Lecture Notes 10
Uniformly Most Powerful Tests (UMP)

1 The Neyman-Pearson Test

Definition 1 *Let $\mathcal{C}_\alpha$ denote all level α tests. A test in $\mathcal{C}_\alpha$ with power function β is uniformly most powerful (UMP) if the following holds: if β' is the power function of any other test in $\mathcal{C}_\alpha$ then $\beta(\theta) \geq \beta'(\theta)$ for all $\theta \in \Theta_1$.*

Let us define a test function which we call $\phi(X^n)$ such that we have for a size α test ϕ with rejection region R,

$$\phi(x^n) \quad = \quad \begin{cases} 1 & \text{when} \quad x^n \in R \\ 0 & \text{when} \quad x^n \in R^c \end{cases}$$

Thus for a *non-randomized test*, $\phi(x^n)$ is simply the indicator function of the critical region R. Now clearly the probability of rejection is

$$E_{\theta_0}[\phi(X^n)] = \int \phi(x^n) f(x^n;\theta) dx^n = P_{\theta_0}(X^n \in R).$$

Recall the general problem is: for some $\alpha \in [0,1]$, we select ϕ so as to maximize the power

$$\begin{aligned} &\text{maximize} \;\; \beta(\theta) = E_\theta \phi(X^n) \quad \text{for} \quad \theta \in \Theta_1 \\ &\text{subject to} \quad \beta(\theta) = E_\theta \phi(X^n) \leq \alpha \text{ for } \quad \theta \in \Theta_0 \end{aligned}$$

with typical choices being $\alpha = .01, .05, .10$. That is, the experimenter controls the Type I error. If this approach is taken, then the experimenter should specify the null and alternative hypotheses so that it is most important to control the Type I Error probability while maximizing the power of rejecting H_0 when H_1 is true.

Theorem 2 *(***Neyman-Pearson Lemma***) Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$. (Simple null and simple alternative), where the pdf or pmf corresponding to θ_i is $f(x, \theta_i), i = 0, 1$. Suppose use a test with rejection region R that satisfies*

$$x^n \in R \qquad \textit{when} \;\; f(x^n;\theta_1) > k f(x^n;\theta_0) \tag{1}$$

$$x^n \in R^c \qquad \textit{when} \;\; f(x^n;\theta_1) < k f(x^n;\theta_0) \tag{2}$$

for some $k \geq 0$, and

$$P_{\theta_0}(X^n \in R) = \alpha.$$

Then this test is a UMP level α test.

Remark 3 *This Lemma and its proof is taken from Theorem 8.3.12 Casella & Berger, Part (a).*

PROOF. We will prove the theorem for the case such that $f(x, \theta_i), i = 0, 1$ are the pdfs of continuous rvs. The proof for discrete rvs will be left as a homework problem. Note that any test satisfying

$$P_{\theta_0}(X^n \in R) = E_{\theta_0}[\phi(X^n)] = \int \phi(x^n) f(x^n; \theta) dx^n = \alpha.$$

is a size α and hence level α test because

$$\sup_{\theta \in \Theta_0} P_\theta(X^n \in R) = P_{\theta_0}(X^n \in R) = \alpha.$$

Let $\phi(x^n)$ be the test function corresponding to (1) and (2). Let $\phi'(x^n)$ be the test function of any other level α test, and $\beta(\theta)$ and $\beta'(\theta)$ be the power functions corresponding to ϕ and $0 \leq \phi' \leq 1$ respectively. We now have for every x^n,

$$(\phi(x^n) - \phi'(x^n))(f(x^n; \theta_1) - k f(x^n, \theta_0)) \geq 0 \tag{3}$$

and hence

$$\begin{aligned} 0 &\leq \int [\phi(x^n) - \phi'(x^n)](f(x^n; \theta_1) - k f(x^n, \theta_0)) dx^n \\ &= \beta(\theta_1) - \beta'(\theta_1) - k(\beta(\theta_0) - \beta'(\theta_0)) \end{aligned} \tag{4}$$

Thus it is clear that

$$\beta(\theta_1) - \beta'(\theta_1) \geq k(\beta(\theta_0) - \beta'(\theta_0)) \geq 0$$

and hence ϕ has greater power than ϕ'. Since ϕ' was an arbitrary level α test and θ_1 the only point in Θ_1, ϕ is the UMP level α test. ■

Exercise 1:

1. Prove this for the discrete case.

2. Verify (3) holds.

3. Expand the integrals and verify each of the β items in (4).

Notes:

1. In general it is hard to find UMP tests. Sometimes they don't even exist. Still, we can find tests with good properties.

2. There is a certain class of problems for which they do exist: One-sided problem with Monotone likelihood ratio, which we will study next time.

2 Randomization in a Neyman-Pearson Test (Optional material)

Now, can you construct size α test for $\alpha = 0.05$ for testing

$$H_0 : \theta_0 = \frac{1}{2} \quad \text{against} \quad H_1 : \theta_1 = \frac{1}{4}?$$

Here is the more complete version of the Neyman-Pearson Lemma. It fixes the problem that is inherent in discrete distributions: without randomization, some α may not be achievable.

Theorem 4 *(***Neyman-Pearson Lemma***) (Optional material) Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$. (Simple null and simple alternative), where the pdf or pmf corresponding to θ_i is $f(x, \theta_i), i = 0, 1$.*

1. *Existence: for each $\alpha \in [0, 1]$ there exists a test ϕ and a constant $\infty \geq k \geq 0$ such that*

$$E_{\theta_0}[\phi(X^n)] = \alpha \quad \textit{and} \tag{5}$$

$$\phi(x^n) = \begin{cases} 1 & \textit{when} \quad f(x^n; \theta_1) > k f(x^n; \theta_0) \\ 0 & \textit{when} \quad f(x^n; \theta_1) < k f(x^n; \theta_0) \end{cases} \tag{6}$$

   where randomization of $\phi(x^n)$ is permitted in case $f(x^n; \theta_1) = k f(x^n; \theta_0)$.

2. *Sufficient condition for a MPT. If a test satisfies (5) and (6) for some k, then it is most powerful test for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ at level α.*

3. *Necessary condition for a MPT. If ϕ is a most powerful test at level α for testing H_0 against H_1, then for some k it satisfies (6) a.e. (except on a set A such that $P_{\theta_0}(A) = P_{\theta_1}(A) = 0$. It also satisfies (5) unless there exists a test of size $< \alpha$ with power 1.*

Notes:

1. Implicitly, the test function for Neyman-Pearson Test is:

$$\phi_{\text{NP}}(x^n) = \begin{cases} 1 & \text{when} \quad f(x^n; \theta_1) > k f(x^n; \theta_0) \\ q & \text{when} \quad f(x^n; \theta_1) = k f(x^n; \theta_0) \\ 0 & \text{when} \quad f(x^n; \theta_1) < k f(x^n; \theta_0) \end{cases}$$

   The interpretation is that we toss a coin with probability of heads $\phi(x^n)$ and reject H_0 iff the coin shows heads. Such randomized tests are not used in practice. They are only used to show that with randomization, likelihood ratio tests are unbeatable no matter what the size of α is.

2. In other words, reject H_0 if $X^n = x^n$ such that x^n satisfies

$$\frac{L_n(\theta_1; x^n)}{L_n(\theta_0; x^n)} = \frac{f(x^n; \theta_1)}{f(x^n; \theta_0)} > k$$

as $R \supset \{x^n : \; f(x^n; \theta_1) > kf(x^n; \theta_0)\}$. Now

$$\begin{aligned} E_{\theta_0}[\phi(X^n)] &= P_{\theta_0}(X^n \in R) = P_{\theta_0}(\text{Reject } \; H_0) \\ &= P_{\theta_0}(f(X^n; \theta_1) > kf(X^n; \theta_0)) + qP_{\theta_0}(f(X^n; \theta_1) = kf(X^n; \theta_0)) \\ &= \alpha. \end{aligned}$$

3 Exponential Family and Monotone Likelihood Ratio Models

Definition 5 *The family of models $\{P_\theta : \theta \in \Theta\}$ with $\Theta \subset R$ is said to be a monotone likelihood ratio (MLR) family in T if for $\theta_1 < \theta_2$, the distribution P_{θ_1} and P_{θ_2} are distinct, and there exists a statistic $T(x)$ such that the ratio $f(x; \theta_2)/f(x; \theta_1)$ is an increasing function of $T(x)$.*

Suppose the family of models $\{P_\theta : \theta \in \Theta\}$ with $\Theta \subset R$ is a monotone likelihood ratio (MLR) family in T. Furthermore, suppose that a univariate random variable $T(X^n)$ is a sufficient statistic for θ, then the pdfs or pmfs $\{g(t; \theta) : \theta \in \Theta\}$ for the sufficient statistic T is a also monotone function of t:

$$\lambda(x^n) = \frac{f(x^n; \theta_2)}{f(x^n; \theta_1)} = \frac{h(x^n)g(T(x^n); \theta_2)}{h(x^n)g(T(x^n); \theta_1)} = \frac{g(T(x^n); \theta_2)}{g(T(x^n); \theta_1)} := \widetilde{\lambda}(t(x^n))$$

by the factorization theorem.

Example 6 *We can rewrite an exponential family in terms of a natural parametrization. We have*

$$f(x; \theta) = h(x) \exp\{\eta(\theta)t(x) - B(\theta))\}$$

define

$$B(\theta) = \log \int h(x) \exp\{\eta(\theta)t(x)\}dx$$

If $\eta(\theta)$ is strictly increasing in $\theta \in \Theta$, then this family is MLR to be defined in Definition 5. For example a Poisson can be written as

$$f(x; \eta(\lambda)) = \exp\{\eta(\lambda)x - \lambda\}/x!$$

where the natural parameter is $\eta(\lambda) = \log \lambda$.

Example 7 *For $X^n = x_n$, we have*

$$f(x^n;\theta) = \prod_i h(x_i) \exp\{\eta(\theta) \sum_{i=1}^{n} t(x_i) - nB(\theta))\}.$$

Write the joint distribution of the i.i.d Bernoulli random variables $X_1, \ldots, X_n$ using the exponential family expression: for $T(X^n) = \sum_{i=1}^n t(X_i) = \sum_{i=1}^n X_i$, we have

$$P_\theta(x^n) = \binom{n}{T} \theta^T (1-\theta)^{n-T} = \binom{n}{T} \exp\left(\log \frac{\theta}{1-\theta} T(x_n) + n \log(1-\theta)\right) \tag{7}$$

We consider the following special case where the Null hypothesis is simple.

Example 8 *Suppose $X_1, \ldots, X_n \sim \text{Bernoulli}(\theta)$.*

$$H_0 : \theta_0 = \frac{1}{2} \qquad H_1 : \theta > \frac{1}{2}.$$

Then $T(X^n) = \sum_{i=1}^n X_i$ then

$$P_\theta(X^n) = \binom{n}{T} \theta^T (1-\theta)^{n-T} = \binom{n}{T} \left(\frac{\theta}{1-\theta}\right)^T (1-\theta)^n \tag{8}$$

Now clearly for $\theta_1 > \theta_0$, the likelihood ratio

$$\frac{P_{\theta_1}(X^n)}{P_{\theta_0}(X^n)} = \left(\frac{\theta_1(1-\theta_0)}{\theta_0(1-\theta_1)}\right)^T \left[\frac{1-\theta_1}{1-\theta_0}\right]^n \tag{9}$$

is an increasing function of T.

The construction of our test does not depend on which particular $\theta \in H_1$ that is chosen; for example, if we fix $H_1 : \theta = 3/4$ or $H_1 : \theta = 3/5$, we would have constructed the same test given the knowledge that the size $\alpha = 1/2^5$ of the test is always:

$$P_{\theta_0} = 1/2^5.$$

Now for $\theta_1 = 3/4$ or $3/5$, we can compute

$$\beta(\theta) = P_{\theta_1}(\sum_{i=1}^{5} X_i = 5) = \theta_1^5.$$

This is an example of the Neyman-Pearson Test, and hence a UMP level α test for $\alpha = 1/2^5$.

Exercise I: Compute k and $\beta(\theta_1)$ in the example above for $\theta_1 = 3/4$ and $\theta_1 = 3/5$ respectively. Convince yourself that when the size of the test is fixed at $\alpha = 1/2^5$, you will have constructed the same test for the following three problems:

1. $H_1 : \theta = 3/4$ or

2. $H_1 : \theta = 3/5$ or

3. $H_1 : \theta > \frac{1}{2}$.

4 The Karlin-Rubin Theorem

We present the theorem for the more general *Right-sided Alternative Hypotheses* problem:
Null hypothesis: $H_0 : \theta \leq \theta_0$
Alternative hypothesis: $H_1 : \theta > \theta_0$

Theorem 9 *(Karlin-Rubin, CB Theorem 8.3.17) Consider testing the* Right-sided Alternative Hypotheses *problem as immediately above. Suppose $T(X)$ is a sufficient statistic for θ and the family of pdfs or pmfs $\{g(t;\theta) : \theta \in \Theta\}$ for the sufficient statistic T has a nondecreasing MLR. Then for any t_0, the test that rejects H_0 iff $T > t_0$ is a UMP level α test, where $\alpha = P_{\theta_0}(T > t_0)$.*

You are required to remember the Karlin-Rubin Theorem, and apply it in your problems.

5 Proof of Karlin Rubin Theorem

We first state the following

Corollary 10 (Corollary 8.3.13) *Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$. Suppose $T(X^n)$ is a sufficient statistic for θ and $g(t;\theta_i)$ is the pdf or pmf of the sufficient statistic T corresponding to θ_i $i = 0, 1$. Then any test based on T with rejection region S, which is a subset of the sample space of T, is a UMP level α test if it satisfies*

$$t \in S \quad when \quad \frac{g(t;\theta_1)}{g(t;\theta_0)} > k \tag{10}$$

$$t \in S^c \quad when \quad \frac{g(t;\theta_1)}{g(t;\theta_0)} < k \tag{11}$$

for some $k \geq 0$, where $\alpha = P_{\theta_0}(T \in S)$.

Proof of Theorem 9. Consider testing the *Right-sided Alternative Hypotheses* problem as stated in Theorem 9.

Let $\beta(\theta) = P_\theta(T > t_0)$ be the power function of the test. Fix $\theta' > \theta_0$. Consider testing

$$H_0' : \theta = \theta_0 \quad \text{against} \quad H_1' : \theta = \theta'$$

Since the family of pdfs or pmfs of T has an MLR, $\beta(\theta)$ is nondecreasing (cf. Exercise 8.34 in the book), so

(i)
$$\sup_{\theta \leq \theta_0} \beta(\theta) = \beta(\theta_0),$$
and this is a level α test.

(ii) If we define
$$k' = \inf_{t \in \mathcal{T}} \frac{g(t;\theta')}{g(t;\theta_0)},$$
where
$$\mathcal{T} = \{t : t > t_0 \text{ and either } \ g(t|\theta') > 0 \text{ or } g(t|\theta_0) > 0\},$$
it follows that
$$T > t_0 \Leftrightarrow \frac{g(t;\theta')}{g(t;\theta_0)} > k'.$$

We now invoke the following Corollary 10 of the Neyman-Pearson Lemma (see proof in the book), and the facts as stated in (i) and (ii), to conclude that

$$\beta(\theta') \geq \beta^*(\theta')$$

where $\beta^*(\theta')$ is the power function for any other level α test of H_0', that is, any test satisfying

$$\beta(\theta_0) \leq \alpha.$$

However, any level α test of H_0 with power function β^* satisfies

$$\beta^*(\theta_0) \leq \sup_{\theta \leq \theta_0} \beta^*(\theta) \leq \alpha.$$

Thus

$$\beta(\theta') \geq \beta^*(\theta')$$

for any level α test of H_0. Since θ' was arbitrary, the test is a UMP level $\alpha = P_{\theta_0}(T > t_0)$. This is the end of the proof of the Karlin Rubin Theorem 9. ■

Remarks:

Finally, because the class of tests with level α for $H_0 : \theta \le \theta_0$ against $H_1 : \theta > \theta_0$ (First problem):

$$\begin{aligned} &\text{maximize} \quad \beta(\theta) = E_\theta \phi(X^n) \quad \text{for} \quad \theta \in \Theta_1 \\ &\text{subject to} \quad \sup_{\theta \in \Theta_0} \beta(\theta) \le \alpha \end{aligned}$$

is contained in the class of tests with level α for $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$ (Second problem) by comparing the set of inequality above with the set of optimization function below:

$$\begin{aligned} &\text{maximize} \quad \beta(\theta) = E_\theta \phi(X^n) \quad \text{for} \quad \theta \in \Theta_1 \\ &\text{subject to} \quad \beta(\theta_0) = E_{\theta_0} \phi(X^n) \le \alpha \end{aligned}$$

and because $\phi(T)$ maximizes the power over this larger class of tests, it is UMP for the First problem also.

6 Review on hypothesis testing

We have gone through the four related problems on hypothesis testing. You can find reference in CB 8.2, 8.3.1, and 8.3.2.

Let $X_1, \dots, X_n \sim N(\theta, 1)$

1. Find the most powerful level α test of

$$H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta = \theta_1$$

   where $\theta_1 > \theta_0$

2. Find the most powerful level α test of

$$H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta > \theta_0$$

3. Find the most powerful level α test of

$$H_0 : \theta \le \theta_0 \quad \text{against} \quad H_1 : \theta > \theta_0$$

4. And the LRT for:

$$H_0 : \theta = \theta_0, \quad \text{against} \quad H_1 : \theta \neq \theta_0.$$

UMP tests exist for the first three problems.

Exercises I:

1. Apply the proof in the Karlin-Rubin Theorem to each of these three settings to convince yourself that they all use the same test statistic and the same rejection region.

2. UMP test does not exist for the last problem. A rigorous argument is given in the text Example 8.3.19. You should read it.