We note that $\det T = \begin{vmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{vmatrix} = -1$

and so the corresponding transformation is orthogonal but left – handed .

7.10 PROPER AND IMPROPER TRANSFORMATIONS

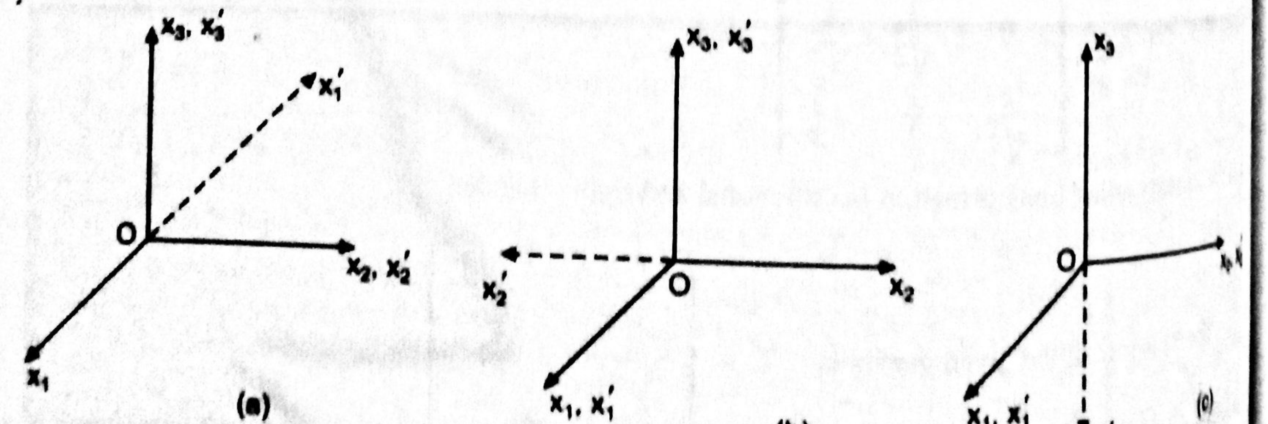
Since for an orthogonal transformation , $\det T = \pm 1$, all coordinate transformations are divide into two classes . One class consists of those transformations for which $\det T = 1$ and are called prope transformations ; the other class consists of transformations for which $\det T = -1$ are called improp transformations . Under a proper transformation , a right – handed (or left – handed) system rema right – handed (or left – handed) after rotation . Under an improper transformation , a right – hande system is changed into a left – handed system and vice–versa . The transformation for which T = 1, th unit matrix , is called the **identity** transformation . Obviously , for the identity transformation

$$x'_1 = x_1 , \quad x'_2 = x_2 , \quad x'_3 = x_3$$

or $$x'_i = x_i \quad (i = 1 , 2 , 3)$$

The proper transformation can be obtained from the identity transformation by continuously rotating th coordinate axes . On the other hand , the improper transformations cannot be obtained by that proces The improper transformation can be obtained from the identity transformation by two types discontinuous or discrete operations .

(I) REFLECTION: This is the operation in which the new coordinate system $Ox'_1 x'_2 x'_3$ i obtained from the original system $Ox_1 x_2 x_3$ by inverting (reversing) the direction of one of the axes the latter , the other two remaining in their original position as shown in figure (7.6) .



In this case , the transformation will be

Figure (7.6)

$$x'_1 = -x_1 , \quad x'_2 = x_2 , \quad x'_3 = x_3$$

or $$x'_1 = x_1 , \quad x'_2 = -x_2 , \quad x'_3 = x_3$$

or $$x'_1 = x_1 , \quad x'_2 = x_2 , \quad x'_3 = -x_3$$

corresponding transformation matrix T will be

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

that for each of these matrices del $T = -1$.

INVERSION: This is the operation in which the coordinate system $Ox'_1\,x'_2\,x'_3$ is obtained from the original system $Ox_1\,x_2\,x_3$ by inverting the directions of all coordinate axes of the latter as shown in figure (7.7). transformation equations in this case will be

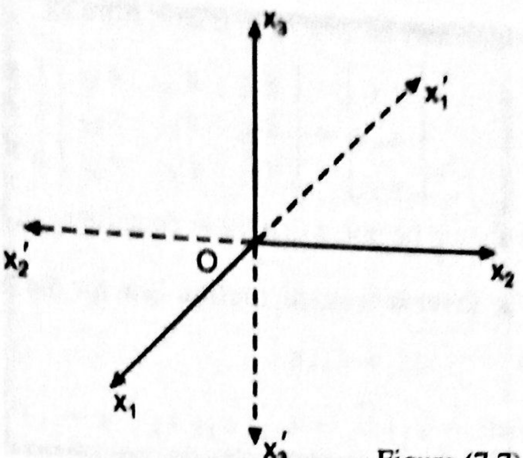


Figure (7.7)

$$x'_1 = -x_1,\quad x'_2 = -x_2,\quad x'_3 = -x_3$$

corresponding transformation matrix T in this case will be

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

that for each of these matrices del $T = -1$.

11 TRANSFORMATION EQUATIONS

TRANSFORMATION EQUATIONS FOR COORDINATES OF A POINT

Consider two rectangular coordinate systems K and K′ having the same origin O as shown in figure (7.8). We first find the transformation equations expressing the coordinates x'_1, x'_2, x'_3 of an arbitrary point P in the system K′ in terms of its coordinates x_1, x_2, x_3 in the system K, and vice versa. Let $\vec{r}$ and $\vec{r}\,'$ be the position vectors of any point P in the systems K and K′ respectively. Then $\vec{r}\,' = \vec{r}$

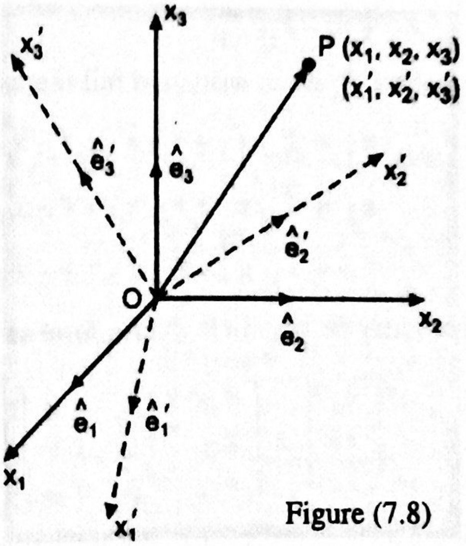


Figure (7.8)

$$x'_1\hat{e}'_1 + x'_2\hat{e}'_2 + x'_3\hat{e}'_3 = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$$

$$x'_j\hat{e}'_j = x_i\hat{e}_i\,,\quad i, j = 1, 2, 3 \qquad (1)$$

For the components x'_j, take the dot product of equation (1) with $\hat{e}'_j$, we have

$$x'_j(\hat{e}'_j \cdot \hat{e}'_j) = x_i(\hat{e}_i \cdot \hat{e}'_j)$$

$$= (\hat{e}'_j \cdot \hat{e}_i)\,x_i$$

$$x'_j = \ell_{ji}\,x_i \qquad (2)$$

where ℓ_{ji} are the direction cosines of the jth – axis of the system K′ with the ith – axis of the system K. Equation (2) is called the equation of transformation for the coordinates of the point P from the system K to the system K′.

Equation (2) when written in full represents the following three equations:

$$x'_1 = \ell_{11} x_1 + \ell_{12} x_2 + \ell_{13} x_3$$

$$x'_2 = \ell_{21} x_1 + \ell_{22} x_2 + \ell_{23} x_3$$

$$x'_3 = \ell_{31} x_1 + \ell_{32} x_2 + \ell_{33} x_3$$

which may be written in matrix form as

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

It should be noted that these equations are formed from the transformation matrix .

The **inverse transformation law** for the coordinates of the point P from the system K' to the system K

is $\quad x_i = \ell_{ji} x'_j \qquad$ (3)

since $\quad \ell_{ji} x'_j = \ell_{ji}(\ell_{jk} x_k) = \ell_{ji}\ell_{jk} x_k = \delta_{ik} x_k = x_i$

Equation (3) can also be obtained by taking the dot product of equation (1) with $\hat{e}_i$, since

$$x'_j(\hat{e}'_j \cdot \hat{e}_i) = x_i(\hat{e}_i \cdot \hat{e}_i)$$

or $\quad x_i = \ell_{ji} x'_j$

Equation (3) can equivalently be written as

$$x_j = \ell_{ij} x'_i$$

Equation (3) when written in full represents the following three equations :

$$x_1 = \ell_{11} x'_1 + \ell_{21} x'_2 + \ell_{31} x'_3$$

$$x_2 = \ell_{12} x'_1 + \ell_{22} x'_2 + \ell_{32} x'_3$$

$$x_3 = \ell_{13} x'_1 + \ell_{23} x'_2 + \ell_{33} x'_3$$

which may be written in matrix form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}$$

(b) TRANSFORMATION EQUATIONS FOR UNIT VECTORS

We next find the transformation equations expressing the unit vectors $\hat{e}'_1, \hat{e}'_2, \hat{e}'_3$ in the system K' in terms of the unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ of the system K and vice versa .

From vector analysis , we know that

$$\begin{aligned} \bar{A} &= (\bar{A}\cdot\hat{e}_1)\hat{e}_1 + (\bar{A}\cdot\hat{e}_2)\hat{e}_2 + (\bar{A}\cdot\hat{e}_3)\hat{e}_3 \\ &= (\bar{A}\cdot\hat{e}_i)\hat{e}_i \end{aligned}$$

Setting $\bar{A} = \hat{e}'_j$, we get $\quad \hat{e}'_j = (\hat{e}'_j \cdot \hat{e}_i)\hat{e}_i = \ell_{ji}\hat{e}_i \qquad$ (4)

ation (4) when written in full represents the following three equations :

$$\hat{e}_1' = \ell_{11}\hat{e}_1 + \ell_{12}\hat{e}_2 + \ell_{13}\hat{e}_3$$

$$\hat{e}_2' = \ell_{21}\hat{e}_1 + \ell_{22}\hat{e}_2 + \ell_{23}\hat{e}_3$$

$$\hat{e}_3' = \ell_{31}\hat{e}_1 + \ell_{32}\hat{e}_2 + \ell_{33}\hat{e}_3$$

ilarly, from $\vec{A} = (\vec{A}\cdot\hat{e}_1')\hat{e}_1' + (\vec{A}\cdot\hat{e}_2')\hat{e}_2' + (\vec{A}\cdot\hat{e}_3')\hat{e}_3'$

$$= (\vec{A}\cdot\hat{e}_i')\hat{e}_i'$$

ing $\vec{A} = \hat{e}_j$, we get $\hat{e}_j = (\hat{e}_j\cdot\hat{e}_i')\hat{e}_i' = \ell_{ij}\hat{e}_i'$ (5)

ation (5) when written in full represents the following three equations:

$$\hat{e}_1 = \ell_{11}\hat{e}_1' + \ell_{21}\hat{e}_2' + \ell_{31}\hat{e}_3'$$

$$\hat{e}_2 = \ell_{12}\hat{e}_1' + \ell_{22}\hat{e}_2' + \ell_{32}\hat{e}_3'$$

$$\hat{e}_3 = \ell_{13}\hat{e}_1' + \ell_{23}\hat{e}_2' + \ell_{33}\hat{e}_3'$$

ations (4) and (5) are the required transformation equations .

AMPLE (10): **A set of axes $Ox_1'x_2'x_3'$ is initially coincident with a set $Ox_1x_2x_3$. The set $Ox_1'x_2'x_3'$ is then rotated through an angle θ in the positive sense about the x_3– axis . Show that**

$$x_1' = x_1\cos\theta + x_2\sin\theta$$

$$x_2' = -x_1\sin\theta + x_2\cos\theta$$

$$x_3' = x_3$$

LUTION: Since the rotation is about x_3 – axis , therefore x_3' – axis coincides with x_3 – axis own in the figure (7.9) . If the angle $x_1'Ox_1 = x_2'Ox_2 = \theta$, then

$\ell_{11} = \cos(x_1'Ox_1) = \cos\theta$;

$\ell_{12} = \cos(x_1'Ox_2) = \cos(90-\theta) = \sin\theta$

$\ell_{13} = \cos(x_1'Ox_3) = \cos 90^\circ = 0$

$\ell_{21} = \cos(x_2'Ox_1) = \cos(90+\theta) = -\sin\theta$

$\ell_{22} = \cos(x_2'Ox_2) = \cos\theta$

$\ell_{23} = \cos(x_2'Ox_3) = \cos 90^\circ = 0$

$\ell_{31} = \cos(x_3'Ox_1) = \cos 90^\circ = 0$

$\ell_{32} = \cos(x_3'Ox_2) = \cos 90^\circ = 0$

$\ell_{33} = \cos(x_3'Ox_3) = \cos 0^\circ = 1$

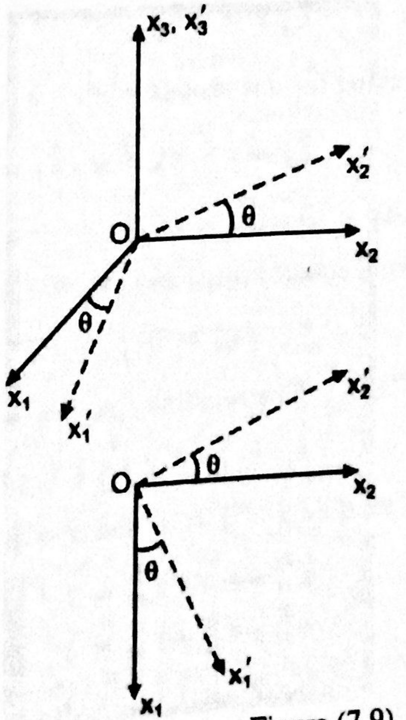


Figure (7.9)

ransformation matrix is given by

$$[\ell_{ij}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and thus the transformation equations for the coordinates become

$$x'_1 = x_1 \cos\theta + x_2 \sin\theta$$

$$x'_2 = -x_1 \sin\theta + x_2 \cos\theta$$

$$x'_3 = x_3$$

Note that effectively we are dealing with a transformation of axes in two–dimensions only and we can write the transformation matrix as

$$[\ell_{ij}] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

7.12 ORTHONORMALITY CONDITIONS

THEOREM (7.5): Prove that $\ell_{ik}\ell_{jk} = \delta_{ij} = \ell_{ki}\ell_{kj}$.

PROOF: We know that $\hat{e}_j' = \ell_{ji}\hat{e}_i$

or $\hat{e}_j' = \ell_{jk}\hat{e}_k$

Taking the dot product with $\hat{e}_i'$, we get

$$\hat{e}_i' \cdot \hat{e}_j' = \ell_{jk}\hat{e}_i' \cdot \hat{e}_k = \ell_{ik}\ell_{jk} \tag{1}$$

Also, $\hat{e}_i' \cdot \hat{e}_j' = \delta_{ij}$ (2)

From equations (1) and (2), we have

$$\ell_{ik}\ell_{jk} = \delta_{ij} \tag{3}$$

Similarly, we know that $\hat{e}_j = \ell_{ij}\hat{e}_i'$

or $\hat{e}_i = \ell_{ki}\hat{e}_k'$

Taking the dot product with $\hat{e}_j$, we get

$$\hat{e}_i \cdot \hat{e}_j = \ell_{ki}\hat{e}_k' \cdot \hat{e}_j = \ell_{ki}\ell_{kj} \tag{4}$$

Also $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ (5)

From equations (4) and (5), we have

$$\ell_{ki}\ell_{kj} = \delta_{ij} \tag{6}$$

NOTE: The relation $\ell_{ik}\ell_{jk} = \delta_{ij}$ implies six orthonormality conditions. Write the relation as

$$\ell_{i1}\ell_{j1} + \ell_{i2}\ell_{j2} + \ell_{i3}\ell_{j3} = \delta_{ij} \tag{A}$$

If we take $i = j = 1$ and $i = j = 2$ and $i = j = 3$ in turn in relation (A), we get

$$\left.\begin{aligned} \ell_{11}^2 + \ell_{12}^2 + \ell_{13}^2 &= 1 \\ \ell_{21}^2 + \ell_{22}^2 + \ell_{32}^2 &= 1 \\ \ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 &= 1 \end{aligned}\right\} \tag{1}$$

e take $i = 1, j = 2$ and $i = 2, j = 3$ and $i = 3, j = 1$ in turn in relation (A), we get

$$\left.\begin{aligned} \ell_{11}\ell_{21}+\ell_{12}\ell_{22}+\ell_{13}\ell_{23} &= 0 \\ \ell_{21}\ell_{31}+\ell_{22}\ell_{32}+\ell_{23}\ell_{33} &= 0 \\ \ell_{31}\ell_{11}+\ell_{32}\ell_{12}+\ell_{33}\ell_{13} &= 0 \end{aligned}\right\} \qquad (2)$$

ilarly, the relation $\ell_{ki}\ell_{kj} = \delta_{ij}$ implies alternative form of orthonormality conditions. Write the tion as

$$\ell_{1i}\ell_{1j}+\ell_{2i}\ell_{2j}+\ell_{3i}\ell_{3j} = \delta_{ij} \qquad (B)$$

e take $i = j = 1$ and $i = j = 2$ and $i = j = 3$ in turn in relation (B), we get

$$\left.\begin{aligned} \ell_{11}^2+\ell_{21}^2+\ell_{31}^2 &= 1 \\ \ell_{12}^2+\ell_{22}^2+\ell_{32}^2 &= 1 \\ \ell_{13}^2+\ell_{23}^2+\ell_{33}^2 &= 1 \end{aligned}\right\} \qquad (3)$$

e take $i = 1, j = 2$ and $i = 2, j = 3$ and $i = 3, j = 1$ in turn in relation (B), we get

$$\left.\begin{aligned} \ell_{11}\ell_{12}+\ell_{21}\ell_{22}+\ell_{31}\ell_{32} &= 0 \\ \ell_{12}\ell_{13}+\ell_{22}\ell_{23}+\ell_{32}\ell_{33} &= 0 \\ \ell_{13}\ell_{11}+\ell_{23}\ell_{21}+\ell_{33}\ell_{31} &= 0 \end{aligned}\right\} \qquad (4)$$

should be observed how the orthonormality conditions given in equations (1), (2), (3), and (4) are ined from the transformation matrix.

11 TRANSLATION AND ROTATION

Let the coordinates of a point P be (x_1, x_2, x_3) the system $Ox_1x_2x_3$ and the coordinates of the point O' this system be (a_1, a_2, a_3). Shift the system x_2x_3 to the positions $O'x_1'x_2'x_3'$ through O' as new gin to form a new rectangular coordinate system as shown Figure (7.10). Let the coordinate of P be (x_1', x_2', x_3') the new system. We know that under the translation , the axes remain parallel through O', so that the ordinates of P are $(x_i - a_i)$, $i = 1, 2, 3$. Under hogonal transformation, these become x_j' so that

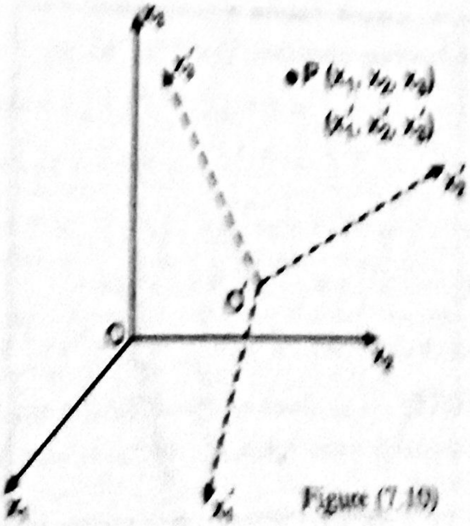


Figure (7.10)

$$\begin{aligned} x_j' &= \ell_{ji}(x_i - a_i) \\ &= \ell_{ji}x_i + c_j \end{aligned} \qquad (5)$$

re $c_j = -\ell_{ji}a_i$

tion (5) illustrates the transformation of coordinates under translation of axes followed by rotation but rving the right - handed rectangular character of the axes.

7.14 INVARIANCE WITH RESPECT TO ROTATION OF AXES

Consider two rectangular coordinate systems K and K' having the same origin O but with axes rotated with respect to each other as shown in figure (7.8). Let (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) be the coordinates of an arbitrary point P in the systems K and K' respectively. Let $\phi(x_1, x_2, x_3)$ be the value of the scalar point function ϕ at P in the system K and $\phi'(x'_1, x'_2, x'_3)$ be the value of this function at the same point in the system K'.

If $\phi(x_1, x_2, x_3) = \phi'(x'_1, x'_2, x'_3)$ where x_1, x_2, x_3 and x'_1, x'_2, x'_3 are related by the transformation equations (2) or (3) on pages (405) and (406), then $\phi(x_1, x_2, x_3)$ is called an **invariant** with respect to the coordinate transformation or rotation of axes.

Similarly, a vector point function $\vec{A}(x_1, x_2, x_3)$ is called **invariant** with respect to rotation of axes if $\vec{A}(x_1, x_2, x_3) = \vec{A}'(x'_1, x'_2, x'_3)$. This will be true if

$$A_1(x_1, x_2, x_3)\hat{e}_1 + A_2(x_1, x_2, x_3)\hat{e}_2 + A_3(x_1, x_2, x_3)\hat{e}_3$$
$$= A'_1(x'_1, x'_2, x'_3)\hat{e}'_1 + A'_2(x'_1, x'_2, x'_3)\hat{e}'_2 + A'_3(x'_1, x'_2, x'_3)\hat{e}'_3$$

EXAMPLE (11): **Show that the quantity $x_1^2 + x_2^2 + x_3^2 = x_i x_i$ is invariant under a rotation of axes.**

SOLUTION: Let (x_1, x_2, x_3) be the coordinates of a point P in the system K and (x'_1, x'_2, x'_3) be the coordinates of the same point in the system K'. Then we know that the transformation equations are :

$$x'_j = \ell_{ji} x_i \qquad (1)$$

Also $$x'_j = \ell_{jk} x_k \qquad (2)$$

Multiplying equations (1) and (2), we get

$$x'_j x'_j = (\ell_{ji} x_i)(\ell_{jk} x_k) = \ell_{ji}\ell_{jk} x_i x_k$$
$$= \delta_{ik} x_i x_k = (\delta_{ik} x_k) x_i = x_i x_i$$

or $$x'_1 x'_1 + x'_2 x'_2 + x'_3 x'_3 = x_1 x_1 + x_2 x_2 + x_3 x_3$$

or $$x'^2_1 + x'^2_2 + x'^2_3 = x_1^2 + x_2^2 + x_3^2$$

which shows that the quantity $x_1^2 + x_2^2 + x_3^2 = x_i x_i$ is invariant with respect to rotation of axes.

NOTE: This result expresses the fact that the distance between the origin O and a point P does not depend upon the system of coordinates used in calculating the distance.

7.15 SCALAR INVARIANT OPERATORS

Let $\mathcal{D}$ denote a linear partial differential operator which involves only the rectangular Cartesian coordinates x_1, x_2, x_3 as independent variables. For example, we might have

$$\mathcal{D} = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3} \quad \text{or} \quad \mathcal{D} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_3}$$

upon chang

$$\mathcal{D}' = \frac{\partial}{\partial x_1'} + \frac{\partial}{\partial x_2'} + 2\frac{\partial}{\partial x_3'}$$

following theorem concerning scalar invariant operators will be required .

EOREM (7.6): Let $\mathcal{D}$ be a scalar invariant operator , and define its operation on a vector field $\bar{A}$ by $\mathcal{D}\bar{A} = \mathcal{D}(A_1, A_2, A_3) = (\mathcal{D}A_1, \mathcal{D}A_2, \mathcal{D}A_3)$ Then prove that $\mathcal{D}\bar{A}$ is a vector field .

OOF: Let A_i and A_j' be the components of $\bar{A}$ in the system $Ox_1x_2x_3$ and $x_2'x_3'$ respectively , then

$$A_j' = \ell_{ji} A_i$$

ing the property of invariance of the form of $\mathcal{D}$ and also the linearity property of $\mathcal{D}$, we have

$$\mathcal{D}' A_j' = \mathcal{D}(\ell_{ji} A_i) = \ell_{ji}\mathcal{D}A_i$$

owing that the components of $\mathcal{D}\bar{A}$ transform according to the vector law under the rotation of the axes.

us it follows that $\mathcal{D}\bar{A}$ is a vector field .

THE LAPLACIAN OPERATOR ∇^2

The most important of the scalar invariant operators is the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}$$

mally , the Laplacian operator is the square of the del operator .

EOREM (7.7): Prove that the Laplacian operator $\nabla^2 = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}$ is invariant under a rotation of the axes .

OOF: The invariance of the Laplacian operator follows from the fact that the components del-operator transforms as a vector . Under a rotation of coordinate axes $Ox_1'x_2'x_3'$, we have

$$\frac{\partial}{\partial x_j'} = \ell_{ji}\frac{\partial}{\partial x_i} \quad \text{and} \quad \frac{\partial}{\partial x_j'} = \ell_{jk}\frac{\partial}{\partial x_k}$$

$$\frac{\partial}{\partial x_j'}\frac{\partial}{\partial x_j'} = \left(\ell_{ji}\frac{\partial}{\partial x_i}\right)\left(\ell_{jk}\frac{\partial}{\partial x_k}\right)$$

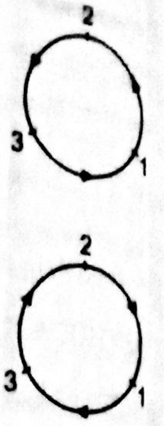
$$= \ell_{ji}\ell_{jk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} = \delta_{ik}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}$$

shows that the Laplacian operator ∇^2 is invariant under a rotation of the axes .

7.16 THE ALTERNATING SYMBOL ϵ_{ijk}

The alternating symbol written ϵ_{ijk} is defined as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ is a cyclic permutation of } 1, 2, 3 \\ -1 & \text{if } i, j, k \text{ is an anticyclic permutation of } 1, 2, 3 \\ 0 & \text{if any two of } i, j, k \text{ are equal} \end{cases}$$



Thus $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$

$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$

$\epsilon_{223} = \epsilon_{131} = \epsilon_{313} = \ldots\ldots = 0$

Figure (7.11)

RELATIONSHIP BETWEEN ALTERNATING SYMBOL AND KRONECKER DELTA

THEOREM (7.8): **Prove that $\epsilon_{ijk}\,\epsilon_{\ell mk} = \delta_{i\ell}\,\delta_{jm} - \delta_{im}\,\delta_{j\ell}$**

PROOF: We know that

$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3, \quad \hat{e}_2 \times \hat{e}_3 = \hat{e}_1, \quad \hat{e}_3 \times \hat{e}_1 = \hat{e}_2 \qquad (1)$$

Using the definition of the alternating symbol we can write equation (1) as

$$\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k = \epsilon_{ijk}$$

Thus

$$\begin{aligned} \epsilon_{ijk}\,\epsilon_{\ell mk} &= (\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k)(\hat{e}_\ell \times \hat{e}_m \cdot \hat{e}_k) \\ &= (\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k)\hat{e}_k \cdot (\hat{e}_\ell \times \hat{e}_m) \\ &= (\hat{e}_i \times \hat{e}_j) \cdot (\hat{e}_\ell \times \hat{e}_m) \quad [\text{since } (\vec{A} \cdot \hat{e}_k)\hat{e}_k = \vec{A}\,] \\ &= (\hat{e}_i \times \hat{e}_j) \times \hat{e}_\ell \cdot \hat{e}_m \\ &= [(\hat{e}_i \cdot \hat{e}_\ell)\hat{e}_j - (\hat{e}_j \cdot \hat{e}_\ell)\hat{e}_i] \cdot \hat{e}_m \\ &= (\hat{e}_i \cdot \hat{e}_\ell)(\hat{e}_j \cdot \hat{e}_m) - (\hat{e}_j \cdot \hat{e}_\ell)(\hat{e}_i \cdot \hat{e}_m) \\ &= \delta_{i\ell}\,\delta_{jm} - \delta_{im}\,\delta_{j\ell} \end{aligned}$$

ALTERNATIVE METHOD

We have to prove

$$\epsilon_{ij1}\,\epsilon_{\ell m1} + \epsilon_{ij2}\,\epsilon_{\ell m2} + \epsilon_{ij3}\,\epsilon_{\ell m3} = \delta_{i\ell}\,\delta_{jm} - \delta_{im}\,\delta_{j\ell} \qquad (A)$$

Case (1) **(a) when $i = j$ but $\ell \neq m$** **(b) when $i \neq j$ but $\ell = m$**

For these cases we may easily verify from equation (A) that L.H.S. $= 0 =$ R.H.S.

Case (2) **When $i \neq j$, $\ell \neq m$ and the pairs (i, j) and (ℓ, m) are different from each other.**

For example, $(i, j) = (1, 2)$ and $(\ell, m) = (1, 3)$, then from equation (A)

$$\epsilon_{121}\,\epsilon_{131} + \epsilon_{122}\,\epsilon_{132} + \epsilon_{123}\,\epsilon_{133} = \delta_{11}\,\delta_{23} - \delta_{13}\,\delta_{21}$$

or L.H.S. $= 0 =$ R.H.S.

(3) When $i \neq j$, $\ell \neq m$ but the pairs (i, j) and (ℓ, m) are identical. Thus i, j and ℓ, m ve the following pairs of values in any order:

$1, 2;\ 1, 3;\ 2, 3;\ 2, 1;\ 3, 1;\ 3, 2$

e first pair gives rise to the following possibilities:

$i = 1, j = 2, \quad \ell = 1, m = 2$

$i = 1, j = 2, \quad \ell = 2, m = 1$

$i = 2, j = 1, \quad \ell = 1, m = 2$

$i = 2, j = 1, \quad \ell = 2, m = 1$

r these possibilities, equation (A) gives

L.H.S. = 1 = R.H.S.

L.H.S. = − 1 = R.H.S.

L.H.S. = − 1 = R.H.S.

L.H.S. = 1 = R.H.S.

e result may easily be seen to be true for the other five pairs also.

ce equation (A) holds for all possible values of i, j, ℓ and m.

OTE: Each side of the theorem is a tensor of order 4, therefore equation (A) is equivalent to a set of $= 81$ scalar equations.

AMPLE (12): **Prove that**

(i) $\epsilon_{ijk}\delta_{jk} = 0$ (ii) $\epsilon_{ijk}\epsilon_{ijk} = 6$

(iii) $\epsilon_{ijk}\epsilon_{\ell jk} = 2\delta_{i\ell}$ (iv) $\epsilon_{ijk}\epsilon_{\ell mk}\delta_{jm} = 2\delta_{i\ell}$

(v) $\epsilon_{iks}\epsilon_{mps} = \epsilon_{sik}\epsilon_{smp} = \epsilon_{ksi}\epsilon_{psm}$ (vi) $\frac{1}{2}\epsilon_{ijk}\epsilon_{ij\ell}A_\ell = A_k$

LUTION: We know that

$$\epsilon_{ijk}\delta_{jk} = \epsilon_{ijj} = \epsilon_{i11} + \epsilon_{i22} + \epsilon_{i33} = 0 + 0 + 0 = 0$$

We know that

$$\epsilon_{ijk}\epsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell} \qquad \text{(A)}$$

g $\ell = i$ and $m = j$ in the above relation

$$\epsilon_{ijk}\epsilon_{ijk} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji} = (3)(3) - \delta_{ii} = 9 - 3 = 6$$

Put $m = j$ in relation (A) we get

$$\epsilon_{ijk}\epsilon_{\ell jk} = \delta_{i\ell}\delta_{jj} - \delta_{ij}\delta_{j\ell} = 3\delta_{i\ell} - \delta_{i\ell} = 2\delta_{i\ell}$$

$$\begin{aligned}\epsilon_{ijk}\epsilon_{\ell mk}\delta_{jm} &= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\delta_{jm} \\ &= \delta_{i\ell}\delta_{jm}\delta_{jm} - \delta_{im}\delta_{j\ell}\delta_{jm} \\ &= \delta_{i\ell}\delta_{mm} - \delta_{im}\delta_{m\ell} = 3\delta_{i\ell} - \delta_{i\ell} = 2\delta_{i\ell}\end{aligned}$$

(v) $\epsilon_{ijs}\epsilon_{mps} = \epsilon_{sik}\epsilon_{smp} = \epsilon_{ksi}\epsilon_{psm}$

A cyclic permutation of the suffixes in ϵ_{ijk} does not change its value .

(vi) $\frac{1}{2}\epsilon_{ijk}\epsilon_{ij\ell}A_\ell = \frac{1}{2}(2\delta_{k\ell})A_\ell = A_k$ (since $\epsilon_{ijk}\epsilon_{\ell jk} = 2\delta_{i\ell}$ from equation (3))

7.17 TENSORS

We know that a scalar is a quantity whose specification (in any coordinate system) requires just one number . On the other hand , a vector is a quantity whose specification in any coordinate system requires three numbers called its **components** . Scalars and vectors are both special cases of a more general object called a **tensor** of order n , whose specification in any given coordinate system requires 3^n numbers, again called the components of the tensor . In fact , scalars are tensors of order 0 , with $3^0 = 1$ component , and vectors are tensors of order 1 with $3^1 = 3$ components . The order or rank of a tensor is effectively the number of suffixes used in it .

ZEROTH – ORDER TENSORS (OR SCALARS)

By a scalar (or zeroth order tensor) is meant a quantity uniquely specified in any coordinate system by a single real number (the component or value of the scalar) which is invariant under changes of the coordinate system i.e. which does not change when the coordinate system is changed . Thus if ϕ is the value of a scalar in the coordinate system K and ϕ' its value in another coordinate system K', then

$$\phi = \phi'$$

or
$$\phi(x_1, x_2, x_3) = \phi'(x'_1, x'_2, x'_3)$$

Thus a tensor of order zero is called a **scalar invariant** .

FIRST ORDER TENSORS (OR VECTORS)

An entity (quantity) representable by a set A_i of three (i.e. 3^1) numbers (called components) relatively to a coordinate system K is called a first order tensor , if its components transform under changes of the coordinate system according to the law

$$A'_j = \ell_{ji}A_i \tag{1}$$

where A'_j are the components of the quantity in the coordinate system K' and ℓ_{ji} is the cosine of the angle between the jth – axis of K' and the ith – axis of K . Equation (1) is called the **transformation law** for the components of first order tensor i.e. vector . Equation (1) represents the following three equations :

$$\begin{aligned} A'_1 &= \ell_{11}A_1 + \ell_{12}A_2 + \ell_{13}A_3 \\ A'_2 &= \ell_{21}A_1 + \ell_{22}A_2 + \ell_{23}A_3 \\ A'_3 &= \ell_{31}A_1 + \ell_{32}A_2 + \ell_{33}A_3 \end{aligned} \tag{2}$$

which may be written in matrix form as

$$\begin{bmatrix} A'_1 \\ A'_2 \\ A'_3 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{3}$$

components

$$A_i = \ell_{ji} A'_j$$

since $\ell_{ji} A'_j = \ell_{ji}(\ell_{jk} A_k) = \ell_{ji}\ell_{jk} A_k = \delta_{ik} A_k = A_i$ (4)

Equation (4) may equivalently be written as

$$A_j = \ell_{ij} A'_i \qquad (5)$$

Equation (4) or (5) represents the following three equations :

$$A_1 = \ell_{11} A'_1 + \ell_{21} A'_2 + \ell_{31} A'_3$$

$$A_2 = \ell_{12} A'_1 + \ell_{22} A'_2 + \ell_{32} A'_3 \qquad (6)$$

$$A_3 = \ell_{13} A'_1 + \ell_{23} A'_2 + \ell_{33} A'_3$$

which may be written in matrix form as

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix} \begin{bmatrix} A'_1 \\ A'_2 \\ A'_3 \end{bmatrix} \qquad (7)$$

SECOND ORDER TENSORS

A quantity representable by a two suffix set A_{ij} of nine (i.e. 3^2) numbers (called components) relatively to a coordinate system K is called a second order tensor , if its components transform under changes of the coordinate system according to the law

$$A'_{mn} = \ell_{mi}\ell_{nj} A_{ij} \qquad (1)$$

where A'_{mn} are the components of the quantity in the coordinate system K' and ℓ_{mi} is the cosine of the angle between the mth – axis of K' and the ith – axis of K . (Similarly for ℓ_{nj})

The **inverse transformation law** expressing the components of the second order tensor in the system K in terms of its components in the system K' is :

$$A_{ij} = \ell_{mi}\ell_{nj} A'_{mn} \qquad (3)$$

since

$$\ell_{mi}\ell_{nj} A'_{mn} = \ell_{mi}\ell_{nj}(\ell_{mr}\ell_{ns} A_{rs}) = (\ell_{mi}\ell_{mr})(\ell_{nj}\ell_{ns}) A_{rs}$$

$$= \delta_{ir}\delta_{js} A_{rs} = (\delta_{ir} A_{rs})\delta_{js}$$

$$= A_{is}\delta_{js} = A_{ij}$$

Equation (3) can be written equivalently as $A_{mn} = \ell_{im}\ell_{jn} A'_{ij}$

NOTE: (i) The nine components of a second order tensor can be written in the form of a 3 x 3 matrix as

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

or briefly $[A_{ij}]$, where A_{ij} is the element in the ith – row and jth – column of the above matrix .

(ii) Equation (1) when written in matrix form becomes

$$\begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ A'_{21} & A'_{22} & A'_{23} \\ A'_{31} & A'_{32} & A'_{33} \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix} \qquad (2)$$

or $[A'] = [T][A][T']$

Equation (2) is easier to use than equation (1) itself

(iii) Given the components of a second order tensor in the coordinate system K, we can use equation (1) to determine its components in another coordinate system K'. In particular, if all the components of a tensor vanish in one coordinate system, they also vanish in any other coordinate system.

EXAMPLE (13): **Prove that if A_i and B_j are two first order tensors i.e. vectors, then their product $A_i B_j$ $(i, j = 1, 2, 3)$ is a second order tensor.**

SOLUTION: Let $C_{ij} = A_i B_j$ (1)

then we have to prove that C_{ij} $(i, j = 1, 2, 3)$ are the components of a second order tensor. Since A_i and B_j are the first order tensors, their equations of transformation from the system K to K' are

$$A'_m = \ell_{mi} A_i \qquad (2)$$

$$B'_n = \ell_{nj} B_j \qquad (3)$$

Multiplying equations (2) and (3), we obtain

$$A'_m B'_n = \ell_{mi} \ell_{nj} A_i B_j \qquad (4)$$

or $$C'_{mn} = \ell_{mi} \ell_{nj} C_{ij} \qquad (5)$$

where $C'_{mn} = A'_m B'_n$

Equation (5) shows that $C_{ij} = A_i B_j$ are the components of a second order tensor.

THEOREM (7.9): **Prove that the Kronecker delta δ_{ij} is a Cartesian tensor of rank 2.**

PROOF: Let δ_{ij} and δ'_{mn} be the components of the Kronecker delta in the systems K and K' respectively. Then $\delta_{ij} = \hat{e}_i . \hat{e}_j$ and $\delta'_{mn} = \hat{e}_m{}' . \hat{e}_n{}'$

Also we know that $\hat{e}_m{}' = \ell_{mi} \hat{e}_i$ and $\hat{e}_n{}' = \ell_{nj} \hat{e}_j$

so
$$\begin{aligned} \delta'_{mn} &= \hat{e}_m{}' . \hat{e}_n{}' \\ &= (\ell_{mi} . \hat{e}_i) . (\ell_{nj} \hat{e}_j) \\ &= \ell_{mi} \ell_{nj} (\hat{e}_i . \hat{e}_j) \\ &= \ell_{mi} \ell_{nj} \delta_{ij} \end{aligned}$$

which shows that δ_{ij} is a second order Cartesian tensor.

NOTE: (i) The nine components of the Kronecker delta tensor δ_{ij} can be written in the form of 3×3 matrix as

$$[\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$