hapter

CARTESIAN TENSORS

INTRODUCTION

Tensor analysis may be regarded as a generalization of vector analysis . It is of great value to a ist or an engineer in two ways . First , it allows complex mathematical and physical relationships to pressed in a compact way and simplifies the mechanics of the development of theory . Second , it ts greater understanding to vector notations and also establishes certain invariance properties with ce and simplicity . It is of great use in mechanics , fluid dynamics , elasticity , differential geometry , magnetic theory , general relativity theory , and numerous other fields of science and engineering . is chapter , we shall discuss Cartesian tensors i.e. tensors which are expressed in terms of components red to rectangular Cartesian coordinate systems . At first sight , the notation of Cartesian tensors is what complicated . The aim of this chapter is to provide a familiarity with the notation which will e the reader to study other texts and applications without difficulty .

e we start the actual studies of Cartesian tensors , it will be important to give some basic ideas ations , definitions , transformations , etc.) which are useful in the study of Cartesian tensors .

SUMMATION CONVENTION

Consider an expression $a_1 x_1 + a_2 x_2 + a_3 x_3$ (1)

which can be written using summation sign as $\sum_{j=1}^{3} a_j x_j$. (2)

omit the summation sign and write it simply as $a_j x_j$ (3)

it is understood that the repeated index (or suffix) j represents the summation from 1 to 3 . that the form (3) is much more convenient than the original form (1) . This situation occurs so ly that it is convenient to adopt a convention which avoids the necessity of writing summation This convention known as the summation convention is as follows :

r a suffix appears twice in the same expression that expression is to be summed over all f the suffix namely , 1 , 2 , 3 .

DUMMY AND FREE INDICES

An index which is repeated in a given expression so that the summation convention applies , is dummy index , while an index occurring only once in a given expression is called a free index and imply any summation. For example, in the expression $A_k B_{jk}$. k is dummy index while j is a dex

EXAMPLE (1): **Write each of the following using summation convention .**

(i) $a_{11}x_1+a_{12}x_2+a_{13}x_3$ **(ii)** $a_{11}b_{11}+a_{12}b_{12}+a_{13}b_{13}$

(iii) $(x_1)^2+(x_2)^2+(x_3)^2$ **(iv)** $\frac{\partial\phi}{\partial x_1}dx_1+\frac{\partial\phi}{\partial x_2}dx_2+\frac{\partial\phi}{\partial x_3}dx_3$

SOLUTION: We have

(i) $a_{11}x_1+a_{12}x_2+a_{13}x_3 = a_{1i}x_i$

(ii) $a_{11}b_{11}+a_{12}b_{12}+a_{13}b_{13} = a_{1i}b_{1i}$

(iii) $(x_1)^2+(x_2)^2+(x_3)^2 = x_1x_1+x_2x_2+x_3x_3 = x_ix_i$

(iv) $\frac{\partial\phi}{\partial x_1}dx_1+\frac{\partial\phi}{\partial x_2}dx_2+\frac{\partial\phi}{\partial x_3}dx_3 = \frac{\partial\phi}{\partial x_i}dx_i$

EXAMPLE (2): **Write out explicitly the following summations and compare the results :**

(i) $a_i(x_i+y_i)$ **(ii)** $a_jx_j+a_ky_k$

SOLUTION: We have

(i) $a_i(x_i+y_i) = a_1(x_1+y_1)+a_2(x_2+y_2)+a_3(x_3+y_3)$

$= a_1x_1+a_1y_1+a_2x_2+a_2y_2+a_3x_3+a_3y_3$

(ii) $a_jx_j+a_ky_k = a_1x_1+a_2x_2+a_3x_3+a_1y_1+a_2y_2+a_3y_3$

The two summations are identical except for the order in which the terms occur .

NOTE: **(i)** A repeated suffix may be replaced by any other suitable symbol not already in use . For example , $a_jb_j = a_kb_k = a_\alpha b_\alpha$ since in each expression summation over the repeated suffix is implied.

(ii) No suffix may occur more than twice in an expression . For example , $a_{ii}x_i$ is ambiguous because of the differences in the three quantities :

$$a_{ij}x_j = a_{i1}x_1+a_{i2}x_2+a_{i3}x_3$$

$$a_{ii}x_j = (a_{11}+a_{22}+a_{33})x_j$$

$$a_{ij}x_i = a_{1j}x_1+a_{2j}x_2+a_{3j}x_3$$

On putting $j = i$ in these equations , we obtain entirely different expressions on the R.H.S.

(iii) An expression of the form $a_i(x_i+y_i)$ is considered well – defined , for it is obtained by composition of the meaningful expressions a_iz_i and $x_i+y_i = z_i$. In other words , the index i is regarded as occurring once in the term (x_i+y_i) .

7.3 DOUBLE SUMS

An expression can involve more than one summation indices . For example , $a_{ij}x_ix_j$ indicates a summation taking place on both i and j simultaneously . If an expression has two summation (dummy) indices , there will be a total of 3^2 terms in the sum ; if there are three indices , there will be 3^3 terms ; and so on .

XAMPLE (3): **Write the terms in the expression $a_{ij}x_ix_j$; $i,j = 1,2,3$.**

OLUTION: The given expression represents the double sum and has 9 terms in it. Its expansion n be written logically by first summing over i, and then over j. Since i varies from 1 to 3, therefore lding j fixed, the given expression is the sum of three terms. That is

$$a_{ij}x_ix_j = a_{1j}x_1x_j + a_{2j}x_2x_j + a_{3j}x_3x_j$$

w each term on the R.H.S. has the repeated index j which implies summation. Hence

$$a_{ij}x_ix_j = a_{11}x_1x_1 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_2x_1 + a_{22}x_2x_2 + a_{23}x_2x_3$$
$$+ a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3x_3$$

e result is the same if one sums over j first, and then over i.

AMPLE (4): **Write the following expression using summation convention.**

$$a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13} + a_{12}b_{21} + a_{22}b_{22} + a_{32}b_{23} + a_{13}b_{31} + a_{23}b_{32} + a_{33}b_{33}$$

OLUTION: The given expression can be written as

$$(a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13}) + (a_{12}b_{21} + a_{22}b_{22} + a_{32}b_{23}) + (a_{13}b_{31} + a_{23}b_{32} + a_{33}b_{33})$$

$$= a_{i1}b_{1i} + a_{i2}b_{2i} + a_{i3}b_{3i} = a_{ij}b_{ji}$$

SUBSTITUTIONS

Suppose it is required to substitute $y_i = a_{ij}x_j$ in the equation $Q = b_{ij}y_ix_j$. Simple stitution would lead to an absurd expression like $Q = b_{ij}a_{ij}x_jx_j$.

e correct procedure is first to identify any dummy indices in the expression to be substituted that ncide with indices occurring in the main expression. Changing these dummy indices to characters not nd in the main expression, one may then carry out the substitution in the usual manner as follows :

(1) $y_i = a_{ij}x_j$, $Q = b_{ij}y_ix_j$. We see that the dummy index j is duplicated.

(2) Change the dummy index from j to r, to get $y_i = a_{ir}x_r$.

(3) Substitute and rearrange to get $Q = b_{ij}(a_{ir}x_r)x_j = a_{ir}b_{ij}x_rx_j$.

AMPLE (5): **If $y_i = a_{ij}x_j$, express the quadratic form $Q = g_{ij}y_iy_j$ in terms of x – variables.**

UTION: First write $y_i = a_{ir}x_r$, $y_j = a_{js}x_s$

by substitution, $Q = g_{ij}(a_{ir}x_r)(a_{js}x_s) = g_{ij}a_{ir}a_{js}x_rx_s$

ALGEBRA AND THE SUMMATION CONVENTION

Certain routine algebraic manipulations in tensors can be easily justified by properties of ordinary However, some care should be taken. The following are several valid identities ; they will be used edly from now on .

$a_{ij}(x_j + y_j) = a_{ij}x_j + a_{ij}y_j$ (2) $a_{ij}x_iy_j = a_{ij}y_jx_i$

$a_{ij}x_ix_j = a_{ji}x_ix_j$ (4) $(a_{ij} + a_{ji})x_ix_j = 2a_{ji}x_ix_j$

$(a_{ij} - a_{ji})x_ix_j = 0$

The following non – identities should be carefully noted :

(1) $a_{ij}(x_i + y_j) \not\equiv a_{ij}x_i + a_{ij}y_j$ (2) $a_{ij}x_i y_j \not\equiv a_{ij}y_i x_j$

(3) $(a_{ij} + a_{ji})x_i y_j \not\equiv 2a_{ij}x_i y_j$

EXAMPLE (6): **Show that , generally , $a_{ijk}(x_i + y_j)z_k \neq a_{ijk}x_i z_k + a_{ijk}y_j z_k$.**

SOLUTION: Simply observe that on the left side there are no free indices , but on the right , j is free for the first term and i is free for the second .

7.6 THE KRONECKER DELTA δ_{ij}

The Kronecker delta or substitution operator written δ_{ij} , is defined as $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Thus $\delta_{11} = \delta_{22} = \delta_{33} = 1$, and $\delta_{12} = \delta_{21} = \delta_{23} = \delta_{32} = \delta_{31} = \delta_{13} = 0$

EXAMPLE (7): **Using the definition of Kronecker delta , calculate $\delta_{ij}x_i x_j$.**

SOLUTION: We have $\delta_{ij}x_i x_j = \delta_{1j}x_1 x_j + \delta_{2j}x_2 x_j + \delta_{3j}x_3 x_j$

$$= \delta_{11}x_1x_1 + \delta_{12}x_1x_2 + \delta_{13}x_1x_3 + \delta_{21}x_2x_1 + \delta_{22}x_2x_2 + \delta_{23}x_2x_3 + \delta_{31}x_3x_1 + \delta_{32}x_3x_2 + \delta_{33}x_3x_3$$

$$= 1x_1x_1 + 0x_1x_2 + 0x_1x_3 + 0x_2x_1 + 1x_2x_2 + 0x_2x_3 + 0x_3x_1 + 0x_3x_2 + 1x_3x_3$$

$$= x_1x_1 + x_2x_2 + x_3x_3 = x_i x_i$$

THEOREM (7.1): **Show that if x_1, x_2, x_3 are independent variables , then $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$**

PROOF: We have , if $i = j$, $\dfrac{\partial x_i}{\partial x_j} = \dfrac{\partial x_i}{\partial x_i} = 1$

If $i \neq j$, $\dfrac{\partial x_i}{\partial x_j} = 0$ since x_i and x_j are independent variables .

Thus $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

THEOREM (7.2): **Prove that $\delta_{ij}A_j = A_i$.**

PROOF: We know that the index j represents summation , therefore

$$\delta_{ij}A_j = \delta_{i1}A_1 + \delta_{i2}A_2 + \delta_{i3}A_3 \quad (i = 1, 2, 3)$$

when $i = 1$, $\delta_{1j}A_j = \delta_{11}A_1 + \delta_{12}A_2 + \delta_{13}A_3 = A_1$, and

when $i = 2$, $\delta_{2j}A_j = \delta_{21}A_1 + \delta_{22}A_2 + \delta_{23}A_3 = A_2$, and

when $i = 3$, $\delta_{3j}A_j = \delta_{31}A_1 + \delta_{32}A_2 + \delta_{33}A_3 = A_3$.

Thus in all cases : $\delta_{ij}A_j = A_i$

That is , δ_{ij} operating on A_j has substituted the free index i for the index j in A_j which gives a justification of the term **substitution operator** .

NOTE: This result is of fundamental importance and will be used frequently in our later discussion .

OREM (7.3): Prove that $\delta_{ik}\delta_{jk} = \delta_{ij}$.

OOF: We have $\delta_{ik}\delta_{jk} = \delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2} + \delta_{i3}\delta_{j3} \quad (i, j = 1, 2, 3)$

$j = 1, \; \delta_{ik}\delta_{1k} = \delta_{i1}$ and

$j = 2, \; \delta_{ik}\delta_{2k} = \delta_{i2}$ and

$j = 3, \; \delta_{ik}\delta_{3k} = \delta_{i3}$

follows that $\delta_{ik}\delta_{jk} = \delta_{ij}$.

AMPLE (8): Show that

(i) $\delta_{ii} = 3$

(ii) $\delta_{ik}\,\delta_{ik} = 3$

(iii) $\delta_{ij}\delta_{jk}\delta_{ki} = 3$

(iv) $\delta_{ij}\delta_{k\ell}A_{ik} = A_{j\ell}$

(v) $\delta_{ij}\,\delta_{jk}\,A_{ik} = A_{ii}$

LUTION: We have (i) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$

$$\begin{aligned} \delta_{ik}\delta_{ik} &= \delta_{1k}\delta_{1k} + \delta_{2k}\delta_{2k} + \delta_{3k}\delta_{3k}, \quad k = 1, 2, 3 \\ &= (\delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13}) + (\delta_{21}\delta_{21} + \delta_{22}\delta_{22} + \delta_{23}\delta_{23}) \\ &\qquad + (\delta_{31}\delta_{31} + \delta_{32}\delta_{32} + \delta_{33}\delta_{33}) \\ &= \delta_{11}\delta_{11} + \delta_{22}\delta_{22} + \delta_{33}\delta_{33} = (1)(1) + (1)(1) + (1)(1) = 1 + 1 + 1 = 3 \end{aligned}$$

$\delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ik}\delta_{ki} = \delta_{ii} = 3$

$\delta_{ij}\delta_{k\ell}A_{ik} = \delta_{ij}A_{i\ell} = A_{j\ell}$

$\delta_{ij}\,\delta_{jk}\,A_{ik} = \delta_{ik}\,A_{ik} = A_{ii}$.

RECTANGULAR COORDINATE SYSTEM

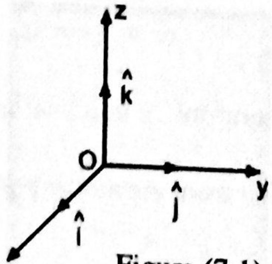

Figure (7.1)

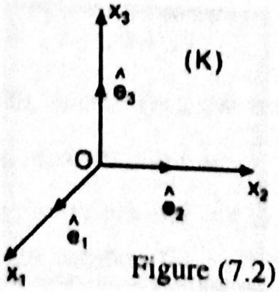

Figure (7.2)

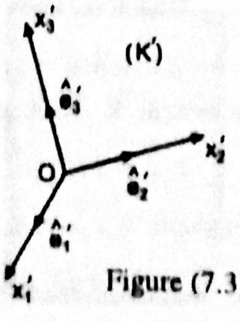

Figure (7.3)

From vector analysis, we are familiar with the coordinate system in which we take Ox, Oy, Oz coordinate axes and $\hat{i}, \hat{j}, \hat{k}$ the unit vectors along coordinate axes respectively, as shown in figure (7.1).

In tensor analysis, in stead of this system we take the in which we have Ox_1, Ox_2, Ox_3 as the coordinate and $\hat{e}_1, \hat{e}_2, \hat{e}_3$ the unit vectors along these axes respectively as shown in figure (7.2). This system of axes will be denoted by K.

In addition to the system K we need another system which will be denoted by K′. In the system take Ox_1', Ox_2', Ox_3' as the coordinate axes and $\hat{e}_3'$ the unit vectors along these axes respectively as figure (7.3)

THEOREM (7.4): Prove that $\hat{e}_i . \hat{e}_j = \delta_{ij}$.

PROOF: From vector analysis, we know that when $j = i$, $\hat{e}_i . \hat{e}_j = \hat{e}_i . \hat{e}_i = 1$, and when $j \neq i$, $\hat{e}_i . \hat{e}_j = 0$.

Therefore $\hat{e}_i . \hat{e}_j = \delta_{ij}$.

NOTE: Similarly, we can prove $\hat{e}_i' . \hat{e}_j' = \delta_{ij}$.

7.8 DIRECTION COSINES

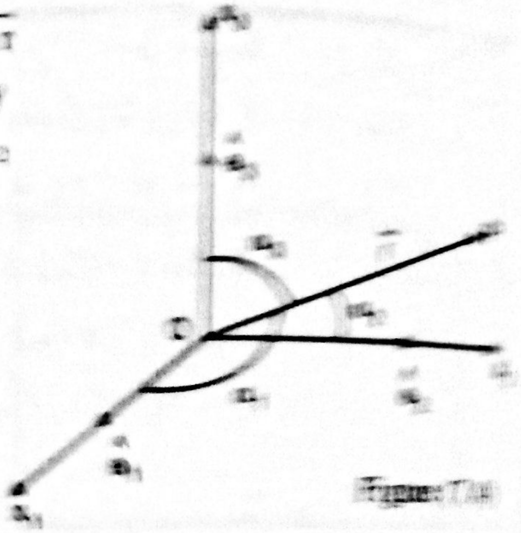

Figure (7.4)

Let $\alpha_1, \alpha_2, \alpha_3$ be the angles which the position vector $\bar{r}$ makes with the positive directions of Ox_1, Ox_2, Ox_3 respectively as shown in figure (7.4). Then the three quantities $\cos\alpha_1, \cos\alpha_2, \cos\alpha_3$ are called the direction cosines of the vector $\bar{r}$.

For convenience, we write

$$\ell_1 = \cos\alpha_1, \quad \ell_2 = \cos\alpha_2, \quad \ell_3 = \cos\alpha_3$$

Now $\quad \bar{r} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3 \qquad (1)$

and $\quad \bar{r} . \hat{e}_1 = x_1 \quad$ or $\quad r\cos\alpha_1 = x_1$

Similarly, $r\cos\alpha_2 = x_2$, and $r\cos\alpha_3 = x_3$

Substitution for x_1, x_2, and x_3 in equation (1), gives

$$\bar{r} = r\cos\alpha_1\hat{e}_1 + r\cos\alpha_2\hat{e}_2 + r\cos\alpha_3\hat{e}_3$$

Therefore, a unit vector in the direction of $\bar{r}$ is $\hat{r} = \frac{\bar{r}}{r} = \cos\alpha_1\hat{e}_1 + \cos\alpha_2\hat{e}_2 + \cos\alpha_3\hat{e}_3$

Also from the above equations, $\cos\alpha_1 = \frac{x_1}{r}$, $\cos\alpha_2 = \frac{x_2}{r}$, and $\cos\alpha_3 = \frac{x_3}{r}$

Hence $\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = \frac{x_1^2 + x_2^2 + x_3^2}{r^2} = \frac{r^2}{r^2} = 1 \quad (\text{since } x_1^2 + x_2^2 + x_3^2 = r^2)$

or $\quad \ell_1^2 + \ell_2^2 + \ell_3^2 = 1$

Thus we have shown that the unit vector in the direction of $\bar{r}$ has its components the direction cosines of $\bar{r}$ and that the sum of the squares of the direction cosines is unity.

NOTE: The direction cosines of the x_1-axis are 1, 0, 0. Similarly, the direction cosines of x_2-axis are 0, 1, 0 and that of x_3-axes are 0, 0, 1.

DEFINITION OF ℓ_{ij}

We define ℓ_{ij} to be the cosine of the angle between the ith-axis of the system X' and jth-axis of the system X, so that

$$\ell_{ij} = \cos(x_i', x_j) = \hat{e}_i' . \hat{e}_j, \quad i, j = 1, 2, 3.$$

For example $\ell_{21} = \cos(x_2', x_1) = \hat{e}_2' . \hat{e}_1 = \cos\theta$

where θ is the angle between x_2'- and x_1-axes.

7.9 ORTHOGONAL ROTATION OF AXES

Consider two right–handed rectangular coordinate systems $Ox_1x_2x_3$ and $Ox'_1x'_2x'_3$ (i.e. systems K and K') having the same origin O and with unit vectors along the coordinate axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$ and $\hat{e}_1', \hat{e}_2', \hat{e}_3'$ as shown in figure (7.5) . Rotate the system $Ox_1x_2x_3$ about O (with Ox_1, Ox_2, Ox_3 always fixed relative to each other) so that it coincides with the system $Ox'_1x'_2x'_3$. Such a movement is called a **rotation of axes** .

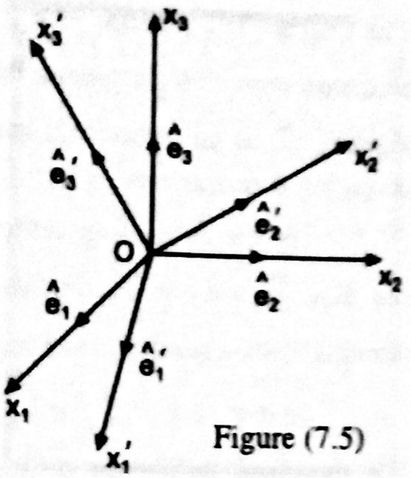

Figure (7.5)

Let the direction cosines of Ox'_1 relative to the axes Ox_1, Ox_2, Ox_3 be ℓ_{11}, ℓ_{12}, ℓ_{13} respectively . Furthermore , denote the direction cosines of Ox'_2 and Ox'_3 by ℓ_{21}, ℓ_{22}, ℓ_{23} and ℓ_{31}, ℓ_{32}, ℓ_{33}, respectively . We may conveniently summarize this in the adjacent table . In this table , the direction cosines of Ox'_1 relative to the axes Ox_1, Ox_2, Ox_3 occur in the first row , the direction cosines of Ox'_2 occur in the second row , and those of Ox'_3 in the third row . Furthermore , reading down the three columns in turn , it is seen that we obtain the direction cosines of the axes Ox_1, Ox_2, Ox_3 relative to the axes Ox'_1, Ox'_2, Ox'_3 respectively .

	Ox_1	Ox_2	Ox_3
Ox'_1	ℓ_{11}	ℓ_{12}	ℓ_{13}
Ox'_2	ℓ_{21}	ℓ_{22}	ℓ_{23}
Ox'_3	ℓ_{31}	ℓ_{32}	ℓ_{33}

The table of direction cosines is called the **transformation matrix** and is written as

$$T = [\ell_{ij}] = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}$$

The transpose of T is $T' = [\ell_{ji}] = \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix}$

and

$$TT' = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix}$$

The element in the ith – row and jth – column of the product matrix $T T'$ is the inner product of the ith – row of T with the jth – column of T', i.e.

$$\begin{aligned} [(TT')_{ij}] &= \ell_{i1}\ell_{j1} + \ell_{i2}\ell_{j2} + \ell_{i3}\ell_{j3} = \ell_{ik}\,\ell_{jk} \\ &= (\hat{e}_i'.\hat{e}_k)(\hat{e}_j'.\hat{e}_k) = (\hat{e}_i'.\hat{e}_k)(\hat{e}_k.\hat{e}_j') \\ &= (\hat{e}_i'.\hat{e}_k)\hat{e}_k.\hat{e}_j' \qquad [\text{since } m(\vec{A}.\vec{B}) = m\vec{A}.\vec{B}] \\ &= \hat{e}_i'.\hat{e}_j' = \delta_{ij} \qquad [\text{since } (\vec{A}.\hat{e}_i)\,\hat{e}_i = \vec{A}] \qquad (1) \end{aligned}$$

Hence

$$TT' = [\delta_{ij}] = I$$

The matrix $T T'$ is the 3×3 unit matrix I, its principle diagonal elements being unity and all other elements zero. This means that the transposed matrix T' is the inverse of T and so, as in matrix algebra, T is an orthogonal matrix. Since T , T' have the same determinants, the relation $TT' = I$ on taking determinants, gives $(\det T)^2 = 1$ so that $\det T = \pm 1$.

In the case of an orthogonal rotation of rectangular coordinate axes as shown in figure (7.5), we see that $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$ and also $\hat{e}_1' \times \hat{e}_2' = \hat{e}_3'$ meaning that a right – handed system $Ox_1x_2x_3$ remains right – handed when rotated to $Ox_1'x_2'x_3'$. Thus in this case,

$$\det T = \hat{e}_1' . \hat{e}_2' \times \hat{e}_3' = \hat{e}_1' . \hat{e}_1' = 1$$

We therefore define an orthogonal transformation to be one whose matrix $T = [\ell_{ij}]$, where $T T' = I$ and for which $\det T = \pm 1$. Such a transformation would leave a right – handed system of axes right – hand and would likewise preserve left – handedness. However, a right – handed orthogonal rotation of coordinate axes specifically requires that $\det \; T = + 1$. This transformation is called **right – handed orthogonal (or proper)** transformation. A transformation which is not right – handed is called a **left – handed (or an improper)** transformation.

Now, when the axes $Ox_1x_2x_3$ and $Ox_1'x_2'x_3'$ coincide, i.e. $x_1' = x_1$, $x_2' = x_2$, $x_3' = x_3$, it is easily seen that the values of the direction cosines in the transformation matrix T are $\ell_{ij} = 1$ when $i = j$ and $\ell_{ij} = 0$ when $i \neq j$; and so for this particular case

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \det (T) = 1$$

Note that if one set of axes is right – handed and the other left – handed, it is impossible to bring them into coincidence by a rotation.

NOTE: Equation (1) can be written in full as :

$$\begin{bmatrix} \ell_{11}^2 + \ell_{12}^2 + \ell_{13}^2 & \ell_{11}\ell_{21} + \ell_{12}\ell_{22} + \ell_{13}\ell_{23} & \ell_{11}\ell_{31} + \ell_{12}\ell_{32} + \ell_{13}\ell_{33} \\ \ell_{21}\ell_{11} + \ell_{22}\ell_{12} + \ell_{23}\ell_{13} & \ell_{21}^2 + \ell_{22}^2 + \ell_{23}^2 & \ell_{21}\ell_{31} + \ell_{22}\ell_{32} + \ell_{23}\ell_{32} \\ \ell_{31}\ell_{11} + \ell_{32}\ell_{12} + \ell_{33}\ell_{13} & \ell_{31}\ell_{21} + \ell_{32}\ell_{22} + \ell_{32}\ell_{23} & \ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which implies the following six equations called the orthonormality conditions :

$$\left. \begin{aligned} \ell_{11}^2 + \ell_{12}^2 + \ell_{13}^2 &= 1 \\ \ell_{21}^2 + \ell_{22}^2 + \ell_{23}^2 &= 1 \\ \ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 &= 1 \end{aligned} \right\} \quad (2) \qquad \left. \begin{aligned} \ell_{11}\ell_{21} + \ell_{12}\ell_{22} + \ell_{13}\ell_{23} &= 0 \\ \ell_{21}\ell_{31} + \ell_{22}\ell_{32} + \ell_{23}\ell_{33} &= 0 \\ \ell_{31}\ell_{11} + \ell_{32}\ell_{12} + \ell_{33}\ell_{13} &= 0 \end{aligned} \right\} \quad (3)$$

AMPLE (9): **Show that the transformation**

(i) $$T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$ **is orthogonal and right – handed .**

On the other hand , the transformation .

(ii) $$T = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix}$$ **is orthogonal but left – handed .**

OLUTION: (i) The transpose of the given matrix T is $T' = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix}$

nd so $$TT' = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

e note that $$\det T = \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = 1$$

d so the corresponding transformation is orthogonal and right – handed .

The transpose of the given matrix is $T' = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$

$$TT' = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$