Quasi-Latin designs for experiments in rectangles

Abstract

Yates, Healy and Rao generalized Latin squares to quasi-Latin squares. A factorial set of treatments is applied to the cells of a square array in such a way that

- no treatment occurs more than once in any row or column and
- the partial confounding with rows and with columns corresponds to standard factorial effects.

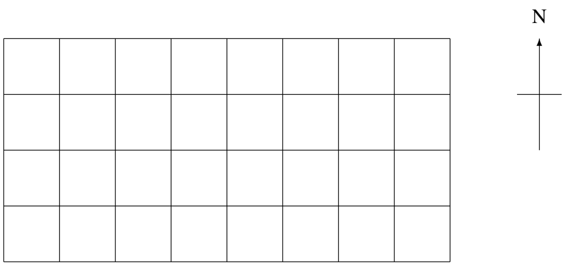
In recent joint work with Chris Brien, Thao Tran and John Tolund (all then at the University of South Australia), I have extended these methods to deal with rectangles, or collections of rectangles, which commonly occur in experiments in glasshouses. The user has considerable freedom over the choice of confounding patterns.

Indian Agricultural Research Institute, Delhi, 2006

Waite Institute, Adelaide, Australia, 2010

Les Serres D'Auteuil, April 2011

A glasshouse



Glasshouses often have their axes aligned North–South and East–West (for example, Les Serres D'Auteuil), so experiments in glasshouses should use rows and columns as blocks.

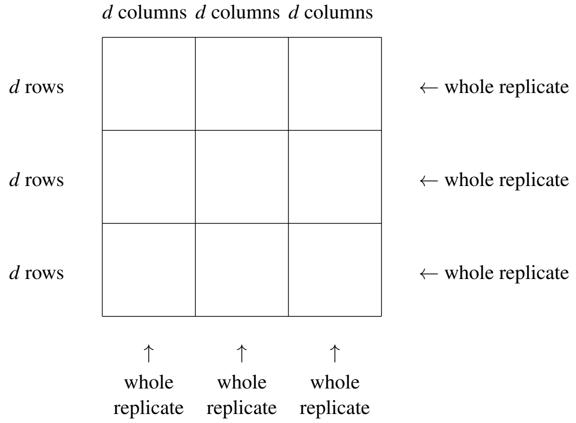
Quasi-Latin square designs

A quasi-Latin square is a square containing one or more complete replicates of the treatments, where no row or column contains all treatments.

1	2	3	4
5	6	7	8
3	8	1	6
7	4	5	2

Suppose that we have a square of size $k \times k$;
that we have v treatments;
that $vr = k^2$;
and that $v = dk$, so that $k = dr$.

Idea of Construction: k rows and k columns, dk treatments



Characters, working modulo 3, for 3×3 factorial

Characters	Treatments								
Factor A	0	0	0	1	1	1	2	2	2
Factor B	0	1	2	0	1	2	0	1	2
$A+B$	0	1	2	1	2	0	2	0	1
$A+2B$	0	2	1	1	0	2	2	1	0
$2A+B$	0	1	2	2	0	1	1	2	0
$2A+2B$	0	2	1	2	1	0	1	0	2
$2A$	0	0	0	2	2	2	1	1	1
$2B$	0	2	1	0	2	1	0	2	1
I	0	0	0	0	0	0	0	0	0

$A \equiv 2A$	main effect of A
$B \equiv 2B$	main effect of B
$A+B \equiv 2A+2B$	2 degrees of freedom for the A-by-B interaction
$A+2B \equiv 2A+B$	2 degrees of freedom for the A-by-B interaction, orthogonal to the previous 2

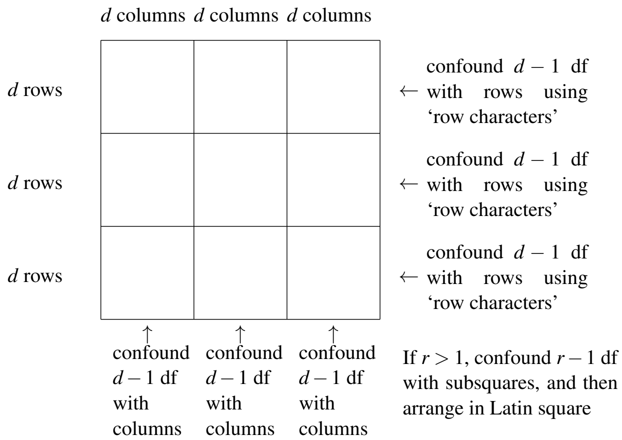
For 3 blocks of size 3, can alias blocks with any character.

Characters, working modulo 2, for $2 \times 2 \times 2$ factorial

Characters	Treatments							
Factor A	0	0	0	0	1	1	1	1
Factor B	0	0	1	1	0	0	1	1
Factor C	0	1	0	1	0	1	0	1
$A+B$	0	0	1	1	1	1	0	0
$A+C$	0	1	0	1	1	0	1	0
$B+C$	0	1	1	0	0	1	1	0
$A+B+C$	0	1	1	0	1	0	0	1
I	0	0	0	0	0	0	0	0

A	main effect of A
B	main effect of B
C	main effect of C
$A+B$	(1 degree of freedom for) the A-by-B interaction
$A+C$	(1 degree of freedom for) the A-by-C interaction
$B+C$	(1 degree of freedom for) the B-by-C interaction
$A+B+C$	(1 degree of freedom for) the A-by-B-by-C interaction

p^t rows and columns, p^m treatments, $d = p^{m-t}$



If $r > 1$, confound $r-1$ df with subsquares, and then arrange in Latin square

A 2^3 factorial experiment in 4 rows and 4 columns

	$A+B=0$	$A+B=1$	$A+C=0$	$A+C=1$
$B+C=0$	$(1,1,1)$	$(1,0,0)$	$(0,0,0)$	$(0,1,1)$
$B+C=1$	$(1,1,0)$	$(1,0,1)$	$(0,1,0)$	$(0,0,1)$
$A+B+C=0$	$(0,0,0)$	$(0,1,1)$	$(1,0,1)$	$(1,1,0)$
$A+B+C=1$	$(0,0,1)$	$(0,1,0)$	$(1,1,1)$	$(1,0,0)$

auxiliary design

$A=1$	$A=0$
$A=0$	$A=1$

Efficiency factors

	A	B	C	$A+B$	$A+C$	$B+C$	$A+B+C$
Rows	0	0	0	0	0	0.5	0.5
Columns	0	0	0	0.5	0.5	0	0
Rows.Columns	1	1	1	0.5	0.5	0.5	0.5

Important condition on characters

	$A+B=0$	$A+B=1$	$A+C=0$	$A+C=1$
$B+C=0$	$(1,1,1)$	$(1,0,0)$	$(0,0,0)$	$(0,1,1)$
$B+C=1$	$(1,1,0)$	$(1,0,1)$	$(0,1,0)$	$(0,0,1)$
$A+B+C=0$	$(0,0,0)$	$(0,1,1)$	$(1,0,1)$	$(1,1,0)$
$A+B+C=1$	$(0,0,1)$	$(0,1,0)$	$(1,1,1)$	$(1,0,0)$

auxiliary design

$A=1$	$A=0$
$A=0$	$A=1$

Row characters $B+C, A+B+C$
Column characters $A+B, A+C$
Unit character A

Any 2 or 3 characters, all of different sorts, must be linearly independent.

A 3^3 factorial experiment in two 9×9 squares

Characters	Square I			Square II		
Rows	$A+C$	$A+B+C$	$A+2B+C$	$B+C$	$A+B+C$	$A+2B+2C$
Columns	$A+2C$	$A+B+2C$	$A+2B+2C$	$B+2C$	$A+2B+C$	$A+B+2C$
Subsquares	B			A		

Efficiency factors

	main effects	A-by-B	A-by-C	B-by-C	A-by-B-by-C
Rows.Columns	1	1	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{2}{3}$

Quasi-Latin rectangles

An (extended) quasi-Latin rectangle is a row-column design with

- k rows and l columns;
- v treatments, each replicated r times, so that $vr = kl$.

We consider the special case where

- $v = p^m$ for some prime p;
- $k = p^t r_1$ and $l = p^u r_2$;
- $t+u \geq m$ so that $r_3 = p^{t+u-m}$ is an integer and $r = r_1 r_2 r_3$.

The previous construction (Rao's) has $t = u$ and $r_1 = r_2 = 1$.

Method 1a: $r_1 = r_2 = 1$

This is similar to the previous construction.

A 2^4 factorial experiment in a 4×8 rectangle.

confound $A+B+C+D$								
confound $A+C$								
	confound $A+B+C$, $A+B+D$ and $C+D$				confound $A+C+D$, $B+C+D$ and $A+B$			

The character to confound with sub-rectangles must not be

- a row character
- a column character
- the sum of a row character and a column character.

The unique possibility is $B+D$.

Method 1b: $r_1 > 1$

A 3^3 factorial experiment in a 12×9 rectangle.
Divide the rectangle into boxes whose size is a power of 3.

3 rows	5	6	4	8	9	7	2	3	1
3 rows	6	4	5	9	7	8	3	1	2
3 rows	8	9	7	2	3	1	5	6	4
3 rows	9	7	8	3	1	2	6	4	5

1. Confound $A+B+C$, $A+2B$, $2A+C$ and $2B+C$ (these will be the column characters) to get 9 "groups" of size 3 (the same size as the boxes).
2. Arrange groups 1–9 in a good 4×9 row–column design ($r_2 = 4$ and this is another type of auxiliary design).
3. Confound other characters in each set of 3 rows: for example, $A+B+2C$, $A+2B+C$, $2A+B+C$, $B+C$ (these will be the row characters).
4. Here $r_3 = 1$ and so there is no need for unit characters or the first type of auxiliary design.

Method 1 in general

$$v = p^m, \quad k = p^t r_1, \quad l = p^u r_2, \quad v r_3 = p^{t+u}.$$

- We need row characters unless $u = m$.
- We need column characters unless $t = m$.
- We need unit characters and the first type of auxiliary design if $r_3 > 1$.
- We need the second sort of auxiliary design if $r_1 > 1$.
- We need the third sort of auxiliary design if $r_2 > 1$.
- It is rare for all three of r_1, r_2 and r_3 to be greater than 1.
- The condition on characters must be satisfied.

Method 2: k divides v, and v divides l

To avoid the difficulties caused by restricted choice of unit character:

1. use column characters as before;
2. the number of columns is a multiple of v, so use the algorithm from Hall's Marriage Theorem to re-arrange the treatments in each column so that each row consists of complete replicates.

A 2^3 experiment in a 4×8 rectangle

confound $A+B+C$		confound $A+B$		confound $A+C$		confound $B+C$	
$(0,0,0)$	$(1,0,0)$	$(0,0,0)$	$(1,0,0)$	$(0,0,0)$	$(1,0,0)$	$(0,0,0)$	$(0,1,0)$
$(1,1,0)$	$(0,1,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,0)$	$(1,1,0)$	$(1,0,0)$	$(1,1,0)$
$(1,0,1)$	$(0,0,1)$	$(1,1,0)$	$(0,1,0)$	$(1,0,1)$	$(0,0,1)$	$(0,1,1)$	$(0,0,1)$
$0,1,1)$	$(1,1,1)$	$(1,1,1)$	$(0,1,1)$	$(1,1,1)$	$(0,1,1)$	$(1,1,1)$	$(1,0,1)$

Re-arrange treatments in each column to make each row a complete replicate:

$(0,0,0)$	$(1,0,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,0)$	$(0,1,1)$	$(1,1,1)$	$(1,1,0)$
$(1,1,0)$	$(1,1,1)$	$(0,0,0)$	$(1,0,0)$	$(1,0,1)$	$(0,0,1)$	$(0,1,1)$	$(0,1,0)$
$(1,0,1)$	$(0,1,0)$	$(1,1,1)$	$(0,1,1)$	$(0,0,0)$	$(1,1,0)$	$(1,0,0)$	$(0,0,1)$
$(0,1,1)$	$(0,0,1)$	$(1,1,0)$	$(0,1,0)$	$(1,1,1)$	$(1,0,0)$	$(0,0,0)$	$(1,0,1)$

Method 3: Segmentation and recursion

- Partition the rows as $k = k_1 + k_2$ and the columns as $l = l_1 + l_2$, in such a way that v divides all $k_i l_j$ and that k_1 and l_1 are divisible by large powers of p.
- Use one of Methods 1, 2 and 3 in each segment.
- Where two segments have rows in common, use the same row characters to compensate for loss of information.
- Where two segments have columns in common, use the same column characters to compensate for loss of information.

A 2^3 factorial experiment in 4 rows and 6 columns

	$B+C$ $=0$	$B+C$ $=1$	$A+B+C$ $=0$	$A+B+C$ $=1$	$A+B+C$ $=0$	$A+B+C$ $=1$	
$A+B=0$	$(1,1,1)$	$(1,1,0)$	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(1,0,0)$	$*$
$A+B=1$	$(1,0,0)$	$(1,0,1)$	$(0,1,1)$	$(0,1,0)$	$(0,0,0)$	$(1,1,1)$	
$A+C=0$	$(0,0,0)$	$(0,1,0)$	$(1,0,1)$	$(1,1,1)$	$(1,1,0)$	$(0,0,1)$	
$A+C=1$	$(0,1,1)$	$(0,0,1)$	$(1,1,0)$	$(1,0,0)$	$(1,0,1)$	$(0,1,0)$	

$*$ the last two columns of the first row have

- $A+B=1$ (to compensate for first four columns)
- $A+C=1$ (to use $A+C$)
- $B+C=0$ (no need for linear independence from a column character used in the first segment).

Extensions

These methods can be adapted to designs for

- several unrelated rectangles

  rectangles/(rows × columns);

- several rectangles with contiguous rows

  rows × (blocks/columns);

- contiguity in both directions

  (big rows/small rows) × (big columns/small columns).

Advantages

- All designs have orthogonal factorial structure (general balance with respect to the factorial decomposition), which means that, within any stratum, estimators of different treatment effects are orthogonal to each other and so the order of fitting does not matter.
- The construction methods are very flexible, allowing the designer of the experiment to decide which treatment effects should have the largest efficiency factors in the bottom stratum.

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