in comparison to the distance between them, the amount of charge on the outer surfaces is negligibly small, and the fringing can be neglected except near the edges. In this case we can assume that the field is uniform in the interior region between the plates, as in Fig. 22.21b, and that the charges are distributed uniformly over the opposing surfaces. To exploit this symmetry, we can use the shaded Gaussian surfaces S_1, S_2, S_3, and S_4. These surfaces are cylinders with flat ends of area A; one end of each surface lies *within* a plate.

22.21 Electric field between oppositely charged parallel plates.



Continued

EXECUTE: The left-hand end of surface S_1 is within the positive plate 1. Since the field is zero within the volume of any solid conductor under electrostatic conditions, there is no electric flux through this end. The electric field between the plates is perpendicular to the right-hand end, so on that end, $E_\perp$ is equal to E and the flux is EA; this is positive, since $\vec{E}$ is directed out of the Gaussian surface. There is no flux through the side walls of the cylinder, since these walls are parallel to $\vec{E}$. So the total flux integral in Gauss's law is EA. The net charge enclosed by the cylinder is σA, so Eq. (22.8) yields $EA = \sigma A/\epsilon_0$; we then have

$$E = \frac{\sigma}{\epsilon_0} \quad \text{(field between oppositely charged conducting plates)}$$

The field is uniform and perpendicular to the plates, and its magnitude is independent of the distance from either plate. The Gaussian surface S_4 yields the same result. Surfaces S_2 and S_3 yield $E = 0$ to the left of plate 1 and to the right of plate 2, respectively. We leave these calculations to you (see Exercise 22.29).

EVALUATE: We obtained the same results in Example 21.11 by using the principle of superposition of electric fields. The fields due to the two sheets of charge (one on each plate) are $\vec{E}_1$ and $\vec{E}_2$; from Example 22.7, both of these have magnitude $\sigma/2\epsilon_0$. The total electric field at any point is the vector sum $\vec{E} = \vec{E}_1 + \vec{E}_2$. At points a and c in Fig. 22.21b, $\vec{E}_1$ and $\vec{E}_2$ point in opposite directions, and their sum is zero. At point b, $\vec{E}_1$ and $\vec{E}_2$ are in the same direction; their sum has magnitude $E = \sigma/\epsilon_0$, just as we found above using Gauss's law.

Example 22.9 Field of a uniformly charged sphere

Positive electric charge Q is distributed uniformly *throughout the volume* of an *insulating* sphere with radius R. Find the magnitude of the electric field at a point P a distance r from the center of the sphere.

SOLUTION

IDENTIFY and SET UP: As in Example 22.5, the system is spherically symmetric. Hence we can use the conclusions of that example about the direction and magnitude of $\vec{E}$. To make use of the spherical symmetry, we choose as our Gaussian surface a sphere with radius r, concentric with the charge distribution.

EXECUTE: From symmetry, the direction of $\vec{E}$ is radial at every point on the Gaussian surface, so $E_\perp = E$ and the field magnitude E is the same at every point on the surface. Hence the total electric flux through the Gaussian surface is the product of E and the total area of the surface $A = 4\pi r^2$—that is, $\Phi_E = 4\pi r^2 E$.

The amount of charge enclosed within the Gaussian surface depends on r. To find E *inside* the sphere, we choose $r < R$. The volume charge density ρ is the charge Q divided by the volume of the entire charged sphere of radius R:

$$\rho = \frac{Q}{4\pi R^3/3}$$

The volume V_{encl} enclosed by the Gaussian surface is $\frac{4}{3}\pi r^3$, so the total charge Q_{encl} enclosed by that surface is

$$Q_{encl} = \rho V_{encl} = \left(\frac{Q}{4\pi R^3/3}\right)\left(\tfrac{4}{3}\pi r^3\right) = Q\frac{r^3}{R^3}$$

Then Gauss's law, Eq. (22.8), becomes

$$4\pi r^2 E = \frac{Q}{\epsilon_0}\frac{r^3}{R^3} \quad \text{or}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3} \quad \text{(field inside a uniformly charged sphere)}$$

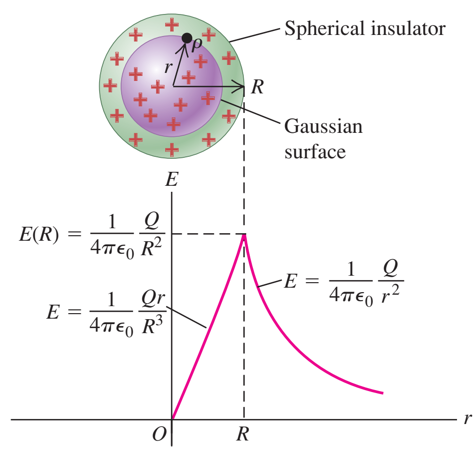
The field magnitude is proportional to the distance r of the field point from the center of the sphere (see the graph of E versus r in Fig. 22.22).

To find E *outside* the sphere, we take $r > R$. This surface encloses the entire charged sphere, so $Q_{encl} = Q$, and Gauss's law gives

$$4\pi r^2 E = \frac{Q}{\epsilon_0} \quad \text{or}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \quad \text{(field outside a uniformly charged sphere)}$$

22.22 The magnitude of the electric field of a uniformly charged insulating sphere. Compare this with the field for a conducting sphere (see Fig. 22.18).



The field outside *any* spherically symmetric charged body varies as $1/r^2$, as though the entire charge were concentrated at the center. This is graphed in Fig. 22.22.

If the charge is *negative*, $\vec{E}$ is radially *inward* and in the expressions for E we interpret Q as the absolute value of the charge.

EVALUATE: Notice that if we set $r = R$ in either expression for E, we get the same result $E = Q/4\pi\epsilon_0 R^2$ for the magnitude of the field at the surface of the sphere. This is because the magnitude E is a *continuous* function of r. By contrast, for the charged conducting sphere of Example 22.5 the electric-field magnitude is *discontinuous* at $r = R$ (it jumps from $E = 0$ just inside the sphere to $E = Q/4\pi\epsilon_0 R^2$ just outside the sphere). In general, the electric field $\vec{E}$ is discontinuous in magnitude, direction, or both wherever there is a *sheet* of charge, such as at the surface of a charged conducting sphere (Example 22.5), at the surface of an infinite charged sheet (Example 22.7), or at the surface of a charged conducting plate (Example 22.8).

The approach used here can be applied to *any* spherically symmetric distribution of charge, even if it is not radially uniform, as it was here. Such charge distributions occur within many atoms and atomic nuclei, so Gauss's law is useful in atomic and nuclear physics.

Example 22.10 Charge on a hollow sphere

A thin-walled, hollow sphere of radius 0.250 m has an unknown charge distributed uniformly over its surface. At a distance of 0.300 m from the center of the sphere, the electric field points radially inward and has magnitude 1.80×10^2 N/C. How much charge is on the sphere?

SOLUTION

IDENTIFY and SET UP: The charge distribution is spherically symmetric. As in Examples 22.5 and 22.9, it follows that the electric field is radial everywhere and its magnitude is a function only of the radial distance r from the center of the sphere. We use a spherical Gaussian surface that is concentric with the charge distribution and has radius $r = 0.300$ m. Our target variable is $Q_{encl} = q$.

EXECUTE: The charge distribution is the same as if the charge were on the surface of a 0.250-m-radius conducting sphere. Hence we can borrow the results of Example 22.5. We note that the electric field here is directed toward the sphere, so that q must be *negative*. Furthermore, the electric field is directed into the Gaussian surface, so that $E_\perp = -E$ and $\Phi_E = \oint E_\perp \, dA = -E(4\pi r^2)$.

By Gauss's law, the flux is equal to the charge q on the sphere (all of which is enclosed by the Gaussian surface) divided by ϵ_0. Solving for q, we find

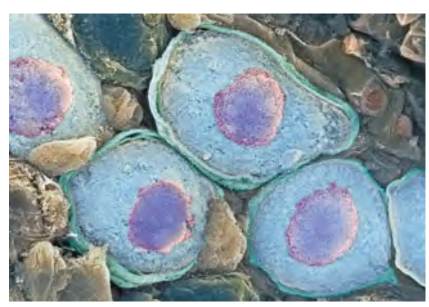
$$q = -E(4\pi\epsilon_0 r^2) = -(1.80 \times 10^2 \text{ N/C})(4\pi) \times (8.854 \times 10^{-12} \text{ C}^2/\text{N}\cdot\text{m}^2)(0.300 \text{ m})^2 = -1.80 \times 10^{-9} \text{ C} = -1.80 \text{ nC}$$

EVALUATE: To determine the charge, we had to know the electric field at *all* points on the Gaussian surface so that we could calculate the flux integral. This was possible here because the charge distribution is highly symmetric. If the charge distribution is irregular or lacks symmetry, Gauss's law is not very useful for calculating the charge distribution from the field, or vice versa.

Test Your Understanding of Section 22.4 You place a known amount of charge Q on the irregularly shaped conductor shown in Fig. 22.17. If you know the size and shape of the conductor, can you use Gauss's law to calculate the electric field at an arbitrary position outside the conductor?

Application Charge Distribution Inside a Nerve Cell

The interior of a human nerve cell contains both positive potassium ions (K^+) and negatively charged protein molecules (Pr^-). Potassium ions can flow out of the cell through the cell membrane, but the much larger protein molecules cannot. The result is that the interior of the cell has a net negative charge. (The fluid outside the cell has a positive charge that balances this.) The fluid within the cell is a good conductor, so the Pr^- molecules distribute themselves on the outer surface of the fluid—that is, on the inner surface of the cell membrane, which is an insulator. This is true no matter what the shape of the cell.

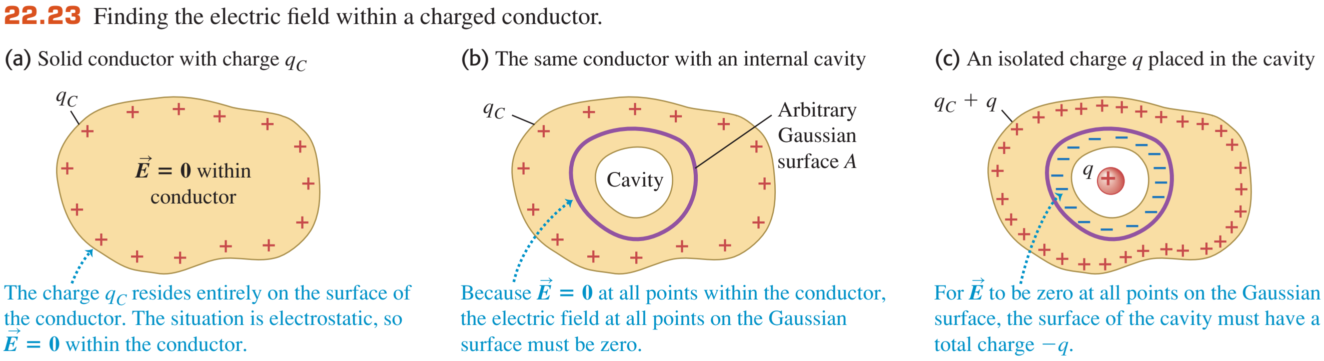


22.5 Charges on Conductors

We have learned that in an electrostatic situation (in which there is no net motion of charge) the electric field at every point within a conductor is zero and that any excess charge on a solid conductor is located entirely on its surface (Fig. 22.23a). But what if there is a *cavity* inside the conductor (Fig. 22.23b)? If there is no charge within the cavity, we can use a Gaussian surface such as A (which lies completely within the material of the conductor) to show that the *net* charge on the *surface of the cavity* must be zero, because $\vec{E} = \mathbf{0}$ everywhere on the Gaussian surface. In fact, we can prove in this situation that there can't be any charge *anywhere* on the cavity surface. We will postpone detailed proof of this statement until Chapter 23.

Suppose we place a small body with a charge q inside a cavity within a conductor (Fig. 22.23c). The conductor is uncharged and is insulated from the charge q. Again $\vec{E} = \mathbf{0}$ everywhere on surface A, so according to Gauss's law the *total* charge inside this surface must be zero. Therefore there must be a charge $-q$ distributed on the surface of the cavity, drawn there by the charge q inside the cavity. The *total* charge on the conductor must remain zero, so a charge $+q$ must appear

22.23 Finding the electric field within a charged conductor.



either on its outer surface or inside the material. But we showed that in an electrostatic situation there can't be any excess charge within the material of a conductor. So we conclude that the charge $+q$ must appear on the outer surface. By the same reasoning, if the conductor originally had a charge q_C, then the total charge on the outer surface must be $q_C + q$ after the charge q is inserted into the cavity.

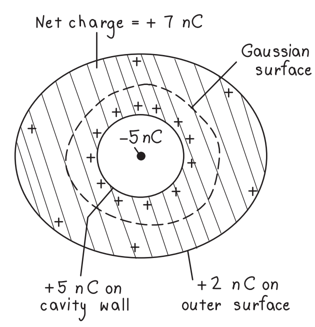
Conceptual Example 22.11 A conductor with a cavity

A solid conductor with a cavity carries a total charge of +7 nC. Within the cavity, insulated from the conductor, is a point charge of −5 nC. How much charge is on each surface (inner and outer) of the conductor?

SOLUTION

Figure 22.24 shows the situation. If the charge in the cavity is $q = -5$ nC, the charge on the inner cavity surface must be $-q = -(-5 \text{ nC}) = +5$ nC. The conductor carries a *total* charge of +7 nC, none of which is in the interior of the material. If +5 nC is on the inner surface of the cavity, then there must be $(+7 \text{ nC}) - (+5 \text{ nC}) = +2$ nC on the outer surface of the conductor.

22.24 Our sketch for this problem. There is zero electric field inside the bulk conductor and hence zero flux through the Gaussian surface shown, so the charge on the cavity wall must be the opposite of the point charge.

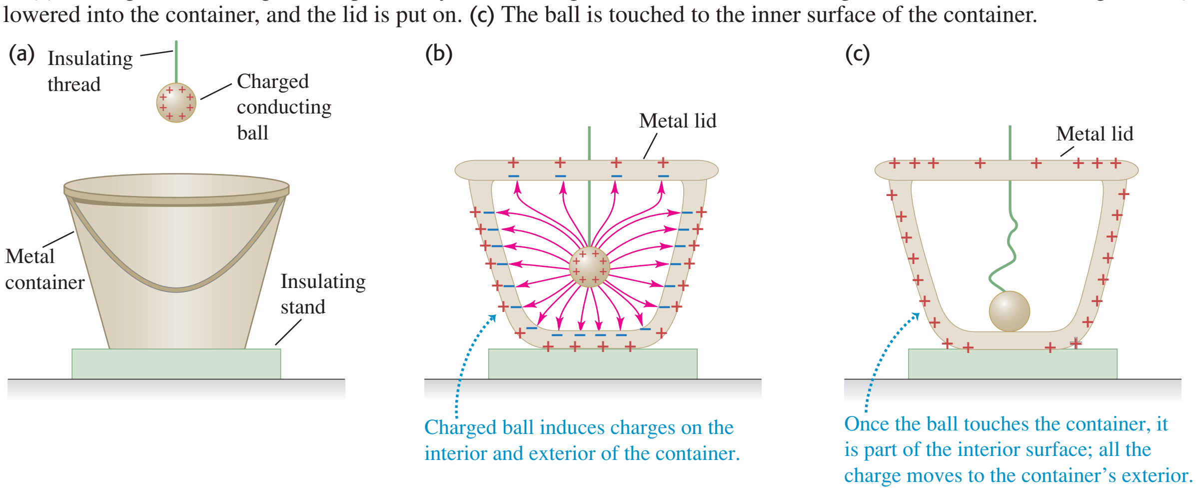


Testing Gauss's Law Experimentally

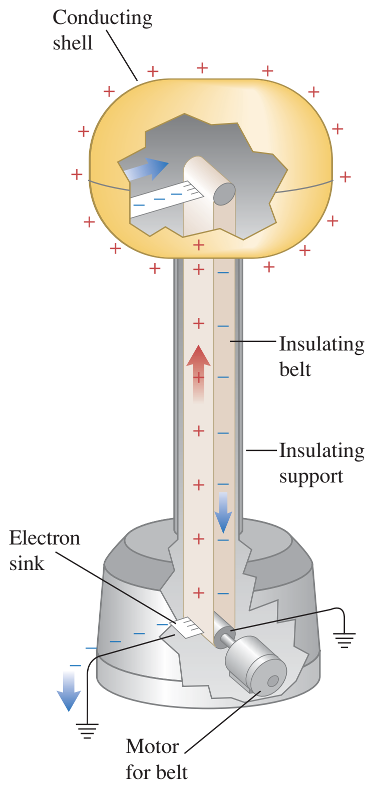
We can now consider a historic experiment, shown in Fig. 22.25. We mount a conducting container on an insulating stand. The container is initially uncharged. Then we hang a charged metal ball from an insulating thread (Fig. 22.25a), lower it into the container, and put the lid on (Fig. 22.25b). Charges are induced on the walls of the container, as shown. But now we let the ball *touch* the inner wall (Fig. 22.25c). The surface of the ball becomes part of the cavity surface. The situation is now the same as Fig. 22.23b; if Gauss's law is correct, the net charge on the cavity surface must be zero. Thus the ball must lose all its charge. Finally, we pull the ball out; we find that it has indeed lost all its charge.

This experiment was performed in the 19th century by the English scientist Michael Faraday, using a metal icepail with a lid, and it is called **Faraday's icepail experiment.** The result confirms the validity of Gauss's law and therefore of

22.25 (a) A charged conducting ball suspended by an insulating thread outside a conducting container on an insulating stand. (b) The ball is lowered into the container, and the lid is put on. (c) The ball is touched to the inner surface of the container.



22.26 Cutaway view of the essential parts of a Van de Graaff electrostatic generator. The electron sink at the bottom draws electrons from the belt, giving it a positive charge; at the top the belt attracts electrons away from the conducting shell, giving the shell a positive charge.

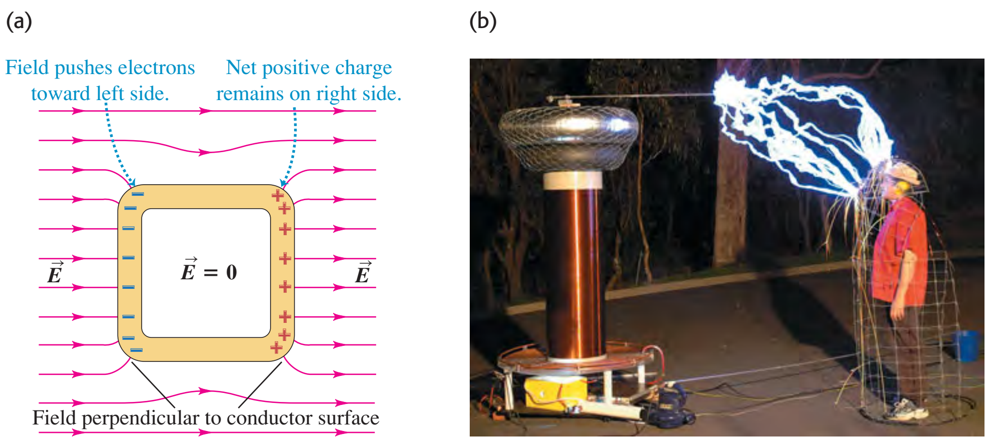


Coulomb's law. Faraday's result was significant because Coulomb's experimental method, using a torsion balance and dividing of charges, was not very precise; it is very difficult to confirm the $1/r^2$ dependence of the electrostatic force by direct force measurements. By contrast, experiments like Faraday's test the validity of Gauss's law, and therefore of Coulomb's law, with much greater precision. Modern versions of this experiment have shown that the exponent 2 in the $1/r^2$ of Coulomb's law does not differ from precisely 2 by more than 10^{-16}. So there is no reason to believe it is anything other than exactly 2.

The same principle behind Faraday's icepail experiment is used in a *Van de Graaff electrostatic generator* (Fig. 22.26). A charged belt continuously carries charge to the inside of a conducting shell. By Gauss's law, there can never be any charge on the inner surface of this shell, so the charge is immediately carried away to the outside surface of the shell. As a result, the charge on the shell and the electric field around it can become very large very rapidly. The Van de Graaff generator is used as an accelerator of charged particles and for physics demonstrations.

This principle also forms the basis for *electrostatic shielding*. Suppose we have a very sensitive electronic instrument that we want to protect from stray electric fields that might cause erroneous measurements. We surround the instrument with a conducting box, or we line the walls, floor, and ceiling of the room with a conducting material such as sheet copper. The external electric field redistributes the free electrons in the conductor, leaving a net positive charge on the outer surface in some regions and a net negative charge in others (Fig. 22.27). This charge distribution causes an additional electric field such that the *total* field at every point inside the box is zero, as Gauss's law says it must be. The charge distribution on the box also alters the shapes of the field lines near the box, as the figure shows. Such a setup is often called a *Faraday cage*. The same physics tells

22.27 (a) A conducting box (a Faraday cage) immersed in a uniform electric field. The field of the induced charges on the box combines with the uniform field to give zero total field inside the box. (b) This person is inside a Faraday cage, and so is protected from the powerful electric discharge.

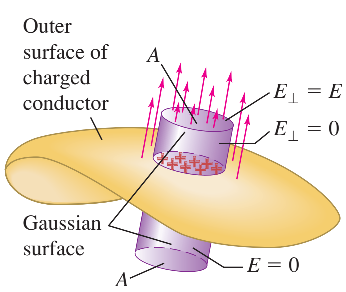


you that one of the safest places to be in a lightning storm is inside an automobile; if the car is struck by lightning, the charge tends to remain on the metal skin of the vehicle, and little or no electric field is produced inside the passenger compartment.

Field at the Surface of a Conductor

Finally, we note that there is a direct relationship between the $\vec{E}$ field at a point just outside any conductor and the surface charge density σ at that point. In general, σ varies from point to point on the surface. We will show in Chapter 23 that at any such point, the direction of $\vec{E}$ is always *perpendicular* to the surface. (You can see this effect in Fig. 22.27a.)

22.28 The field just outside a charged conductor is perpendicular to the surface, and its perpendicular component $E_\perp$ is equal to σ/ϵ_0.



To find a relationship between σ at any point on the surface and the perpendicular component of the electric field at that point, we construct a Gaussian surface in the form of a small cylinder (Fig. 22.28). One end face, with area A, lies within the conductor and the other lies just outside. The electric field is zero at all points within the conductor. Outside the conductor the component of $\vec{E}$ perpendicular to the side walls of the cylinder is zero, and over the end face the perpendicular component is equal to $E_\perp$. (If σ is positive, the electric field points out of the conductor and $E_\perp$ is positive; if σ is negative, the field points inward and $E_\perp$ is negative.) Hence the total flux through the surface is $E_\perp A$. The charge enclosed within the Gaussian surface is σA, so from Gauss's law,

$$E_\perp A = \frac{\sigma A}{\epsilon_0} \quad \text{and} \quad E_\perp = \frac{\sigma}{\epsilon_0} \quad \text{(field at the surface of a conductor)} \tag{22.10}$$

We can check this with the results we have obtained for spherical, cylindrical, and plane surfaces.

We showed in Example 22.8 that the field magnitude between two infinite flat oppositely charged conducting plates also equals σ/ϵ_0. In this case the field magnitude E is the same at *all* distances from the plates, but in all other cases E decreases with increasing distance from the surface.

Conceptual Example 22.12 Field at the surface of a conducting sphere

Verify Eq. (22.10) for a conducting sphere with radius R and total charge q.

SOLUTION

In Example 22.5 (Section 22.4) we showed that the electric field just outside the surface is

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}$$

The surface charge density is uniform and equal to q divided by the surface area of the sphere:

$$\sigma = \frac{q}{4\pi R^2}$$

Comparing these two expressions, we see that $E = \sigma/\epsilon_0$, which verifies Eq. (22.10).

Example 22.13 Electric field of the earth

The earth (a conductor) has a net electric charge. The resulting electric field near the surface has an average value of about 150 N/C, directed toward the center of the earth. (a) What is the corresponding surface charge density? (b) What is the *total* surface charge of the earth?

SOLUTION

IDENTIFY and SET UP: We are given the electric-field magnitude at the surface of the conducting earth. We can calculate the surface charge density σ using Eq. (22.10). The total charge Q on the earth's surface is then the product of σ and the earth's surface area.

EXECUTE: (a) The direction of the field means that σ is negative (corresponding to $\vec{E}$ being directed *into* the surface, so $E_\perp$ is negative). From Eq. (22.10),

$$\sigma = \epsilon_0 E_\perp = (8.85 \times 10^{-12}\ \mathrm{C^2/N\cdot m^2})(-150\ \mathrm{N/C})$$
$$= -1.33 \times 10^{-9}\ \mathrm{C/m^2} = -1.33\ \mathrm{nC/m^2}$$

(b) The earth's surface area is $4\pi R_E^2$, where $R_E = 6.38 \times 10^6$ m is the radius of the earth (see Appendix F). The total charge Q is the product $4\pi R_E^2\sigma$, or

$$Q = 4\pi(6.38 \times 10^6\ \mathrm{m})^2(-1.33 \times 10^{-9}\ \mathrm{C/m^2})$$
$$= -6.8 \times 10^5\ \mathrm{C} = -680\ \mathrm{kC}$$

EVALUATE: You can check our result in part (b) using the result of Example 22.5. Solving for Q, we find

$$Q = 4\pi\epsilon_0 R^2 E_\perp$$
$$= \frac{1}{9.0 \times 10^9\ \mathrm{N\cdot m^2/C^2}}(6.38 \times 10^6\ \mathrm{m})^2(-150\ \mathrm{N/C})$$
$$= -6.8 \times 10^5\ \mathrm{C}$$

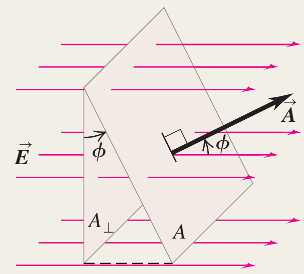
One electron has a charge of -1.60×10^{-19} C. Hence this much excess negative electric charge corresponds to there being $(-6.8 \times 10^5\ \mathrm{C})/(-1.60 \times 10^{-19}\ \mathrm{C}) = 4.2 \times 10^{24}$ excess electrons on the earth, or about 7 moles of excess electrons. This is compensated by an equal *deficiency* of electrons in the earth's upper atmosphere, so the combination of the earth and its atmosphere is electrically neutral.

Test Your Understanding of Section 22.5 A hollow conducting sphere has no net charge. There is a positive point charge q at the center of the spherical cavity within the sphere. You connect a conducting wire from the outside of the sphere to ground. Will you measure an electric field outside the sphere? ❙

CHAPTER 22 SUMMARY

Electric flux: Electric flux is a measure of the "flow" of electric field through a surface. It is equal to the product of an area element and the perpendicular component of $\vec{E}$, integrated over a surface. (See Examples 22.1–22.3.)

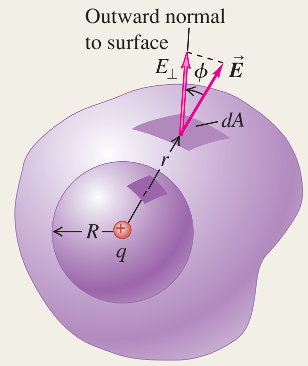
$$\Phi_E = \int E\cos\phi \, dA = \int E_\perp \, dA = \int \vec{E}\cdot d\vec{A} \quad (22.5)$$



Gauss's law: Gauss's law states that the total electric flux through a closed surface, which can be written as the surface integral of the component of $\vec{E}$ normal to the surface, equals a constant times the total charge Q_{encl} enclosed by the surface. Gauss's law is logically equivalent to Coulomb's law, but its use greatly simplifies problems with a high degree of symmetry. (See Examples 22.4–22.10.)

When excess charge is placed on a conductor and is at rest, it resides entirely on the surface, and $\vec{E} = \mathbf{0}$ everywhere in the material of the conductor. (See Examples 22.11–22.13.)

$$\Phi_E = \oint E\cos\phi \, dA = \oint E_\perp \, dA = \oint \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0} \quad (22.8), (22.9)$$



Electric field of various symmetric charge distributions: The following table lists electric fields caused by several symmetric charge distributions. In the table, q, Q, λ, and σ refer to the *magnitudes* of the quantities.

Charge Distribution	Point in Electric Field	Electric Field Magnitude
Single point charge q	Distance r from q	$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$
Charge q on surface of conducting sphere with radius R	Outside sphere, $r > R$	$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$
	Inside sphere, $r < R$	$E = 0$
Infinite wire, charge per unit length λ	Distance r from wire	$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$
Infinite conducting cylinder with radius R, charge per unit length λ	Outside cylinder, $r > R$	$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$
	Inside cylinder, $r < R$	$E = 0$
Solid insulating sphere with radius R, charge Q distributed uniformly throughout volume	Outside sphere, $r > R$	$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$
	Inside sphere, $r < R$	$E = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}$
Infinite sheet of charge with uniform charge per unit area σ	Any point	$E = \frac{\sigma}{2\epsilon_0}$
Two oppositely charged conducting plates with surface charge densities $+\sigma$ and $-\sigma$	Any point between plates	$E = \frac{\sigma}{\epsilon_0}$
Charged conductor	Just outside the conductor	$E = \frac{\sigma}{\epsilon_0}$

BRIDGING PROBLEM Electric Field Inside a Hydrogen Atom

A hydrogen atom is made up of a proton of charge $+Q = 1.60 \times 10^{-19}$ C and an electron of charge $-Q = -1.60 \times 10^{-19}$ C. The proton may be regarded as a point charge at $r = 0$, the center of the atom. The motion of the electron causes its charge to be "smeared out" into a spherical distribution around the proton, so that the electron is equivalent to a charge per unit volume of $\rho(r) = -(Q/\pi a_0^3)e^{-2r/a_0}$, where $a_0 = 5.29 \times 10^{-11}$ m is called the *Bohr radius.* (a) Find the total amount of the hydrogen atom's charge that is enclosed within a sphere with radius r centered on the proton. (b) Find the electric field (magnitude and direction) caused by the charge of the hydrogen atom as a function of r. (c) Make a graph as a function of r of the ratio of the electric-field magnitude E to the magnitude of the field due to the proton alone.

SOLUTION GUIDE

See MasteringPhysics® study area for a Video Tutor solution.

IDENTIFY and SET UP

1. The charge distribution in this problem is spherically symmetric, just as in Example 22.9, so you can solve it using Gauss's law.
2. The charge within a sphere of radius r includes the proton charge $+Q$ plus the portion of the electron charge distribution that lies within the sphere. The difference from Example 22.9 is that the electron charge distribution is *not* uniform, so the charge enclosed within a sphere of radius r is *not* simply the charge density multiplied by the volume $4\pi r^3/3$ of the sphere. Instead, you'll have to do an integral.
3. Consider a thin spherical shell centered on the proton, with radius r' and infinitesimal thickness dr'. Since the shell is so thin, every point within the shell is at essentially the same radius from the proton. Hence the amount of electron charge within this shell *is* equal to the electron charge density $\rho(r')$ at this radius multiplied by the volume dV of the shell. What is dV in terms of r'?
4. The total electron charge within a radius r equals the integral of $\rho(r')dV$ from $r' = 0$ to $r' = r$. Set up this integral (but don't solve it yet), and use it to write an expression for the total charge (including the proton) within a sphere of radius r.

EXECUTE

5. Integrate your expression from step 4 to find the charge within radius r. *Hint:* Integrate by substitution: Change the integration variable from r' to $x = 2r'/a_0$. You can calculate the integral $\int x^2 e^{-x}\,dx$ using integration by parts, or you can look it up in a table of integrals or on the World Wide Web.
6. Use Gauss's law and your results from step 5 to find the electric field at a distance r from the proton.
7. Find the ratio referred to in part (c) and graph it versus r. (You'll actually find it simplest to graph this function versus the quantity r/a_0.)

EVALUATE

8. How do your results for the enclosed charge and the electric-field magnitude behave in the limit $r \rightarrow 0$? In the limit $r \rightarrow \infty$? Explain your results.

Problems

For instructor-assigned homework, go to www.masteringphysics.com

•, ••, •••: Problems of increasing difficulty. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q22.1 A rubber balloon has a single point charge in its interior. Does the electric flux through the balloon depend on whether or not it is fully inflated? Explain your reasoning.

Q22.2 Suppose that in Fig. 22.15 both charges were positive. What would be the fluxes through each of the four surfaces in the example?

Q22.3 In Fig. 22.15, suppose a third point charge were placed outside the purple Gaussian surface C. Would this affect the electric flux through any of the surfaces A, B, C, or D in the figure? Why or why not?

Q22.4 A certain region of space bounded by an imaginary closed surface contains no charge. Is the electric field always zero everywhere on the surface? If not, under what circumstances is it zero on the surface?

Q22.5 A spherical Gaussian surface encloses a point charge q. If the point charge is moved from the center of the sphere to a point away from the center, does the electric field at a point on the surface change? Does the total flux through the Gaussian surface change? Explain.

Q22.6 You find a sealed box on your doorstep. You suspect that the box contains several charged metal spheres packed in insulating material. How can you determine the total net charge inside the box without opening the box? Or isn't this possible?

Q22.7 A solid copper sphere has a net positive charge. The charge is distributed uniformly over the surface of the sphere, and the electric field inside the sphere is zero. Then a negative point charge outside the sphere is brought close to the surface of the sphere. Is all the net charge on the sphere still on its surface? If so, is this charge still distributed uniformly over the surface? If it is not uniform, how is it distributed? Is the electric field inside the sphere still zero? In each case justify your answers.

Q22.8 If the electric field of a point charge were proportional to $1/r^3$ instead of $1/r^2$, would Gauss's law still be valid? Explain your reasoning. (*Hint:* Consider a spherical Gaussian surface centered on a single point charge.)

Q22.9 In a conductor, one or more electrons from each atom are free to roam throughout the volume of the conductor. Does this contradict the statement that any excess charge on a solid conductor must reside on its surface? Why or why not?

Q22.10 You charge up the van de Graaff generator shown in Fig. 22.26, and then bring an identical but uncharged hollow conducting sphere near it, without letting the two spheres touch. Sketch the distribution of charges on the second sphere. What is the net flux through the second sphere? What is the electric field inside the second sphere?

Q22.11 A lightning rod is a rounded copper rod mounted on top of a building and welded to a heavy copper cable running down into the ground. Lightning rods are used to protect houses and barns from lightning; the lightning current runs through the copper rather than through the building. Why? Why should the end of the rod be rounded?

Q22.12 A solid conductor has a cavity in its interior. Would the presence of a point charge inside the cavity affect the electric field outside the conductor? Why or why not? Would the presence of a point charge outside the conductor affect the electric field inside the cavity? Again, why or why not?

Q22.13 Explain this statement: "In a static situation, the electric field at the surface of a conductor can have no component parallel to the surface because this would violate the condition that the charges on the surface are at rest." Would this same statement be valid for the electric field at the surface of an *insulator*? Explain your answer and the reason for any differences between the cases of a conductor and an insulator.

Q22.14 In a certain region of space, the electric field $\vec{E}$ is uniform. (a) Use Gauss's law to prove that this region of space must be electrically neutral; that is, the volume charge density ρ must be zero. (b) Is the converse true? That is, in a region of space where there is no charge, must $\vec{E}$ be uniform? Explain.

Q22.15 (a) In a certain region of space, the volume charge density ρ has a uniform positive value. Can $\vec{E}$ be uniform in this region? Explain. (b) Suppose that in this region of uniform positive ρ there is a "bubble" within which $\rho = 0$. Can $\vec{E}$ be uniform within this bubble? Explain.

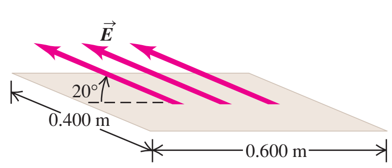
EXERCISES

Section 22.2 Calculating Electric Flux

22.1 • A flat sheet of paper of area 0.250 m^2 is oriented so that the normal to the sheet is at an angle of 60° to a uniform electric field of magnitude 14 N/C. (a) Find the magnitude of the electric flux through the sheet. (b) Does the answer to part (a) depend on the shape of the sheet? Why or why not? (c) For what angle ϕ between the normal to the sheet and the electric field is the magnitude of the flux through the sheet (i) largest and (ii) smallest? Explain your answers.

22.2 •• A flat sheet is in the shape of a rectangle with sides of lengths 0.400 m and 0.600 m. The sheet is immersed in a uniform electric field of magnitude 75.0 N/C that is directed at 20° from the plane of the sheet (Fig. E22.2). Find the magnitude of the electric flux through the sheet.

Figure **E22.2**



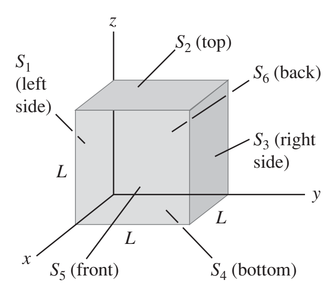
22.3 • You measure an electric field of 1.25×10^6 N/C at a distance of 0.150 m from a point charge. There is no other source of electric field in the region other than this point charge. (a) What is the electric flux through the surface of a sphere that has this charge at its center and that has radius 0.150 m? (b) What is the magnitude of this charge?

22.4 • It was shown in Example 21.11 (Section 21.5) that the electric field due to an infinite line of charge is perpendicular to the line and has magnitude $E = \lambda/2\pi\epsilon_0 r$. Consider an imaginary cylinder with radius $r = 0.250$ m and length $l = 0.400$ m that has an infinite line of positive charge running along its axis. The charge per unit length on the line is $\lambda = 3.00\ \mu$C/m. (a) What is the electric flux through the cylinder due to this infinite line of charge? (b) What is the flux through the cylinder if its radius is increased to $r = 0.500$ m? (c) What is the flux through the cylinder if its length is increased to $l = 0.800$ m?

22.5 •• A hemispherical surface with radius r in a region of uniform electric field $\vec{E}$ has its axis aligned parallel to the direction of the field. Calculate the flux through the surface.

22.6 • The cube in Fig. E22.6 has sides of length L =10.0 cm. The electric field is uniform, has magnitude $E = 4.00 \times 10^3$ N/C, and is parallel to the xy-plane at an angle of 53.1° measured from the $+x$-axis toward the $+y$-axis. (a) What is the electric flux through each of the six cube faces S_1, S_2, S_3, S_4, S_5, and S_6? (b) What is the total electric flux through all faces of the cube?

Figure **E22.6**

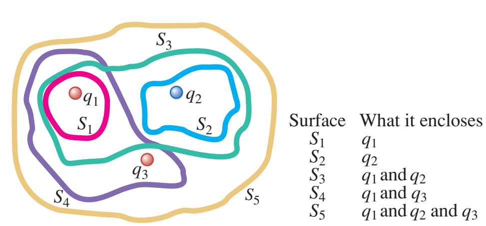


Section 22.3 Gauss's Law

22.7 • **BIO** As discussed in Section 22.5, human nerve cells have a net negative charge and the material in the interior of the cell is a good conductor. If a cell has a net charge of −8.65 pC, what are the magnitude and direction (inward or outward) of the net flux through the cell boundary?

22.8 • The three small spheres shown in Fig. E22.8 carry charges $q_1 = 4.00$ nC, $q_2 = -7.80$ nC, and $q_3 = 2.40$ nC. Find the net electric flux through each of the following closed surfaces shown in cross section in the figure: (a) S_1; (b) S_2; (c) S_3; (d) S_4; (e) S_5. (f) Do your answers to parts (a)–(e) depend on how the charge is distributed over each small sphere? Why or why not?

Figure **E22.8**



22.9 •• A charged paint is spread in a very thin uniform layer over the surface of a plastic sphere of diameter 12.0 cm, giving it a charge of −35.0 μC. Find the electric field (a) just inside the paint layer; (b) just outside the paint layer; (c) 5.00 cm outside the surface of the paint layer.

22.10 • A point charge $q_1 = 4.00$ nC is located on the x-axis at $x = 2.00$ m, and a second point charge $q_2 = -6.00$ nC is on the y-axis at $y = 1.00$ m. What is the total electric flux due to these two point charges through a spherical surface centered at the origin and with radius (a) 0.500 m, (b) 1.50 m, (c) 2.50 m?

22.11 • A 6.20-μC point charge is at the center of a cube with sides of length 0.500 m. (a) What is the electric flux through one of the six faces of the cube? (b) How would your answer to part (a) change if the sides were 0.250 m long? Explain.

22.12 • **Electric Fields in an Atom.** The nuclei of large atoms, such as uranium, with 92 protons, can be modeled as spherically symmetric spheres of charge. The radius of the uranium nucleus is approximately 7.4×10^{-15} m. (a) What is the electric field this nucleus produces just outside its surface? (b) What magnitude of electric field does it produce at the distance of the electrons, which is about 1.0×10^{-10} m? (c) The electrons can be modeled as forming a uniform shell of negative charge. What net electric field do they produce at the location of the nucleus?

22.13 • A point charge of $+5.00$ μC is located on the x-axis at $x = 4.00$ m, next to a spherical surface of radius 3.00 m centered at the origin. (a) Calculate the magnitude of the electric field at $x = 3.00$ m. (b) Calculate the magnitude of the electric field at $x = -3.00$ m. (c) According to Gauss's law, the net flux through the sphere is zero because it contains no charge. Yet the field due to the external charge is much stronger on the near side of the sphere (i.e., at $x = 3.00$ m) than on the far side (at $x = -3.00$ m). How, then, can the flux into the sphere (on the near side) equal the flux out of it (on the far side)? Explain. A sketch will help.

Section 22.4 Applications of Gauss's Law and Section 22.5 Charges on Conductors

22.14 •• A solid metal sphere with radius 0.450 m carries a net charge of 0.250 nC. Find the magnitude of the electric field (a) at a point 0.100 m outside the surface of the sphere and (b) at a point inside the sphere, 0.100 m below the surface.

22.15 •• Two very long uniform lines of charge are parallel and are separated by 0.300 m. Each line of charge has charge per unit length $+5.20$ μC/m. What magnitude of force does one line of charge exert on a 0.0500-m section of the other line of charge?

22.16 • Some planetary scientists have suggested that the planet Mars has an electric field somewhat similar to that of the earth, producing a net electric flux of 3.63×10^{16} N $\cdot$ m^2/C at the planet's surface, directed toward the center of the planet. Calculate: (a) the total electric charge on the planet; (b) the electric field at the planet's surface (refer to the astronomical data inside the back cover); (c) the charge density on Mars, assuming all the charge is uniformly distributed over the planet's surface.

22.17 •• How many excess electrons must be added to an isolated spherical conductor 32.0 cm in diameter to produce an electric field of 1150 N/C just outside the surface?

22.18 •• The electric field 0.400 m from a very long uniform line of charge is 840 N/C. How much charge is contained in a 2.00-cm section of the line?

22.19 •• A very long uniform line of charge has charge per unit length 4.80 μC/m and lies along the x-axis. A second long uniform line of charge has charge per unit length -2.40 μC/m and is parallel to the x-axis at $y = 0.400$ m. What is the net electric field (magnitude and direction) at the following points on the y-axis: (a) $y = 0.200$ m and (b) $y = 0.600$ m?

22.20 • (a) At a distance of 0.200 cm from the center of a charged conducting sphere with radius 0.100 cm, the electric field is 480 N/C. What is the electric field 0.600 cm from the center of the sphere? (b) At a distance of 0.200 cm from the axis of a very long charged conducting cylinder with radius 0.100 cm, the electric field is 480 N/C. What is the electric field 0.600 cm from the axis of the cylinder? (c) At a distance of 0.200 cm from a large uniform sheet of charge, the electric field is 480 N/C. What is the electric field 1.20 cm from the sheet?

22.21 •• A hollow, conducting sphere with an outer radius of 0.250 m and an inner radius of 0.200 m has a uniform surface charge density of $+6.37 \times 10^{-6}$ C/m^2. A charge of -0.500 μC is now introduced into the cavity inside the sphere. (a) What is the new charge density on the outside of the sphere? (b) Calculate the strength of the electric field just outside the sphere. (c) What is the electric flux through a spherical surface just inside the inner surface of the sphere?

22.22 •• A point charge of -2.00 μC is located in the center of a spherical cavity of radius 6.50 cm inside an insulating charged solid. The charge density in the solid is $\rho = 7.35 \times 10^{-4}$ C/m^3. Calculate the electric field inside the solid at a distance of 9.50 cm from the center of the cavity.

22.23 •• The electric field at a distance of 0.145 m from the surface of a solid insulating sphere with radius 0.355 m is 1750 N/C. (a) Assuming the sphere's charge is uniformly distributed, what is the charge density inside it? (b) Calculate the electric field inside the sphere at a distance of 0.200 m from the center.

22.24 •• **CP** A very small object with mass 8.20×10^{-9} kg and positive charge 6.50×10^{-9} C is projected directly toward a very large insulating sheet of positive charge that has uniform surface charge density 5.90×10^{-8} C/m^2. The object is initially 0.400 m from the sheet. What initial speed must the object have in order for its closest distance of approach to the sheet to be 0.100 m?

22.25 •• **CP** At time $t = 0$ a proton is a distance of 0.360 m from a very large insulating sheet of charge and is moving parallel to the sheet with speed 9.70×10^2 m/s. The sheet has uniform surface charge density 2.34×10^{-9} C/m^2. What is the speed of the proton at $t = 5.00 \times 10^{-8}$ s?

22.26 •• **CP** An electron is released from rest at a distance of 0.300 m from a large insulating sheet of charge that has uniform surface charge density $+2.90 \times 10^{-12}$ C/m^2. (a) How much work is done on the electron by the electric field of the sheet as the electron moves from its initial position to a point 0.050 m from the sheet? (b) What is the speed of the electron when it is 0.050 m from the sheet?

22.27 ••• **CP CALC** An insulating sphere of radius $R = 0.160$ m has uniform charge density $\rho = +7.20 \times 10^{-9}$ C/m^3. A small object that can be treated as a point charge is released from rest just outside the surface of the sphere. The small object has positive charge $q = 3.40 \times 10^{-6}$ C. How much work does the electric field of the sphere do on the object as the object moves to a point very far from the sphere?

22.28 • A conductor with an inner cavity, like that shown in Fig. 22.23c, carries a total charge of $+5.00$ nC. The charge within the cavity, insulated from the conductor, is -6.00 nC. How much charge is on (a) the inner surface of the conductor and (b) the outer surface of the conductor?

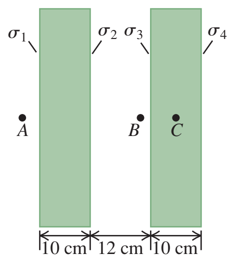
22.29 • Apply Gauss's law to the Gaussian surfaces S_2, S_3, and S_4 in Fig. 22.21b to calculate the electric field between and outside the plates.

22.30 • A square insulating sheet 80.0 cm on a side is held horizontally. The sheet has 7.50 nC of charge spread uniformly over its area. (a) Calculate the electric field at a point 0.100 mm above the center of the sheet. (b) Estimate the electric field at a point 100 m above the center of the sheet. (c) Would the answers to parts (a) and (b) be different if the sheet were made of a conducting material? Why or why not?

22.31 • An infinitely long cylindrical conductor has radius R and uniform surface charge density σ. (a) In terms of σ and R, what is the charge per unit length λ for the cylinder? (b) In terms of σ, what is the magnitude of the electric field produced by the charged cylinder at a distance $r > R$ from its axis? (c) Express the result of part (b) in terms of λ and show that the electric field outside the cylinder is the same as if all the charge were on the axis. Compare your result to the result for a line of charge in Example 22.6 (Section 22.4).

22.32 • Two very large, nonconducting plastic sheets, each 10.0 cm thick, carry uniform charge densities σ_1, σ_2, σ_3, and σ_4 on their surfaces, as shown in Fig. E22.32. These surface charge densities have the values $\sigma_1 = -6.00\ \mu\text{C/m}^2$, $\sigma_2 = +5.00\ \mu\text{C/m}^2$, $\sigma_3 = +2.00\ \mu\text{C/m}^2$, and $\sigma_4 = +4.00\ \mu\text{C/m}^2$. Use Gauss's law to find the magnitude and direction of the electric field at the following points, far from the edges of these sheets: (a) point A, 5.00 cm from the left face of the left-hand sheet; (b) point B, 1.25 cm from the inner surface of the right-hand sheet; (c) point C, in the middle of the right-hand sheet.

Figure **E22.32**



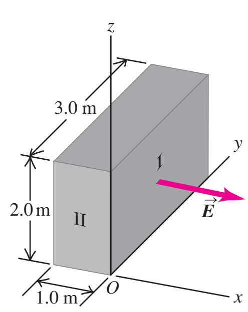
22.33 • A negative charge $-Q$ is placed inside the cavity of a hollow metal solid. The outside of the solid is grounded by connecting a conducting wire between it and the earth. (a) Is there any excess charge induced on the inner surface of the piece of metal? If so, find its sign and magnitude. (b) Is there any excess charge on the outside of the piece of metal? Why or why not? (c) Is there an electric field in the cavity? Explain. (d) Is there an electric field within the metal? Why or why not? Is there an electric field outside the piece of metal? Explain why or why not. (e) Would someone outside the solid measure an electric field due to the charge $-Q$? Is it reasonable to say that the grounded conductor has *shielded* the region from the effects of the charge $-Q$? In principle, could the same thing be done for gravity? Why or why not?

PROBLEMS

22.34 •• A cube has sides of length $L = 0.300$ m. It is placed with one corner at the origin as shown in Fig. E22.6. The electric field is not uniform but is given by $\vec{E} = (-5.00\ \text{N/C}\cdot\text{m})x\hat{\imath} + (3.00\ \text{N/C}\cdot\text{m})z\hat{k}$. (a) Find the electric flux through each of the six cube faces S_1, S_2, S_3, S_4, S_5, and S_6. (b) Find the total electric charge inside the cube.

22.35 • The electric field $\vec{E}$ in Fig. P22.35 is everywhere parallel to the x-axis, so the components E_y and E_z are zero. The x-component of the field E_x depends on x but not on y and z. At points in the yz-plane (where $x = 0$), $E_x = 125$ N/C. (a) What is the electric flux through surface I in Fig. P22.35? (b) What is the electric flux through surface II? (c) The volume shown in the figure is a small section of a very large insulating slab 1.0 m thick. If there is a total charge of -24.0 nC within the volume shown, what are the magnitude and direction of $\vec{E}$ at the face opposite surface I? (d) Is the electric field produced only by charges within the slab, or is the field also due to charges outside the slab? How can you tell?

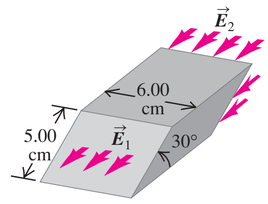
Figure **P22.35**



22.36 •• **CALC** In a region of space there is an electric field $\vec{E}$ that is in the z-direction and that has magnitude $E = (964\ \text{N/(C}\cdot\text{m}))x$. Find the flux for this field through a square in the xy-plane at $z = 0$ and with side length 0.350 m. One side of the square is along the $+x$-axis and another side is along the $+y$-axis.

22.37 •• The electric field $\vec{E}_1$ at one face of a parallelepiped is uniform over the entire face and is directed out of the face. At the opposite face, the electric field $\vec{E}_2$ is also uniform over the entire face and is directed into that face (Fig. P22.37). The two faces in question are inclined at 30.0° from the horizontal, while $\vec{E}_1$ and $\vec{E}_2$ are both horizontal; $\vec{E}_1$ has a magnitude of 2.50×10^4 N/C, and $\vec{E}_2$ has a magnitude of 7.00×10^4 N/C. (a) Assuming that no other electric field lines cross the surfaces of the parallelepiped, determine the net charge contained within. (b) Is the electric field produced only by the charges within the parallelepiped, or is the field also due to charges outside the parallelepiped? How can you tell?

Figure **P22.37**



22.38 • A long line carrying a uniform linear charge density $+50.0\ \mu\text{C/m}$ runs parallel to and 10.0 cm from the surface of a large, flat plastic sheet that has a uniform surface charge density of $-100\ \mu\text{C/m}^2$ on one side. Find the location of all points where an α particle would feel no force due to this arrangement of charged objects.

22.39 • **The Coaxial Cable.** A long coaxial cable consists of an inner cylindrical conductor with radius a and an outer coaxial cylinder with inner radius b and outer radius c. The outer cylinder is mounted on insulating supports and has no net charge. The inner cylinder has a uniform positive charge per unit length λ. Calculate the electric field (a) at any point between the cylinders a distance r from the axis and (b) at any point outside the outer cylinder. (c) Graph the magnitude of the electric field as a function of the distance r from the axis of the cable, from $r = 0$ to $r = 2c$. (d) Find the charge per unit length on the inner surface and on the outer surface of the outer cylinder.

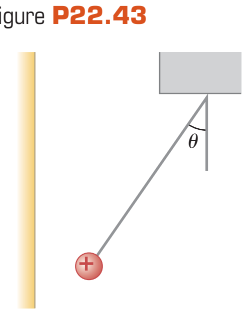
22.40 • A very long conducting tube (hollow cylinder) has inner radius a and outer radius b. It carries charge per unit length $+\alpha$, where α is a positive constant with units of C/m. A line of charge lies along the axis of the tube. The line of charge has charge per unit length $+\alpha$. (a) Calculate the electric field in terms of α and the distance r from the axis of the tube for (i) $r < a$; (ii) $a < r < b$; (iii) $r > b$. Show your results in a graph of E as a function of r. (b) What is the charge per unit length on (i) the inner surface of the tube and (ii) the outer surface of the tube?

22.41 • Repeat Problem 22.40, but now let the conducting tube have charge per unit length $-\alpha$. As in Problem 22.40, the line of charge has charge per unit length $+\alpha$.

22.42 • A very long, solid cylinder with radius R has positive charge uniformly distributed throughout it, with charge per unit volume ρ. (a) Derive the expression for the electric field inside the volume at a distance r from the axis of the cylinder in terms of the charge density ρ. (b) What is the electric field at a point outside the volume in terms of the charge per unit length λ in the cylinder? (c) Compare the answers to parts (a) and (b) for $r = R$. (d) Graph the electric-field magnitude as a function of r from $r = 0$ to $r = 3R$.

22.43 •• **CP** A small sphere with a mass of 4.00×10^{-6} kg and carrying a charge of 5.00×10^{-8} C hangs from a thread near a very large, charged insulating

Figure **P22.43**



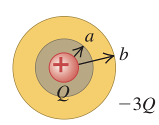
sheet, as shown in Fig. P22.43. The charge density on the surface of the sheet is uniform and equal to $2.50 \times 10^{-9}\ \text{C/m}^2$. Find the angle of the thread.

22.44 • **A Sphere in a Sphere.** A solid conducting sphere carrying charge q has radius a. It is inside a concentric hollow conducting sphere with inner radius b and outer radius c. The hollow sphere has no net charge. (a) Derive expressions for the electric-field magnitude in terms of the distance r from the center for the regions $r < a$, $a < r < b$, $b < r < c$, and $r > c$. (b) Graph the magnitude of the electric field as a function of r from $r = 0$ to $r = 2c$. (c) What is the charge on the inner surface of the hollow sphere? (d) On the outer surface? (e) Represent the charge of the small sphere by four plus signs. Sketch the field lines of the system within a spherical volume of radius $2c$.

22.45 • A solid conducting sphere with radius R that carries positive charge Q is concentric with a very thin insulating shell of radius $2R$ that also carries charge Q. The charge Q is distributed uniformly over the insulating shell. (a) Find the electric field (magnitude and direction) in each of the regions $0 < r < R$, $R < r < 2R$, and $r > 2R$. (b) Graph the electric-field magnitude as a function of r.

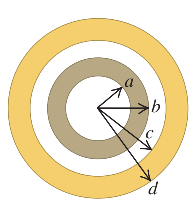
22.46 • A conducting spherical shell with inner radius a and outer radius b has a positive point charge Q located at its center. The total charge on the shell is $-3Q$, and it is insulated from its surroundings (Fig. P22.46). (a) Derive expressions for the electric-field magnitude in terms of the distance r from the center for the regions $r < a$, $a < r < b$, and $r > b$. (b) What is the surface charge density on the inner surface of the conducting shell? (c) What is the surface charge density on the outer surface of the conducting shell? (d) Sketch the electric field lines and the location of all charges. (e) Graph the electric-field magnitude as a function of r.

Figure P22.46



22.47 • **Concentric Spherical Shells.** A small conducting spherical shell with inner radius a and outer radius b is concentric with a larger conducting spherical shell with inner radius c and outer radius d (Fig. P22.47). The inner shell has total charge $+2q$, and the outer shell has charge $+4q$. (a) Calculate the electric field (magnitude and direction) in terms of q and the distance r from the common center of the two shells for (i) $r < a$; (ii) $a < r < b$; (iii) $b < r < c$; (iv) $c < r < d$; (v) $r > d$. Show your results in a graph of the radial component of $\vec{E}$ as a function of r. (b) What is the total charge on the (i) inner surface of the small shell; (ii) outer surface of the small shell; (iii) inner surface of the large shell; (iv) outer surface of the large shell?

Figure P22.47



22.48 • Repeat Problem 22.47, but now let the outer shell have charge $-2q$. As in Problem 22.47, the inner shell has charge $+2q$.

22.49 • Repeat Problem 22.47, but now let the outer shell have charge $-4q$. As in Problem 22.47, the inner shell has charge $+2q$.

22.50 • A solid conducting sphere with radius R carries a positive total charge Q. The sphere is surrounded by an insulating shell with inner radius R and outer radius $2R$. The insulating shell has a uniform charge density ρ. (a) Find the value of ρ so that the net charge of the entire system is zero. (b) If ρ has the value found in part (a), find the electric field (magnitude and direction) in each of the regions $0 < r < R$, $R < r < 2R$, and $r > 2R$. Show your results in a graph of the radial component of $\vec{E}$ as a function of r. (c) As a general rule, the electric field is discontinuous only at locations where there is a thin sheet of charge. Explain how your results in part (b) agree with this rule.

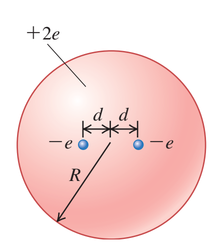
22.51 • Negative charge $-Q$ is distributed uniformly over the surface of a thin spherical insulating shell with radius R. Calculate the force (magnitude and direction) that the shell exerts on a positive point charge q located (a) a distance $r > R$ from the center of the shell (outside the shell) and (b) a distance $r < R$ from the center of the shell (inside the shell).

22.52 •• (a) How many excess electrons must be distributed uniformly within the volume of an isolated plastic sphere 30.0 cm in diameter to produce an electric field of 1390 N/C just outside the surface of the sphere? (b) What is the electric field at a point 10.0 cm outside the surface of the sphere?

22.53 ••• **CALC** An insulating hollow sphere has inner radius a and outer radius b. Within the insulating material the volume charge density is given by $\rho(r) = \frac{\alpha}{r}$, where α is a positive constant. (a) In terms of α and a, what is the magnitude of the electric field at a distance r from the center of the shell, where $a < r < b$? (b) A point charge q is placed at the center of the hollow space, at $r = 0$. In terms of α and a, what value must q have (sign and magnitude) in order for the electric field to be constant in the region $a < r < b$, and what then is the value of the constant field in this region?

22.54 •• **CP** **Thomson's Model of the Atom.** In the early years of the 20th century, a leading model of the structure of the atom was that of the English physicist J. J. Thomson (the discoverer of the electron). In Thomson's model, an atom consisted of a sphere of positively charged material in which were embedded negatively charged electrons, like chocolate chips in a ball of cookie dough. Consider such an atom consisting of one electron with mass m and charge $-e$, which may be regarded as a point charge, and a uniformly charged sphere of charge $+e$ and radius R. (a) Explain why the equilibrium position of the electron is at the center of the nucleus. (b) In Thomson's model, it was assumed that the positive material provided little or no resistance to the motion of the electron. If the electron is displaced from equilibrium by a distance less than R, show that the resulting motion of the electron will be simple harmonic, and calculate the frequency of oscillation. (*Hint:* Review the definition of simple harmonic motion in Section 14.2. If it can be shown that the net force on the electron is of this form, then it follows that the motion is simple harmonic. Conversely, if the net force on the electron does not follow this form, the motion is not simple harmonic.) (c) By Thomson's time, it was known that excited atoms emit light waves of only certain frequencies. In his model, the frequency of emitted light is the same as the oscillation frequency of the electron or electrons in the atom. What would the radius of a Thomson-model atom have to be for it to produce red light of frequency 4.57×10^{14} Hz? Compare your answer to the radii of real atoms, which are of the order of 10^{-10} m (see Appendix F for data about the electron). (d) If the electron were displaced from equilibrium by a distance greater than R, would the electron oscillate? Would its motion be simple harmonic? Explain your reasoning. (*Historical note:* In 1910, the atomic nucleus was discovered, proving the Thomson model to be incorrect. An atom's positive charge is not spread over its volume as Thomson supposed, but is concentrated in the tiny nucleus of radius 10^{-14} to 10^{-15} m.)

Figure P22.55



22.55 • **Thomson's Model of the Atom, Continued.** Using Thomson's (outdated) model of the atom described in Problem 22.54, consider an atom consisting of two electrons, each of charge $-e$, embedded in a sphere of charge $+2e$ and radius R. In equilibrium, each electron is a distance d from the center of the atom (Fig. P22.55). Find the distance d in terms of the other properties of the atom.

22.56 • **A Uniformly Charged Slab.** A slab of insulating material has thickness $2d$ and is oriented so that its faces are parallel to the yz-plane and given by the planes $x = d$ and $x = -d$. The y- and z-dimensions of the slab are very large compared to d and may be treated as essentially infinite. The slab has a uniform positive charge density ρ. (a) Explain why the electric field due to the slab is zero at the center of the slab $(x = 0)$. (b) Using Gauss's law, find the electric field due to the slab (magnitude and direction) at all points in space.

22.57 • CALC **A Nonuniformly Charged Slab.** Repeat Problem 22.56, but now let the charge density of the slab be given by $\rho(x) = \rho_0(x/d)^2$, where ρ_0 is a positive constant.

22.58 • CALC A nonuniform, but spherically symmetric, distribution of charge has a charge density $\rho(r)$ given as follows:

$$\rho(r) = \rho_0(1 - 4r/3R) \quad \text{for } r \le R$$
$$\rho(r) = 0 \quad \text{for } r \ge R$$

where ρ_0 is a positive constant. (a) Find the total charge contained in the charge distribution. (b) Obtain an expression for the electric field in the region $r \ge R$. (c) Obtain an expression for the electric field in the region $r \le R$. (d) Graph the electric-field magnitude E as a function of r. (e) Find the value of r at which the electric field is maximum, and find the value of that maximum field.

22.59 • CP CALC **Gauss's Law for Gravitation.** The gravitational force between two point masses separated by a distance r is proportional to $1/r^2$, just like the electric force between two point charges. Because of this similarity between gravitational and electric interactions, there is also a Gauss's law for gravitation. (a) Let $\vec{g}$ be the acceleration due to gravity caused by a point mass m at the origin, so that $\vec{g} = -(Gm/r^2)\hat{r}$. Consider a spherical Gaussian surface with radius r centered on this point mass, and show that the flux of $\vec{g}$ through this surface is given by

$$\oint \vec{g} \cdot d\vec{A} = -4\pi Gm$$

(b) By following the same logical steps used in Section 22.3 to obtain Gauss's law for the electric field, show that the flux of $\vec{g}$ through *any* closed surface is given by

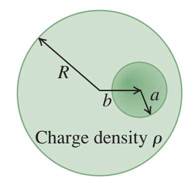
$$\oint \vec{g} \cdot d\vec{A} = -4\pi GM_{\text{encl}}$$

where M_{encl} is the total mass enclosed within the closed surface.

22.60 • CP **Applying Gauss's Law for Gravitation.** Using Gauss's law for gravitation (derived in part (b) of Problem 22.59), show that the following statements are true: (a) For any spherically symmetric mass distribution with total mass M, the acceleration due to gravity outside the distribution is the same as though all the mass were concentrated at the center. (*Hint:* See Example 22.5 in Section 22.4.) (b) At any point inside a spherically symmetric shell of mass, the acceleration due to gravity is zero. (*Hint:* See Example 22.5.) (c) If we could drill a hole through a spherically symmetric planet to its center, and if the density were uniform, we would find that the magnitude of $\vec{g}$ is directly proportional to the distance r from the center. (*Hint:* See Example 22.9 in Section 22.4.) We proved these results in Section 13.6 using some fairly strenuous analysis; the proofs using Gauss's law for gravitation are *much* easier.

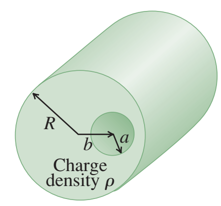
22.61 • (a) An insulating sphere with radius a has a uniform charge density ρ. The sphere is not centered at the origin but at $\vec{r} = \vec{b}$. Show that the electric field inside the sphere is given by $\vec{E} = \rho(\vec{r} - \vec{b})/3\epsilon_0$. (b) An insulating sphere of radius R has a spherical hole of radius a located within its volume and centered a distance b from the center of the sphere, where $a < b < R$ (a cross section of the sphere is shown in Fig. P22.61). The solid part of the sphere has a uniform volume charge density ρ. Find the magnitude and direction of the electric field $\vec{E}$ inside the hole, and show that $\vec{E}$ is uniform over the entire hole. [*Hint:* Use the principle of superposition and the result of part (a).]

Figure **P22.61**



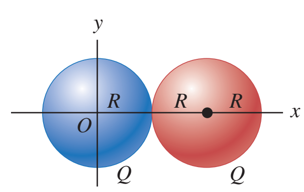
22.62 • A very long, solid insulating cylinder with radius R has a cylindrical hole with radius a bored along its entire length. The axis of the hole is a distance b from the axis of the cylinder, where $a < b < R$ (Fig. P22.62). The solid material of the cylinder has a uniform volume charge density ρ. Find the magnitude and direction of the electric field $\vec{E}$ inside the hole, and show that $\vec{E}$ is uniform over the entire hole. (*Hint:* See Problem 22.61.)

Figure **P22.62**



22.63 • Positive charge Q is distributed uniformly over each of two spherical volumes with radius R. One sphere of charge is centered at the origin and the other at $x = 2R$ (Fig. P22.63). Find the magnitude and direction of the net electric field due to these two distributions of charge at the following points on the x-axis: (a) $x = 0$; (b) $x = R/2$; (c) $x = R$; (d) $x = 3R$.

Figure **P22.63**



22.64 • Repeat Problem 22.63, but now let the left-hand sphere have positive charge Q and let the right-hand sphere have negative charge $-Q$.

22.65 •• CALC A nonuniform, but spherically symmetric, distribution of charge has a charge density $\rho(r)$ given as follows:

$$\rho(r) = \rho_0(1 - r/R) \quad \text{for } r \le R$$
$$\rho(r) = 0 \quad \text{for } r \ge R$$

where $\rho_0 = 3Q/\pi R^3$ is a positive constant. (a) Show that the total charge contained in the charge distribution is Q. (b) Show that the electric field in the region $r \ge R$ is identical to that produced by a point charge Q at $r = 0$. (c) Obtain an expression for the electric field in the region $r \le R$. (d) Graph the electric-field magnitude E as a function of r. (e) Find the value of r at which the electric field is maximum, and find the value of that maximum field.

CHALLENGE PROBLEMS

22.66 ••• CP CALC A region in space contains a total positive charge Q that is distributed spherically such that the volume charge density $\rho(r)$ is given by

$$\rho(r) = \alpha \quad \text{for } r \le R/2$$
$$\rho(r) = 2\alpha(1 - r/R) \quad \text{for } R/2 \le r \le R$$
$$\rho(r) = 0 \quad \text{for } r \ge R$$

Here α is a positive constant having units of C/m^3. (a) Determine α in terms of Q and R. (b) Using Gauss's law, derive an expression for the magnitude of $\vec{E}$ as a function of r. Do this separately for all three regions. Express your answers in terms of the total charge Q. Be sure to check that your results agree on the boundaries of the regions. (c) What fraction of the total charge is contained within the region $r \le R/2$? (d) If an electron with charge $q' = -e$ is oscillating back and forth about $r = 0$ (the center of the distribution) with an amplitude less than $R/2$, show that the motion is simple harmonic. (*Hint:* Review the discussion of simple harmonic motion in Section 14.2. If, and only if, the net force on the electron is proportional to its displacement from equilibrium, then the motion is simple harmonic.) (e) What is the period of the motion in part (d)? (f) If the amplitude of the motion described in part (e) is greater than $R/2$, is the motion still simple harmonic? Why or why not?

22.67 ••• **CP CALC** A region in space contains a total positive charge Q that is distributed spherically such that the volume charge density $\rho(r)$ is given by

$$\rho(r) = 3\alpha r/(2R) \qquad \text{for } r \le R/2$$
$$\rho(r) = \alpha[1 - (r/R)^2] \qquad \text{for } R/2 \le r \le R$$
$$\rho(r) = 0 \qquad \text{for } r \ge R$$

Here α is a positive constant having units of C/m^3. (a) Determine α in terms of Q and R. (b) Using Gauss's law, derive an expression for the magnitude of the electric field as a function of r. Do this separately for all three regions. Express your answers in terms of the total charge Q. (c) What fraction of the total charge is contained within the region $R/2 \le r \le R$? (d) What is the magnitude of $\vec{E}$ at $r = R/2$? (e) If an electron with charge $q' = -e$ is released from rest at any point in any of the three regions, the resulting motion will be oscillatory but not simple harmonic. Why? (See Challenge Problem 22.66.)

Answers

Chapter Opening Question ?

No. The electric field inside a cavity within a conductor is zero, so there is no electric effect on the child. (See Section 22.5.)

Test Your Understanding Questions

22.1 Answer: (iii) Each part of the surface of the box will be three times farther from the charge $+q$, so the electric field will be $(\frac{1}{3})^2 = \frac{1}{9}$ as strong. But the area of the box will increase by a factor of $3^2 = 9$. Hence the electric flux will be multiplied by a factor of $(\frac{1}{9})(9) = 1$. In other words, the flux will be unchanged.

22.2 Answer: (iv), (ii), (i), (iii) In each case the electric field is uniform, so the flux is $\Phi_E = \vec{E} \cdot \vec{A}$. We use the relationships for the scalar products of unit vectors: $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = 1$, $\hat{\imath} \cdot \hat{\jmath} = 0$. In case (i) we have $\Phi_E = (4.0\ \text{N/C})(6.0\ \text{m}^2)\hat{\imath} \cdot \hat{\jmath} = 0$ (the electric field and vector area are perpendicular, so there is zero flux). In case (ii) we have $\Phi_E [(4.0\ \text{N/C})\hat{\imath} + (2.0\ \text{N/C})\hat{\jmath}] \cdot (3.0\ \text{m}^2)\hat{\jmath} = (2.0\ \text{N/C}) \cdot (3.0\ \text{m}^2) = 6.0\ \text{N} \cdot \text{m}^2/\text{C}$. Similarly, in case (iii) we have $\Phi_E = [(4.0\ \text{N/C})\hat{\imath} - (2.0\ \text{N/C})\hat{\jmath}] \cdot [(3.0\ \text{m}^2)\hat{\imath} + (7.0\ \text{m}^2)\hat{\jmath}] = (4.0\ \text{N/C})(3.0\ \text{m}^2) - (2.0\ \text{N/C})(7.0\ \text{m}^2) = -2\ \text{N} \cdot \text{m}^2/\text{C}$, and in case (iv) we have $\Phi_E = [(4.0\ \text{N/C})\hat{\imath} - (2.0\ \text{N/C})\hat{\jmath}] \cdot [(3.0\ \text{m}^2)\hat{\imath} - (7.0\ \text{m}^2)\hat{\jmath}] = (4.0\ \text{N/C})(3.0\ \text{m}^2) + (2.0\ \text{N/C}) \cdot (7.0\ \text{m}^2) = 26\ \text{N} \cdot \text{m}^2/\text{C}$.

22.3 Answer: S_2, S_5, S_4, S_1 and S_3 (tie) Gauss's law tells us that the flux through a closed surface is proportional to the amount of charge enclosed within that surface. So an ordering of these surfaces by their fluxes is the same as an ordering by the amount of enclosed charge. Surface S_1 encloses no charge, surface S_2 encloses $9.0\ \mu\text{C} + 5.0\ \mu\text{C} + (-7.0\ \mu\text{C}) = 7.0\ \mu\text{C}$, surface S_3 encloses $9.0\ \mu\text{C} + 1.0\ \mu\text{C} + (-10.0\ \mu\text{C}) = 0$, surface S_4 encloses $8.0\ \mu\text{C} + (-7.0\ \mu\text{C}) = 1.0\ \mu\text{C}$, and surface S_5 encloses $8.0\ \mu\text{C} + (-7.0\ \mu\text{C}) + (-10.0\ \mu\text{C}) + (1.0\ \mu\text{C}) + (9.0\ \mu\text{C}) + (5.0\ \mu\text{C}) = 6.0\ \mu\text{C}$.

22.4 Answer: no You might be tempted to draw a Gaussian surface that is an enlarged version of the conductor, with the same shape and placed so that it completely encloses the conductor. While you know the flux through this Gaussian surface (by Gauss's law, it's $\Phi_E = Q/\epsilon_0$), the direction of the electric field need not be perpendicular to the surface and the magnitude of the field need not be the same at all points on the surface. It's not possible to do the flux integral $\oint E_\perp\, dA$, and we can't calculate the electric field. Gauss's law is useful for calculating the electric field only when the charge distribution is *highly* symmetric.

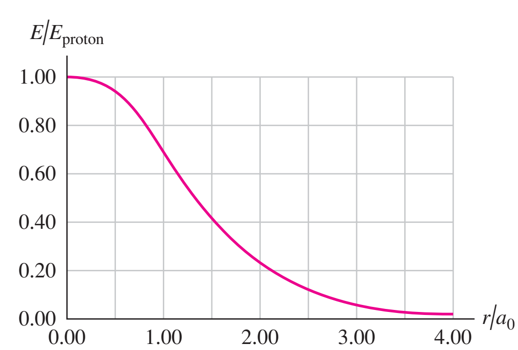
22.5 Answer: no Before you connect the wire to the sphere, the presence of the point charge will induce a charge $-q$ on the inner surface of the hollow sphere and a charge q on the outer surface (the net charge on the sphere is zero). There will be an electric field outside the sphere due to the charge on the outer surface. Once you touch the conducting wire to the sphere, however, electrons will flow from ground to the outer surface of the sphere to neutralize the charge there (see Fig. 21.7c). As a result the sphere will have no charge on its outer surface and no electric field outside.

Bridging Problem

Answers: **(a)** $Q(r) = Qe^{-2r/a_0}[2(r/a_0)^2 + 2(r/a_0) + 1]$

(b) $$E = \frac{kQe^{-2r/a_0}}{r^2}[2(r/a_0)^2 + 2(r/a_0) + 1]$$

(c)



23 ELECTRIC POTENTIAL

? In one type of welding, electric charge flows between the welding tool and the metal pieces that are to be joined together. This produces a glowing arc whose high temperature fuses the pieces together. Why must the tool be held close to the pieces being welded?

LEARNING GOALS

By studying this chapter, you will learn:

- How to calculate the electric potential energy of a collection of charges.
- The meaning and significance of electric potential.
- How to calculate the electric potential that a collection of charges produces at a point in space.
- How to use equipotential surfaces to visualize how the electric potential varies in space.
- How to use electric potential to calculate the electric field.

This chapter is about energy associated with electrical interactions. Every time you turn on a light, listen to an MP3 player, or talk on a mobile phone, you are using electrical energy, an indispensable ingredient of our technological society. In Chapters 6 and 7 we introduced the concepts of *work* and *energy* in the context of mechanics; now we'll combine these concepts with what we've learned about electric charge, electric forces, and electric fields. Just as we found for many problems in mechanics, using energy ideas makes it easier to solve a variety of problems in electricity.

When a charged particle moves in an electric field, the field exerts a force that can do *work* on the particle. This work can always be expressed in terms of electric potential energy. Just as gravitational potential energy depends on the height of a mass above the earth's surface, electric potential energy depends on the position of the charged particle in the electric field. We'll describe electric potential energy using a new concept called *electric potential,* or simply *potential.* In circuits, a difference in potential from one point to another is often called *voltage*. The concepts of potential and voltage are crucial to understanding how electric circuits work and have equally important applications to electron beams used in cancer radiotherapy, high-energy particle accelerators, and many other devices.

23.1 Electric Potential Energy

The concepts of work, potential energy, and conservation of energy proved to be extremely useful in our study of mechanics. In this section we'll show that these concepts are just as useful for understanding and analyzing electrical interactions.