The Sequential Probability Ratio Test

Sequential testing

We have set up the testing problem as if we were forced to take a decision: accept or reject the null hypothesis.

But suppose that we in fact rather would say that *the matter is not clear and we would wich to collect additional observations.*

The idea behind the *sequential testing* is that we collect observations one at a time; when observation $X_i = x_i$ has been made, we choose between the following options:

- Accept the null hypothesis and stop observation;
- Reject the null hypothesis and stop observation;

- Defer decision until we have collected another piece of information as X_{i+1}.

The challenge is now to find out when to choose which of the above options. We would want to control the two types of error

$$\alpha = P\{\text{Deciding for } H_A \text{ when } H_0 \text{ is true}\}$$

and

$$\beta = P\{\text{Deciding for } H_0 \text{ when } H_A \text{ is true}\}.$$

Note that it is traditional in this context to treat H_A and H_0 symmetrically.

The sequential probability ratio test

We consider a simple hypothesis $H_0 : \theta = \theta_0$ against a simple alternative $H_1 : \theta = \theta_1$.

Recall that the standard LRT has critical region of the form

$$\Lambda_n = \lambda(X_1, \ldots, X_n) = \log \frac{L(\theta_1; X_1, \ldots, X_n)}{L(\theta_0; X_1, \ldots, X_n)} > K.$$

Wald's *Sequential Probability Ratio Test* (SPRT) has the following form:

- If $\Lambda_n > B$, decide that H_1 is true and stop;
- If $\Lambda_n < A$, decide that H_0 is true and stop;

- If $A < \Lambda_n < B$, collect another observation to obtain Λ_{n+1}.

It can be shown that *the SPRT is optimal* in the sense that it minimizes the *average sample size* before a decision is made among all sequential tests which do not have larger error probabilities than the SPRT.

It can also be shown that *the boundaries A and B can be calculated as* with very good approximation as

$$A = \log \frac{\beta}{1-\alpha}, \quad B = \log \frac{1-\beta}{\alpha},$$

so the SPRT is really very simple to apply in practice.