of Coordinates

EINSTEIN'S SUMMETION CONVENTION:-

Consider An Expression

$$a_1 x^1 + a_2 x^2 + \cdots + a_N x^N$$

Index Notation is Indicial Convention

In Summetion Form

$$\sum_{j=1}^{N} a_j x^j$$

In Shorter Notation

$$a_j x^j$$

This Shows that If any index Repeated in a given term, then a Summation with Respect to that Index over the Range from 1 to N is implied. This Conversion is called Einstein's Summetion Convention.

DUMMY INDEX:- (OR UMBRAL INDEX)

An Index which is Repeated in a given term so that The Summation Conversion implied, is called a Dummy Index.

FREE INDEX:- (OR REAL INDEX)

An Index which occurring only once in a given term is is called a Free index.

For example,

consider An Expression "$A_k B_{ik}$" then "k" is a Dummy index while "i" is a free index

Scalars or Invariants:- (Tensor of Rank Zero)

Suppose ϕ is a function of the coordinates x^k and let $\bar{\phi}$ denote the functional value of $\bar{\phi}$ under a transformation to another set of coordinates $\bar{x}^k$. If $\phi = \bar{\phi}$ then the function "ϕ" is called Scalar or Invariant or Tensor of Rank Zero.

Contravariant and Covariant Vectors:-

or (Contravariant and Covariant Tensor of Rank One)

If N quantities $A^1, A^2, \ldots, A^N$ in one coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N other quantities $\bar{A}^1, \bar{A}^2, \ldots, \bar{A}^N$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations

$$\bar{A}^p = \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} A^q$$

OR

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^q} A^q$$

then the quantities A^q are called components of a contravariant vector or contravariant tensor of the first rank or first order.

If N quantities $A_1, A_2, \ldots, A_N$ in one coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N other quantities $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_N$ in other coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations

$$\bar{A}_p = \sum_{q=1}^{N} \frac{\partial x^q}{\partial \bar{x}^p} A_q$$

OR

$$\bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p} A_q$$

Contravriant, Covriant and Mixed Tensors:

→ If N^2 quantities A^{qs} in one coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N^2 other quantities $\bar{A}^{pr}$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations

$$\bar{A}^{pr} = \sum_{s=1}^{N} \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs} \qquad p, r = 1, 2, \ldots, N$$

OR

$$\bar{A}^{pr} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs}$$

Then the quantities A^{qs} are called components of a contravariant tensor of the second rank or rank two

→ If N^2 quantities A_{qs} in one coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N^2 other quantities $\bar{A}_{pr}$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations

$$\frac{\partial x^q}{\partial \bar{x}^p} \frac{\partial x^s}{\partial \bar{x}^r} A_{qs} = \bar{A}_{pr} = \sum_{s=1}^{N} \sum_{q=1}^{N} \frac{\partial x^q}{\partial \bar{x}^p} \frac{\partial x^s}{\partial \bar{x}^r} A_{qs}, \quad p, r = 1, 2, \ldots, N$$

Then the quantities A_{qs} are called components of a covariant tensor of the second rank or rank two

→ If N^2 quantities A^q_s in one coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N^2 other quantities $\bar{A}^p_r$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations

$$\bar{A}^p_r = \sum_{s=1}^{N} \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial x^s}{\partial \bar{x}^r} A^q_s \qquad p, r = 1, 2, \ldots, N$$

OR $$\bar{A}^p_r = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial x^s}{\partial \bar{x}^r} A^q_s$$

(Mixed Tensor of Rank 2)

The Kronecker Delta is denoted by δ^p_q and defined as

$$\delta^p_q = \begin{cases} 0 & \text{if } p \neq q \\ 1 & \text{if } p = q \end{cases}$$

The Kronecker Delta is a Mixed Tensor of Second Rank

TENSOR OF RANK GREATER THAN TWO.

→ If N^i quantities $A^{q_1 q_2 \cdots q_i}$ in one coordinate system $(x^1, x^2, \ldots, x^N)$ are Related to N^i other quantities $\bar{A}^{p_1 p_2 \cdots p_i}$ in another Coordinate System $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the Transformation Equations

$$\bar{A}^{p_1 p_2 \cdots p_i} = \frac{\partial \bar{x}^{p_1}}{\partial x^{q_1}} \frac{\partial \bar{x}^{p_2}}{\partial x^{q_2}} \cdots \frac{\partial \bar{x}^{p_i}}{\partial x^{q_i}} A^{q_1 q_2 \cdots q_i}$$

Then the quantities $A^{q_1 q_2 \cdots q_i}$ are called Components of a Contravariant Tensor of the Rank "i" or the Rank

→ If N^i quantities $A_{q_1 q_2 \cdots q_i}$ in one Coordinate System $(x^1, x^2, \ldots, x^N)$ are Related to N^i other quantities $\bar{A}_{p_1 p_2 \cdots p_i}$ in another Coordinate System $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the Transformation Equations

$$\bar{A}_{p_1 p_2 \cdots p_i} = \frac{\partial x^{q_1}}{\partial \bar{x}^{p_1}} \cdot \frac{\partial x^{q_2}}{\partial \bar{x}^{p_2}} \cdots \frac{\partial x^{q_i}}{\partial \bar{x}^{p_i}} A_{q_1 q_2 \cdots q_i}$$

Then the quantities $A_{q_1 q_2 \cdots q_i}$ are called Components of the Rank "i" or the Rank of a Covariant Tensor of the Rank

→ If N^{i+j} quantities $A^{q_1 q_2 \cdots q_i}_{s_1 s_2 \cdots s_j}$ in one coordinate System $(x^1, x^2, \ldots, x^N)$ are Related to N^{i+j} other quantities $\bar{A}^{p_1 p_2 \cdots p_i}_{r_1 r_2 \cdots r_j}$ in another Coordinate System $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the Transformation Equations

$$\bar{A}^{p_1 p_2 \cdots p_i}_{r_1 r_2 \cdots r_j} = \frac{\partial x^{p_1}}{\partial x^{q_1}} \cdot \frac{\partial x^{p_2}}{\partial x^{q_2}} \cdots \frac{\partial x^{p_i}}{\partial x^{q_i}} \frac{\partial x^{r_1}}{\partial x^{s_1}} \frac{\partial x^{r_2}}{\partial x^{s_2}} \cdots \frac{\partial x^{r_j}}{\partial x^{s_j}} A^{q_1 q_2 \cdots q_i}_{s_1 s_2 \cdots s_j}$$

Then the quantities $A^{q_1 q_2 \cdots q_i}_{s_1 s_2 \cdots s_j}$ are called Components of a Mixed

TENSOR CALCULUS

DIFFERENTIATION OF TENSORS

THEOREM (7.19): If $A_{i_1 i_2 \cdots i_n}$ is a tensor of order n, then its partial derivative w.r.t. x_p, i.e. $\frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n}$ is also a tensor of order $n+1$.

PROOF: The law of transformation for the given tensor is

$$A'_{j_1 j_2 \cdots j_n} = \ell_{j_1 i_1} \ell_{j_2 i_2} \cdots \ell_{j_n i_n} A_{i_1 i_2 \cdots i_n} \qquad (1)$$

where all the symbols have the usual meanings. Differentiating both sides of equation (1) w.r.t. x'_k we get

$$\frac{\partial}{\partial x'_k} A'_{j_1 j_2 \cdots j_n} = \ell_{j_1 i_1} \ell_{j_2 i_2} \cdots \ell_{j_n i_n} \frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n} \frac{\partial x_p}{\partial x'_k} \text{ where } p \text{ is dummy}.$$

Also we know that $x'_k = \ell_{kp} x_p$ or $x_p = \ell_{kp} x'_k$

so that $\frac{\partial x_p}{\partial x'_k} = \ell_{kp}$

Hence $$\frac{\partial}{\partial x'_k} A'_{j_1 j_2 \cdots j_n} = \ell_{j_1 i_1} \cdots \ell_{j_n i_n} \ell_{kp} \frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n} \qquad (2)$$

which shows that $\frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n}$ is a tensor of order $n+1$.

NOTE: (i) If the partial derivative of $A_{i_1 i_2 \cdots i_n}$ w.r.t. x_p is denoted by $A_{i_1 i_2 \cdots i_n, p}$, then equation (2) can be written in the form

$$A'_{j_1 \cdots j_n, k} = \ell_{j_1 i_1} \cdots \ell_{j_n i_n} \ell_{kp} A_{i_1 \cdots i_n, p} \qquad (3)$$

Differentiating both sides of equation (3) w.r.t. x'_m we can show

$$A'_{j_1 j_2 \cdots j_n, km} = \ell_{j_1 i_1} \cdots \ell_{j_n i_n} \ell_{kp} \ell_{mq} A_{i_1 \cdots i_n, pq}$$

(where $A_{i_1 \cdots i_n, pq} = \frac{\partial^2}{\partial x_q \partial x_p} A_{i_1 \cdots i_n}$) which shows that $A_{i_1 i_2 \cdots i_n, pq}$ is a tensor of order $n+2$.

(ii) If ϕ is a scalar, then $\frac{\partial \phi}{\partial x_i}$ or $\phi_{,i}$ is a tensor of order 1 i.e. a vector.

INTEGRATION OF TENSORS

Integration of a tensor with respect to the coordinate direction yields a tensor of one order higher unless integration is combined with a contraction. For example,

$$\int A'_{mn} dx'_p = \int (\ell_{mi} \ell_{nj} A_{ij}) \ell_{pk} dx_k = \ell_{mi} \ell_{nj} \ell_{pk} \left(\int A_{ij} dx_k\right) \text{ and thus } \left(\int A_{ij} dx_k\right)$$

is a tensor of order 3. However $\left(\int A_{ij} dx_j\right)$ is a contraction of $\left(\int A_{ij} dx_k\right)$ and is thus of order 1 i.e. one less than A. Integration of a tensor w.r.t. a scalar, for example volume or surface, can be shown to yield a tensor of the same order.

7.27 APPLICATION TO VECTOR ANALYSIS

DOT PRODUCT

Let $\vec{A}$ and $\vec{B}$ be two vectors with components A_1, A_2, A_3 and B_1, B_2, B_3 respectively, then $\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3 = A_iB_i$.

THEOREM (7.20): Prove that if A_i and B_j are the components of two first order tensors $\vec{A}$ and $\vec{B}$ respectively, then A_iB_i is a zeroth order tensor.

PROOF: Since A_i and B_j are the components of first order tensors, therefore under transformation law from the system K to K′, we have

$$A'_m = \ell_{mi}A_i \tag{1}$$

$$B'_n = \ell_{nj}B_j \tag{2}$$

Multiplying equations (1) and (2), we get

$$A'_mB'_n = \ell_{mi}\ell_{nj}A_iB_j \tag{3}$$

Setting $m = n$ in equation (3), we have

$$A'_mB'_m = \ell_{mi}\ell_{mj}A_iB_j = \delta_{ij}A_iB_j = A_iB_i \tag{4}$$

which shows that A_iB_i is a scalar or zeroth order tensor.

NOTE: (i) Equation (4) can be written as

$$A'_1B'_1 + A'_2B'_2 + A'_3B'_3 = A_1B_1 + A_2B_2 + A_3B_3$$

showing that the scalar product of two vectors is invariant under the orthogonal rotation of axes.

(ii) We have already proved that A_iB_j are the components of a second order tensor, whereas in the above theorem we have seen that A_iB_i is a zeroth order tensor. So the difference between A_iB_j and A_iB_i must be carefully observed.

CROSS PRODUCT

Let $\vec{A}$ and $\vec{B}$ be two vectors with components A_1, A_2, A_3 and B_1, B_2, B_3 respectively then $\vec{C} = \vec{A}\times\vec{B} = (A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1)$.

We now show that the components of $\vec{C} = \vec{A}\times\vec{B}$ are given by

$$C_i = \epsilon_{ijk}A_jB_k \quad \text{for} \quad i = 1, 2, 3 \tag{1}$$

Now $C_1 = \epsilon_{1jk}A_jB_k = \epsilon_{123}A_2B_3 + \epsilon_{132}A_3B_2 = A_2B_3 - A_3B_2 = (\vec{A}\times\vec{B})_1$

$C_2 = \epsilon_{2jk}A_jB_k = \epsilon_{231}A_3B_1 + \epsilon_{213}A_1B_3 = A_3B_1 - A_1B_3 = (\vec{A}\times\vec{B})_2$

$C_3 = \epsilon_{3jk}A_jB_k = \epsilon_{312}A_1B_2 + \epsilon_{321}A_2B_1 = A_1B_2 - A_2B_1 = (\vec{A}\times\vec{B})_3$

Thus $(\vec{A}\times\vec{B})_i = \epsilon_{ijk}A_jB_k$

NOTE: From equation (1), we have $C_i \hat{e}_i = \epsilon_{ijk} A_j B_k \hat{e}_i$

$\vec{C} = \epsilon_{ijk} A_j B_k \hat{e}_i$

$\vec{A} \times \vec{B} = \epsilon_{ijk} A_j B_k \hat{e}_i$ where now the summation is over all the indices.

THEOREM (7.21): Prove that

(i) **the components of $\vec{A} \times \vec{B}$ i.e. $C_i = \epsilon_{ijk} A_j B_k$ transform as the components of a vector under a rotation of the coordinate axes.**

(ii) **$\vec{A} \times \vec{B}$ is invariant under the rotation of coordinate axes.**

PROOF: (i) Let ϵ_{ijk}, A_j, B_k be the components of a third order, and two first order tensors in the system $Ox_1x_2x_3$ and ϵ'_{mnp}, A'_n, B'_p be their corresponding components in the system $Ox'_1x'_2x'_3$. Then the laws of transformation are

$$\epsilon'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk}\epsilon_{ijk} \tag{1}$$

$$A'_n = \ell_{nj} A_j = \ell_{nr} A_r \tag{2}$$

$$B'_p = \ell_{pk} B_k = \ell_{ps} B_s \tag{3}$$

From equations (1), (2), and (3), we get

$$\begin{aligned}
\epsilon'_{mnp} A'_n B'_p &= \ell_{mi}\ell_{nj}\ell_{pk}\epsilon_{ijk}\,\ell_{nr}A_r\,\ell_{ps}B_s \\
&= \ell_{mi}(\ell_{nj}\ell_{nr})(\ell_{pk}\ell_{ps})\epsilon_{ijk}A_rB_s \\
&= \ell_{mi}\delta_{jr}\delta_{ks}\epsilon_{ijk}A_rB_s \\
&= \ell_{mi}\epsilon_{ijk}(\delta_{jr}A_r)(\delta_{ks}B_s) \\
&= \ell_{mi}\epsilon_{ijk}A_jB_k
\end{aligned}$$

$$C'_m = \ell_{mi} C_i \tag{4}$$

where $C'_m = \epsilon'_{mnp} A'_n B'_p$

Equations (4) shows that the components of $\vec{A} \times \vec{B}$ transform as the components of a vector.

(ii) $\vec{A} \times \vec{B} = \epsilon_{ijk} A_j B_k \hat{e}_i$

Now

$$\epsilon'_{mnp} A'_n B'_p = \ell_{mi}\epsilon_{ijk}A_jB_k \tag{5}$$

also

$$\hat{e}'_m = \ell_{mi}\hat{e}_i = \ell_{mr}\hat{e}_r \tag{6}$$

From equations (5) and (6), we get

$$\begin{aligned}
\epsilon'_{mnp} A'_n B'_p \hat{e}'_m &= \ell_{mi}\epsilon_{ijk}A_jB_k\,\ell_{mr}\hat{e}_r \\
&= \ell_{mi}\ell_{mr}\epsilon_{ijk}A_jB_k\hat{e}_r \\
&= \delta_{ir}\epsilon_{ijk}A_jB_k\hat{e}_r \\
&= \epsilon_{ijk}A_jB_k(\delta_{ir}\hat{e}_r) = \epsilon_{ijk}A_jB_k\hat{e}_i
\end{aligned}$$

which shows that $\vec{A} \times \vec{B}$ is invariant under the rotation of coordinate axes.

SCALAR TRIPLE PRODUCT

Considering $\vec{A} \cdot \vec{B} \times \vec{C}$ as the scalar product of $\vec{A}$ and $\vec{B} \times \vec{C}$, we get

$$\vec{A} \cdot \vec{B} \times \vec{C} = A_i (B \times C)_i = A_i \epsilon_{ijk} B_j C_k = \epsilon_{ijk} A_i B_j C_k \quad (1)$$

THEOREM (7.22): Prove that

(i) $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A} = \vec{C} \cdot \vec{A} \times \vec{B}$

(ii) $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C}$

PROOF: (i) Since $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$, therefore,

$$\epsilon_{ijk} A_i B_j C_k = \epsilon_{jki} B_j C_k A_i = \epsilon_{kij} C_k A_i B_j$$

Using equation (1) above, we get

$$\vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A} = \vec{C} \cdot \vec{A} \times \vec{B}$$

(ii) $$\vec{A} \cdot \vec{B} \times \vec{C} = \epsilon_{ijk} A_i B_j C_k = (\epsilon_{ijk} A_i B_j) C_k = (\vec{A} \times \vec{B})_k C_k = \vec{A} \times \vec{B} \cdot \vec{C}$$

VECTOR TRIPLE PRODUCT

THEOREM (7.23): Prove that $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$

PROOF: We have

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \epsilon_{ijk} A_j (\vec{B} \times \vec{C})_k$$

$$= \epsilon_{ijk} A_j \epsilon_{k\ell m} B_\ell C_m = \epsilon_{ijk} \epsilon_{\ell mk} A_j B_\ell C_m$$

$$= (\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}) A_j B_\ell C_m$$

$$= \delta_{i\ell} \delta_{jm} A_j B_\ell C_m - \delta_{im} \delta_{j\ell} A_j B_\ell C_m$$

$$= A_j B_i C_j - A_j B_j C_i = (A_j C_j) B_i - (A_j B_j) C_i$$

$$= (\vec{A} \cdot \vec{C}) B_i - (\vec{A} \cdot \vec{B}) C_i$$

which gives the three components of the required formula for $i = 1, 2, 3$.

Hence $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$

THE DEL – OPERATOR

In Cartesian tensors, the del – operator denoted by ∇ is defined as

$$\nabla = \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} = \hat{e}_i \frac{\partial}{\partial x_i}$$