...the vector B_j is arbitrary, the vector B'_q is also arbitrary so that $B'_q \neq 0$ and the above relation is ...only when

$$A'_{pq} - \ell_{pi}\ell_{qj}A_{ij} = 0$$

$$A'_{pq} = \ell_{pi}\ell_{qj}A_{ij}$$

...ing that the 2 – suffix set A_{ij} is a tensor of rank 2 .

GENERALIZATION

If $A_{i_1 i_2 \cdots i_m j_1 j_2 \cdots j_n} B_{j_1 j_2 \cdots j_n}$ is a tensor of order m , where $B_{j_1 j_2 \cdots j_n}$ is an ...ary tensor of order n , then prove that $A_{i_1 i_2 \cdots i_m j_1 j_2 \cdots j_n}$ is a tensor of order m + n .

...2 SYMMETRIC AND ANTI – SYMMETRIC TENSORS

A tensor $A_{i_1 i_2 \cdots i_n}$ is said to be symmetric in a pair of indices i_1 and i_2 (say) if

$$A_{i_1 i_2 \cdots i_n} = A_{i_2 i_1 \cdots i_n} \qquad (1)$$

...le it is said to be anti – symmetric in the indices i_1 and i_2 if

$$A_{i_1 i_2 \cdots i_n} = -A_{i_2 i_1 \cdots i_n} \qquad (2)$$

...ensor is said to be symmetric (anti – symmetric) if it is symmetric (anti – symmetric) in all possible ...s of indices . Symmetric and anti – symmetric tensors occur frequently in mathematics and physics . ...r example , the interia tensor , the stress tensor , the strain tensor and the rate of strain tensor are all ...metric , while the spin tensor is an example of an anti – symmetric tensor .

...EOREM (7.13): Prove that the Kronecker tensor δ_{ij} is a second order symmetric tensor and the alternating tensor ϵ_{ijk} is a third order anti – symmetric tensor .

...ROOF: We have $\delta_{ij} = \hat{e}_i \cdot \hat{e}_j = \hat{e}_j \cdot \hat{e}_i = \delta_{ji}$

...ich shows that δ_{ij} is a symmetric tensor.

...so, $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$

...ich shows that ϵ_{ijk} is an anti – symmetric tensor .

...OTE: A symmetric second order tensor A_{ij} can be written as a matrix in the form

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

...ile an anti – symmetric second order tensor has a matrix of the form

$$[A_{ij}] = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix}$$

...s a symmetric second order tensor has only 6 independent components , while an anti – symmetric ...ond order tensor has only 3 independent components . Also in an anti – symmetric tensor the ...ponents on the leading diagonal are all zero . [$A_{ii} = -A_{ii}$ or $2A_{ii} = 0$ or $A_{ii} = 0$]

$$A'_{pqr}B'_{p\ldots} \ldots A_{ijk}B_{mn}$$

$$= \ell_{qj}\ell_{rk}\ell_{tn}(\ell_{pi}\ell_{pm})A_{ijk}B_{mn}$$

$$= \ell_{qj}\ell_{rk}\ell_{tn}\delta_{im}A_{ijk}B_{mn}$$

$$= \ell_{qj}\ell_{rk}\ell_{tn}A_{ijk}B_{in}$$

which shows that $C_{jkn} = A_{ijk}B_{in}$ called the inner product of A_{ijk} and B_{mn} is a tensor of rank 3. By contracting w.r.t. j and n or k and m in the product $A_{ijk}B_{mn}$, we can similarly show that any inner product is a tensor of rank 3.

COROLLARY: The inner product A_iB_i of two vectors A_i and B_j (i.e. $\bar{A}$ and $\bar{B}$) is a tensor of rank zero i.e. a scalar. For this reason A_iB_i is called the scalar or dot product of $\bar{A}$ and $\bar{B}$.

GENERALIZATION

If $A_{i_1i_2\cdots i_m}$ and $B_{j_1j_2\cdots j_n}$ are two tensors of rank m and n respectively, then any of their inner products is a tensor of rank $m+n-2$.

7.21 QUOTIENT THEOREM

With the help of this theorem we can decide whether a quantity representable by a multi-suffix set is a tensor or not.

THEOREM (7.12): If an inner product of a quantity X with an arbitrary tensor is itself a tensor, then X is also a tensor.

To illustrate this theorem we consider the following example:

EXAMPLE (19): If $A_{ij}B_j$ is a vector where B_j is an arbitrary vector, then prove that the 2 – suffix set A_{ij} is also a tensor of rank 2.

SOLUTION: Let $C_i = A_{ij}B_j$, $C'_p = A'_{pq}B'_q$ where A_{ij}, B_j, C_i and A'_{pq}, B'_q, C'_p are the components of the 2 – suffix set and the two vectors in the systems K and K' respectively.

Now since C_i is a vector, therefore

$$C'_p = \ell_{pi}C_i$$

or $\quad A'_{pq}B'_q = \ell_{pi}A_{ij}B_j \qquad (1)$

Also B_j being an arbitrary, we have $\quad B'_q = \ell_{qj}B_j$

or $\quad B_j = \ell_{qj}B'_q \qquad (2)$

From equations (1) and (2), we get

$$A'_{pq}B'_q = \ell_{pi}\ell_{qj}A_{ij}B'_q$$

or $\quad (A'_{pq} - \ell_{pi}\ell_{qj}A_{ij})B'_q = 0$

THEOREM (7.14): Prove that every second order tensor can be represented uniquely as the sum of a symmetric and an anti – symmetric tensor.

PROOF: Let A_{ij} be a second order tensor, then we can write

$$A_{ij} = \frac{1}{2}(A_{ij}+A_{ji})+\frac{1}{2}(A_{ij}-A_{ji})$$
$$= B_{ij}+C_{ij}$$

where $B_{ij} = \frac{1}{2}(A_{ij}+A_{ji}) = B_{ji}$ is symmetric

and $C_{ij} = \frac{1}{2}(A_{ij}-A_{ji}) = -\frac{1}{2}(A_{ji}-A_{ij}) = -C_{ji}$, is anti – symmetric.

7.23 INVARIANCE OF SYMMETRIC AND ANTI–SYMMETRIC PROPERTY OF A TENSOR

THEOREM (7.15): If a tensor is symmetric (anti – symmetric) w.r.t. a pair of indices in one coordinate system, then it has the same property in any other coordinate system.

PROOF: We prove this theorem for the tensor A_{ijk}

If A_{ijk} is symmetric in i and j, then

$$A_{ijk} = A_{jik} \tag{1}$$

Also $A'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk}A_{ijk}$

$$= \ell_{mi}\ell_{nj}\ell_{pk}A_{jik} \quad [\text{ using equation (1) }]$$

$$= \ell_{nj}\ell_{mi}\ell_{pk}A_{jik} = A'_{nmp}$$

Thus $A'_{mnp} = A'_{nmp}$ showing that the tensor is symmetric w.r.t. the same pair of indices in the new coordinate system as well.

Similarly, in case the tensor A_{ijk} is anti – symmetric in i and j we can show $A'_{mnp} = -A'_{nmp}$.

7.24 FUNDAMENTAL PROPERTY OF TENSOR EQUATIONS

THEOREM (7.16): A tensor equation which holds in one coordinate system holds in every coordinate system i.e. the form of a tensor equation remains the same in every rectangular coordinate system.

PROOF: We prove this property for the simple tensor equation.

Let $A_i B_{ijk} = C_{jk}$ (1)

be a tensor equation where A_i, B_{ijk}, C_{jk} represent the components of the three tensors w.r.t. the system K. We will prove that equation (1) has the same form in another coordinate system K'.

Multiplying both sides of equation (1) with $\ell_{mj}\ell_{nk}$, we get

$$\ell_{mj}\ell_{nk}A_i B_{ijk} = \ell_{mj}\ell_{nk}C_{jk} \tag{2}$$

where i, j, k are dummies.

... B'_{pmn}, and C'_{mn} be the components of the same tensors w.r.t. the system K', then

$$C'_{mn} = \ell_{mj}\ell_{nk}C_{jk} \qquad (3)$$

... since $A_i B_{ijk} = \delta_{iq} A_q B_{ijk}$

$$= \ell_{pi}\ell_{pq} A_q B_{ijk} \qquad (4)$$

... equations (3) and (4), equation (2) becomes

$$\ell_{mj}\ell_{nk}\ell_{pi}\ell_{pq} A_q B_{ijk} = C'_{mn}$$

$$A'_p = \ell_{pq} A_q \text{ and } B'_{pmn} = \ell_{pi}\ell_{mj}\ell_{nk} B_{ijk}$$

... therefore $A'_p B'_{pmn} = C'_{mn}$ (5)

... is of the same form as equation (1).

...OLLARY: From equations (1) and (5) writing $D_{jk} = A_i B_{ijk} - C_{jk}$, $D'_{mn} = A'_p B'_{pmn} - C'_{mn}$, ... have $D_{jk} = 0$ and $D'_{mn} = 0$ for $m, n = 1, 2, 3$ which shows that if the components of a tensor ... coordinate system are all zero, then the components in every coordinate system are also zero.

ZERO TENSOR

A tensor whose components relatively to one coordinate system and, therefore, also relatively to ... coordinate system are all zero is known as a zero tensor.

...2 ISOTROPIC TENSORS

A tensor is said to be isotropic if its components remain the same in all rectangular Cartesian ... coordinate systems under orthogonal rotation of axes.

... that tensors of order zero (i.e. scalars) are all isotropic. Since there are no isotropic tensors of ... 1, therefore, we will discuss the isotropic tensors of second and third orders, which are of particular ... importance in tensor analysis.

...THEOREM (7.17): Prove that the Kronecker tensor δ_{ij} is an isotropic tensor of order 2.

...PROOF:

We know that the equation of transformation for the second order tensor A_{ij} is

$$A'_{mn} = \ell_{mi}\ell_{nj} A_{ij} \qquad (1)$$

... $A_{ij} = \delta_{ij}$ in equation (1), then

$$A'_{mn} = \ell_{mi}\ell_{nj}\delta_{ij} = \ell_{mi}\ell_{ni} = \delta_{mn}$$

... shows that the components δ_{ij} transform into themselves under the tensor rotation law. Thus δ_{ij} ... isotropic tensor of order 2. This tensor is the most important of all the isotropic tensors. Note that ... only isotropic tensor of order 2 is a scalar multiple of δ_{ij}.

...THEOREM (7.18): Prove that the alternating tensor ϵ_{ijk} is an isotropic tensor of order 3.

...PROOF:

The equation of transformation for the third order tensor is

$$A'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk} A_{ijk} \qquad (1)$$

7.20 TENSOR CALCULUS

DIFFERENTIATION OF TENSORS

THEOREM (7.19): **If $A_{i_1 i_2 \cdots i_n}$ is a tensor of order n, then its partial derivative w.r.t. x_p, i.e. $\frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n}$ is also a tensor of order $n+1$.**

PROOF: The law of transformation for the given tensor is

$$A'_{j_1 j_2 \cdots j_n} = \ell_{j_1 i_1} \ell_{j_2 i_2} \cdots \ell_{j_n i_n} A_{i_1 i_2 \cdots i_n} \tag{1}$$

where all the symbols have the usual meanings. Differentiating both sides of equation (1) w.r.t. x'_k we get

$$\frac{\partial}{\partial x'_k} A'_{j_1 j_2 \cdots j_n} = \ell_{j_1 i_1} \ell_{j_2 i_2} \cdots \ell_{j_n i_n} \frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n} \frac{\partial x_p}{\partial x'_k} \text{ where } p \text{ is dummy.}$$

Also we know that $x'_k = \ell_{kp} x_p$ or $x_p = \ell_{kp} x'_k$

so that $\frac{\partial x_p}{\partial x'_k} = \ell_{kp}$

Hence $$\frac{\partial}{\partial x'_k} A'_{j_1 j_2 \cdots j_n} = \ell_{j_1 i_1} \cdots \ell_{j_n i_n} \ell_{kp} \frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n} \tag{2}$$

which shows that $\frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n}$ is a tensor of order $n+1$.

NOTE: (i) If the partial derivative of $A_{i_1 i_2 \cdots i_n}$ w.r.t. x_p is denoted by $A_{i_1 i_2 \cdots i_n, p}$, then equation (2) can be written in the form

$$A'_{j_1 \cdots j_n, k} = \ell_{j_1 i_1} \cdots \ell_{j_n i_n} \ell_{kp} A_{i_1 \cdots i_n, p} \tag{3}$$

Differentiating both sides of equation (3) w.r.t. x'_m we can show

$$A'_{j_1 j_2 \cdots j_n, km} = \ell_{j_1 i_1} \cdots \ell_{j_n i_n} \ell_{kp} \ell_{mq} A_{i_1 \cdots i_n, pq}$$

$\left(\text{where } A_{i_1 \cdots i_n, pq} = \frac{\partial^2}{\partial x_q \partial x_p} A_{i_1 \cdots i_n}\right)$ which shows that $A_{i_1 i_2 \cdots i_n, pq}$ is a tensor of order $n+2$.

(ii) If ϕ is a scalar, then $\frac{\partial \phi}{\partial x_i}$ or $\phi_{,i}$ is a tensor of order 1 i.e. a vector.

INTEGRATION OF TENSORS

Integration of a tensor with respect to the coordinate direction yields a tensor of one order higher unless integration is combined with a contraction. For example,

$$\int A'_{mn} dx'_p = \int (\ell_{mi} \ell_{nj} A_{ij}) \ell_{pk} dx_k = \ell_{mi} \ell_{nj} \ell_{pk} \left(\int A_{ij} dx_k\right) \text{ and thus } \left(\int A_{ij} dx_k\right)$$

is a tensor of order 3. However $\left(\int A_{ij} dx_j\right)$ is a contraction of $\left(\int A_{ij} dx_k\right)$ and is thus of order 1 i.e. one less than A. Integration of a tensor w.r.t. a scalar, for example volume or surface, can be shown to yield a tensor of the same order.

7.27 APPLICATION TO VECTOR ANALYSIS

DOT PRODUCT

Let $\vec{A}$ and $\vec{B}$ be two vectors with components A_1, A_2, A_3 and B_1, B_2, B_3 respectively, then $\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = A_i B_i$.

THEOREM (7.20): Prove that if A_i and B_j are the components of two first order tensors $\vec{A}$ and $\vec{B}$ respectively, then $A_i B_i$ is a zeroth order tensor.

PROOF: Since A_i and B_j are the components of first order tensors, therefore under the transformation law from the system K to K′, we have

$$A'_m = \ell_{mi} A_i \quad (1)$$

$$B'_n = \ell_{nj} B_j \quad (2)$$

Multiplying equations (1) and (2), we get

$$A'_m B'_n = \ell_{mi} \ell_{nj} A_i B_j \quad (3)$$

Setting $m = n$ in equation (3), we have

$$A'_m B'_m = \ell_{mi} \ell_{mj} A_i B_j = \delta_{ij} A_i B_j = A_i B_i \quad (4)$$

which shows that $A_i B_i$ is a scalar or zeroth order tensor.

NOTE: (i) Equation (4) can be written as

$$A'_1 B'_1 + A'_2 B'_2 + A'_3 B'_3 = A_1 B_1 + A_2 B_2 + A_3 B_3$$

showing that the scalar product of two vectors is invariant under the orthogonal rotation of axes.

(ii) We have already proved that $A_i B_j$ are the components of a second order tensor, whereas in the above theorem we have seen that $A_i B_i$ is a zeroth order tensor. So the difference between $A_i B_j$ and $A_i B_i$ must be carefully observed.

CROSS PRODUCT

Let $\vec{A}$ and $\vec{B}$ be two vectors with components A_1, A_2, A_3 and B_1, B_2, B_3 respectively then $\vec{C} = \vec{A} \times \vec{B} = (A_2 B_3 - A_3 B_2, A_3 B_1 - A_1 B_3, A_1 B_2 - A_2 B_1)$.

We now show that the components of $\vec{C} = \vec{A} \times \vec{B}$ are given by

$$C_i = \epsilon_{ijk} A_j B_k \quad \text{for} \quad i = 1, 2, 3 \quad (1)$$

Now $C_1 = \epsilon_{1jk} A_j B_k = \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2 = A_2 B_3 - A_3 B_2 = (\vec{A} \times \vec{B})_1$

$C_2 = \epsilon_{2jk} A_j B_k = \epsilon_{231} A_3 B_1 + \epsilon_{213} A_1 B_3 = A_3 B_1 - A_1 B_3 = (\vec{A} \times \vec{B})_2$

$C_3 = \epsilon_{3jk} A_j B_k = \epsilon_{312} A_1 B_2 + \epsilon_{321} A_2 B_1 = A_1 B_2 - A_2 B_1 = (\vec{A} \times \vec{B})_3$

Thus $(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$

NOTE: From equation (1), we have $C_i \hat{e}_i = \epsilon_{ijk} A_j B_k \hat{e}_i$

$\vec{C} = \epsilon_{ijk} A_j B_k \hat{e}_i$

$\vec{A} \times \vec{B} = \epsilon_{ijk} A_j B_k \hat{e}_i$ where now the summation is over all the indices.

THEOREM (7.21): Prove that

(i) **the components of $\vec{A} \times \vec{B}$ i.e. $C_i = \epsilon_{ijk} A_j B_k$ transform as the components of a vector under a rotation of the coordinate axes.**

(ii) **$\vec{A} \times \vec{B}$ is invariant under the rotation of coordinate axes.**

PROOF: (i) Let ϵ_{ijk}, A_j, B_k be the components of a third order, and two first order tensors in the system $Ox_1x_2x_3$ and ϵ'_{mnp}, A'_n, B'_p be their corresponding components in the system $Ox'_1x'_2x'_3$. Then the laws of transformation are

$$\epsilon'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk}\epsilon_{ijk} \tag{1}$$

$$A'_n = \ell_{nj}A_j = \ell_{nr}A_r \tag{2}$$

$$B'_p = \ell_{pk}B_k = \ell_{ps}B_s \tag{3}$$

From equations (1), (2), and (3), we get

$$\begin{aligned}
\epsilon'_{mnp}A'_nB'_p &= \ell_{mi}\ell_{nj}\ell_{pk}\epsilon_{ijk}\ell_{nr}A_r\ell_{ps}B_s \\
&= \ell_{mi}(\ell_{nj}\ell_{nr})(\ell_{pk}\ell_{ps})\epsilon_{ijk}A_rB_s \\
&= \ell_{mi}\delta_{jr}\delta_{ks}\epsilon_{ijk}A_rB_s \\
&= \ell_{mi}\epsilon_{ijk}(\delta_{jr}A_r)(\delta_{ks}B_s) \\
&= \ell_{mi}\epsilon_{ijk}A_jB_k
\end{aligned}$$

$$C'_m = \ell_{mi}C_i \tag{4}$$

where $C'_m = \epsilon'_{mnp}A'_nB'_p$

Equations (4) shows that the components of $\vec{A} \times \vec{B}$ transform as the components of a vector.

(ii) $\vec{A} \times \vec{B} = \epsilon_{ijk}A_jB_k\hat{e}_i$

Also $\epsilon'_{mnp}A'_nB'_p = \ell_{mi}\epsilon_{ijk}A_jB_k$ (5)

and $\hat{e}'_m = \ell_{mi}\hat{e}_i = \ell_{mr}\hat{e}_r$ (6)

From equations (5) and (6), we get

$$\begin{aligned}
\epsilon'_{mnp}A'_nB'_p\hat{e}'_m &= \ell_{mi}\epsilon_{ijk}A_jB_k\ell_{mr}\hat{e}_r \\
&= \ell_{mi}\ell_{mr}\epsilon_{ijk}A_jB_k\hat{e}_r \\
&= \delta_{ir}\epsilon_{ijk}A_jB_k\hat{e}_r \\
&= \epsilon_{ijk}A_jB_k(\delta_{ir}\hat{e}_r) = \epsilon_{ijk}A_jB_k\hat{e}_i
\end{aligned}$$

which shows that $\vec{A} \times \vec{B}$ is invariant under the rotation of coordinate axes.

SCALAR TRIPLE PRODUCT

Considering $\vec{A} \cdot \vec{B} \times \vec{C}$ as the scalar product of $\vec{A}$ and $\vec{B} \times \vec{C}$, we get

$$\vec{A} \cdot \vec{B} \times \vec{C} = A_i (B \times C)_i$$
$$= A_i \epsilon_{ijk} B_j C_k = \epsilon_{ijk} A_i B_j C_k \qquad (1)$$

THEOREM (7.22): Prove that

(i) $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A} = \vec{C} \cdot \vec{A} \times \vec{B}$

(ii) $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C}$

PROOF: (i) Since $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$, therefore,

$$\epsilon_{ijk} A_i B_j C_k = \epsilon_{jki} B_j C_k A_i = \epsilon_{kij} C_k A_i B_j$$

Using equation (1) above, we get

$$\vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A} = \vec{C} \cdot \vec{A} \times \vec{B}$$

(ii) $\vec{A} \cdot \vec{B} \times \vec{C} = \epsilon_{ijk} A_i B_j C_k$

$$= (\epsilon_{ijk} A_i B_j) C_k = (\vec{A} \times \vec{B})_k C_k = \vec{A} \times \vec{B} \cdot \vec{C}$$

VECTOR TRIPLE PRODUCT

THEOREM (7.23): Prove that $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$

PROOF: We have

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \epsilon_{ijk} A_j (\vec{B} \times \vec{C})_k$$
$$= \epsilon_{ijk} A_j \epsilon_{k\ell m} B_\ell C_m = \epsilon_{ijk} \epsilon_{\ell mk} A_j B_\ell C_m$$
$$= (\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}) A_j B_\ell C_m$$
$$= \delta_{i\ell} \delta_{jm} A_j B_\ell C_m - \delta_{im} \delta_{j\ell} A_j B_\ell C_m$$
$$= A_j B_i C_j - A_j B_j C_i = (A_j C_j) B_i - (A_j B_j) C_i$$
$$= (\vec{A} \cdot \vec{C}) B_i - (\vec{A} \cdot \vec{B}) C_i$$

which gives the three components of the required formula for $i = 1, 2, 3$.

Hence $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$

THE DEL – OPERATOR

In Cartesian tensors, the del – operator denoted by ∇ is defined as

$$\nabla = \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} = \hat{e}_i \frac{\partial}{\partial x_i}$$

THEOREM

(i) the components of the del–operator $\nabla \left(\text{i.e.} \dfrac{\partial}{\partial x_i} \right)$ transform as the components of a vector under a rotation of the coordinate axes.

(ii) the vector del–operator ∇ is invariant under the rotation of the coordinate axes.

PROOF: (i) Let $\dfrac{\partial}{\partial x_i}$ and $\dfrac{\partial}{\partial x'_j}$ be the components of the del – operator ∇ in the system $Ox_1x_2x_3$ and $Ox'_1x'_2x'_3$ respectively. Let x_i and x'_j be the coordinates of a point in these systems, then we know that

$$x_1 = \ell_{11} x'_1 + \ell_{21} x'_2 + \ell_{31} x'_3$$

$$x_2 = \ell_{12} x'_1 + \ell_{22} x'_2 + \ell_{32} x'_3$$

$$x_3 = \ell_{13} x'_1 + \ell_{23} x'_2 + \ell_{33} x'_3$$

Using the chain rule, we have

$$\frac{\partial}{\partial x'_1} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x'_1} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial x'_1} + \frac{\partial}{\partial x_3}\frac{\partial x_3}{\partial x'_1}$$

$$= \ell_{11}\frac{\partial}{\partial x_1} + \ell_{12}\frac{\partial}{\partial x_2} + \ell_{13}\frac{\partial}{\partial x_3}$$

$$= \ell_{1i}\frac{\partial}{\partial x_i} \qquad (1)$$

Similarly,

$$\frac{\partial}{\partial x'_2} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x'_2} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial x'_2} + \frac{\partial}{\partial x_3}\frac{\partial x_3}{\partial x'_2}$$

$$= \ell_{21}\frac{\partial}{\partial x_1} + \ell_{22}\frac{\partial}{\partial x_2} + \ell_{23}\frac{\partial}{\partial x_3}$$

$$= \ell_{2i}\frac{\partial}{\partial x_i} \qquad (2)$$

and

$$\frac{\partial}{\partial x'_3} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x'_3} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial x'_3} + \frac{\partial}{\partial x_3}\frac{\partial x_3}{\partial x'_3}$$

$$= \ell_{31}\frac{\partial}{\partial x_1} + \ell_{32}\frac{\partial}{\partial x_2} + \ell_{33}\frac{\partial}{\partial x_3}$$

$$= \ell_{3i}\frac{\partial}{\partial x_i} \qquad (3)$$

From equations (1), (2), and (3), we get

$$\frac{\partial}{\partial x'_j} = \ell_{ji}\frac{\partial}{\partial x_i} \qquad (4)$$

which shows that under a rotation of the coordinate axes, the components of the del operator ∇ transform as components of a vector. It is often called a vector operator.

(ii) Let $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j'}$ be the components of the vector del – operator ∇ in the system $Ox_1x_2x_3$ and $Ox_1'x_2'x_3'$ respectively. Then we know that

$$\frac{\partial}{\partial x_j'} = \ell_{ji}\frac{\partial}{\partial x_i} \tag{1}$$

Also $\hat{e}_j' = \ell_{ji}\hat{e}_i = \ell_{jk}\hat{e}_k$ (2)

From equations (1) and (2), we get

$$\hat{e}_j'\frac{\partial}{\partial x_j'} = (\ell_{jk}\hat{e}_k)\left(\ell_{ji}\frac{\partial}{\partial x_i}\right)$$

$$= \ell_{jk}\ell_{ji}\hat{e}_k\frac{\partial}{\partial x_i}$$

$$= \delta_{ki}\hat{e}_k\frac{\partial}{\partial x_i} = \hat{e}_i\frac{\partial}{\partial x_i}$$

or $$\hat{e}_1'\frac{\partial}{\partial x_1'} + \hat{e}_2'\frac{\partial}{\partial x_2'} + \hat{e}_3'\frac{\partial}{\partial x_3'} = \hat{e}_1\frac{\partial}{\partial x_2} + \hat{e}_2\frac{\partial}{\partial x_2} + \hat{e}_3\frac{\partial}{\partial x_3}$$

or $\nabla' = \nabla$

which shows that the vector del – operator ∇ is invariant under the rotation of the coordinate axes.

GRADIENT

Let $\phi(x_1, x_2, x_3)$ be a scalar point function, then we know that

$$\nabla\phi = \hat{e}_1\frac{\partial\phi}{\partial x_1} + \hat{e}_2\frac{\partial\phi}{\partial x_2} + \hat{e}_3\frac{\partial\phi}{\partial x_3} \tag{1}$$

In tensor notation, equation (1) becomes

$$\nabla\phi = \hat{e}_i\frac{\partial\phi}{\partial x_i}$$

The components of $\nabla\phi$ are given by

$$(\nabla\phi)_i = \frac{\partial\phi}{\partial x_i}, \quad i = 1, 2, 3.$$

NOTE: (i) From equation (1) it is clear that the operator ∇ is given by

$$\nabla = \hat{e}_1\frac{\partial}{\partial x_1} + \hat{e}_2\frac{\partial}{\partial x_2} + \hat{e}_3\frac{\partial}{\partial x_3}$$

$$= \hat{e}_i\frac{\partial}{\partial x_i} \tag{2}$$

(ii) For any arbitrary vector $\bar{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$ we write

$$\bar{A}.\nabla = A_1\frac{\partial}{\partial x_1} + A_2\frac{\partial}{\partial x_2} + A_3\frac{\partial}{\partial x_3}$$

$$= A_j\frac{\partial}{\partial x_j} \tag{3}$$