In matrix notation , equations (5) can be written as

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_z \end{bmatrix} \quad (6)$$

6.21 EXPRESSIONS FOR ARC LENGTH , AREA , AND VOLUME ELEMENTS IN CYLINDERICAL POLAR COORDINATES

In cylinderical polar coordinates , we have

$u_1 = r, \quad u_2 = \theta, \quad u_3 = z, \quad h_1 = 1, \quad h_2 = r, \quad h_3 = 1$

ARC LENGTH ELEMENT

In orthogonal curvilinear coordinates , the element of arc length is determined from

$$(ds)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2$$

In cylinderical polar coordinates , this becomes

$$(ds)^2 = (1)^2(dr)^2 + (r)^2(d\theta)^2 + (1)^2(dz)^2 = (dr)^2 + r^2(d\theta)^2 + (dz)^2$$

ALTERNATIVE METHOD

In cylinderical polar coordinates , $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial\theta}d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial\theta}d\theta = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$dz = dz$$

Then

$$\begin{aligned} ds^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2 + (dz)^2 \\ &= (\cos^2\theta + \sin^2\theta)(dr)^2 + (r^2\sin^2\theta + r^2\cos^2\theta)(d\theta)^2 \\ &\qquad - 2r\sin\theta\cos\theta\, dr\, d\theta + 2r\sin\theta\cos\theta\, dr\, d\theta + (dz^2)^2 \\ &= (dr)^2 + r^2(d\theta)^2 + (dz)^2 \end{aligned}$$

AREA ELEMENT

We know that the elements of area in orthogonal curvilinear coordinates are :

$dA_1 = h_2 h_3\, du_2\, du_3, \quad dA_2 = h_1 h_3\, du_1\, du_3$ and $dA_3 = h_1 h_2\, du_1\, du_2$

In cylinderical polar coordinates , these becomes

$$dA_1 = (r)(1)\, d\theta\, dz = r\, d\theta\, dz$$

$$dA_2 = (1)(1)\, dr\, dz = dr\, dz$$

and

$$dA_3 = (1)(r)\, dr\, d\theta = r\, dr\, d\theta.$$

VOLUME ELEMENT

The volume element in orthogonal curvilinear coordinates is :

$$dV = h_1 h_2 h_3\, du_1\, du_2\, du_3$$

In cylindrical polar coordinates, this becomes

$J = (1)(r)(1)\, dr\, d\theta\, dz = r\, dr\, d\theta\, dz$

EXPRESSION FOR JACOBIAN IN CYLINDERICAL POLAR COORDINATES

We know that the Jacobian in orthogonal curvilinear coordinates u_1, u_2, u_3 is given by

$$J = J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = h_1 h_2 h_3$$

In cylindrical coordinates $h_1 = 1$, $h_2 = r$, $h_3 = 1$ and $u_1 = r$, $u_2 = \theta$, $u_3 = z$

$$J\left(\frac{x, y, z}{r, \theta, z}\right) = (1)(r)(1) = r.$$

EXPRESSIONS FOR $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ IN CYLINDERICAL COORDINATES

We know that $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $z = z$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{r\sin\theta}{r} = \sin\theta\,,$$

$$\frac{\partial r}{\partial z} = 0$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}\,,$$

$$\frac{\partial \theta}{\partial z} = 0$$

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0, \quad \frac{\partial z}{\partial z} = 1$$

By the chain rule, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} + (0)\frac{\partial}{\partial z}$$

$$= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \qquad (1)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial z}\frac{\partial z}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} + (0)\frac{\partial}{\partial z}$$

$$= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \qquad (2)$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial}{\partial z}\frac{\partial z}{\partial z} = (0)\frac{\partial}{\partial r} + (0)\frac{\partial}{\partial \theta} + (1)\frac{\partial}{\partial z}$$

$$= \frac{\partial}{\partial z} \qquad (3)$$

(1), (2), and (3) are the required expressions in terms of cylinderical polar coordinates.

6.24 EXPRESSIONS FOR GRADIENT , DIVERGENCE , CURL , AND LAPLACIAN IN CYLINDERICAL POLAR COORDINATES

We know that for cylinderical polar coordinates (r , θ , z) ,

$$u_1 = r, \quad u_2 = \theta, \quad u_3 = z; \quad \hat{e}_1 = \hat{e}_r, \quad \hat{e}_2 = \hat{e}_\theta, \quad \hat{e}_3 = \hat{e}_z$$

$$h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_z = 1; \quad A_1 = A_r, \quad A_2 = A_\theta, \quad A_3 = A_z$$

EXPRESSION FOR GRADIENT

In orthogonal curvilinear coordinates , we have

$$\nabla \psi = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial \psi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial \psi}{\partial u_3}\hat{e}_3$$

In cylinderical polar coordinates , this becomes

$$\nabla \psi = \frac{1}{1}\frac{\partial \psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{e}_\theta + \frac{1}{1}\frac{\partial \psi}{\partial z}\hat{e}_z$$

$$= \frac{\partial \psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{e}_\theta + \frac{\partial \psi}{\partial z}\hat{e}_z \qquad (1)$$

EXPRESSION FOR DIVERGENCE

In orthogonal curvilinear coordinates , we have

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

In cylinderical polar coordinates , this becomes

$$\nabla \cdot \vec{A} = \frac{1}{(1)(r)(1)}\left[\frac{\partial}{\partial r}\big((r)(1)A_r\big) + \frac{\partial}{\partial \theta}\big((1)(1)A_\theta\big) + \frac{\partial}{\partial z}\big((1)(r)A_z\big)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_r) + \frac{\partial A_\theta}{\partial \theta} + \frac{\partial}{\partial z}(rA_z)\right] \qquad (2)$$

EXPRESSION FOR CURL

In orthogonal curvilinear coordinates , we have

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

In cylinderical polar coordinates , this becomes

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$

$$= \frac{1}{r}\left[\left(\frac{\partial A_z}{\partial \theta} - \frac{\partial}{\partial z}(rA_\theta)\right)\hat{e}_r + \left(r\frac{\partial A_r}{\partial z} - r\frac{\partial A_z}{\partial r}\right)\hat{e}_\theta + \left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{e}_z\right] \qquad (3)$$

EXPRESSION FOR LAPLACIAN

In orthogonal curvilinear coordinates, we have

$$\nabla^2\psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

In cylindrical polar coordinates, this becomes

$$\nabla^2\psi = \frac{1}{(1)(r)(1)}\left[\frac{\partial}{\partial r}\left(\frac{(r)(1)}{(1)}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{(1)(1)}{(r)}\frac{\partial\psi}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\frac{(1)(r)}{(1)}\frac{\partial\psi}{\partial z}\right)\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$$

ALTERNATIVE METHOD USING TRANSFORMATION EQUATIONS

We know that in cylinderical polar coordinates,

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \qquad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}, \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

$$A_1 = A_r\cos\theta - A_\theta\sin\theta, \qquad A_2 = A_r\sin\theta + A_\theta\cos\theta, \qquad A_3 = A_z$$

$$\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta, \qquad \hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta, \qquad \hat{k} = \hat{e}_z$$

EXPRESSION FOR GRADIENT

We know that $\nabla\psi = \frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j} + \frac{\partial\psi}{\partial z}\hat{k}$ (1)

$$\frac{\partial\psi}{\partial x}\hat{i} = \left(\cos\theta\frac{\partial\psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}\right)(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta)$$

$$= \cos^2\theta\frac{\partial\psi}{\partial r}\hat{e}_r - \frac{\sin\theta\cos\theta}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_r - \sin\theta\cos\theta\frac{\partial\psi}{\partial r}\hat{e}_\theta + \frac{\sin^2\theta}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta \quad (2)$$

$$\frac{\partial\psi}{\partial y}\hat{j} = \left(\sin\theta\frac{\partial\psi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\psi}{\partial\theta}\right)(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta)$$

$$= \sin^2\theta\frac{\partial\psi}{\partial r}\hat{e}_r + \frac{\sin\theta\cos\theta}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_r + \sin\theta\cos\theta\frac{\partial\psi}{\partial r}\hat{e}_\theta + \frac{\cos^2\theta}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta \quad (3)$$

$$\frac{\partial\psi}{\partial z}\hat{k} = \frac{\partial\psi}{\partial z}\hat{e}_z \quad (4)$$

Using equations (2), (3), and (4) in equation (1), we get

$$\nabla\psi = (\cos^2\theta + \sin^2\theta)\frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}(\sin^2\theta + \cos^2\theta)\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{\partial\psi}{\partial z}\hat{e}_z$$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{\partial\psi}{\partial z}\hat{e}_z \quad (5)$$

EXPRESSION FOR DIVERGENCE

We know that $\nabla\cdot\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$ (6)

Now
$$\frac{\partial A_1}{\partial x} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)(A_r\cos\theta - A_\theta\sin\theta)$$
$$= \cos^2\theta\frac{\partial A_r}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial A_r}{\partial\theta} + \frac{\sin^2\theta}{r}A_r$$
$$- \sin\theta\cos\theta\frac{\partial A_\theta}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\sin\theta\cos\theta}{r}A_\theta \tag{7}$$

Now
$$\frac{\partial A_2}{\partial y} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)(A_r\sin\theta + A_\theta\cos\theta)$$
$$= \sin^2\theta\frac{\partial A_r}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial A_r}{\partial\theta} + \frac{\cos^2\theta}{r}A_r$$
$$+ \sin\theta\cos\theta\frac{\partial A_\theta}{\partial r} + \frac{\cos^2\theta}{r}\frac{\partial A_\theta}{\partial\theta} - \frac{\sin\theta\cos\theta}{r}A_\theta \tag{8}$$
$$\frac{\partial A_3}{\partial z} = \frac{\partial A_z}{\partial z} \tag{9}$$

Substituting equations (7), (8), and (9) in equation (6), we get
$$\nabla\cdot\vec{A} = (\cos^2\theta + \sin^2\theta)\frac{\partial A_r}{\partial r} + \frac{1}{r}(\sin^2\theta + \cos^2\theta)A_r + \frac{1}{r}(\sin^2\theta + \cos^2\theta)\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}$$

or
$$\nabla\cdot\vec{A} = \frac{\partial A_r}{\partial r} + \frac{1}{r}A_r + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z} \tag{10}$$

EXPRESSION FOR CURL

We know that
$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$
$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k} \tag{11}$$

Now
$$\frac{\partial A_3}{\partial y} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)A_z = \sin\theta\frac{\partial A_z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial A_z}{\partial\theta}$$
$$\frac{\partial A_2}{\partial z} = \frac{\partial}{\partial z}(A_r\sin\theta + A_\theta\cos\theta) = \sin\theta\frac{\partial A_r}{\partial z} + \cos\theta\frac{\partial A_\theta}{\partial z}$$
$$\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} = \left(\sin\theta\frac{\partial A_z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial A_z}{\partial\theta} - \sin\theta\frac{\partial A_r}{\partial z} - \cos\theta\frac{\partial A_\theta}{\partial z}\right)(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta)$$
$$= \sin\theta\cos\theta\frac{\partial A_z}{\partial r}\hat{e}_r + \frac{\cos^2\theta}{r}\frac{\partial A_z}{\partial\theta}\hat{e}_r - \sin\theta\cos\theta\frac{\partial A_r}{\partial z}\hat{e}_r - \cos^2\theta\frac{\partial A_\theta}{\partial z}\hat{e}_r$$
$$- \sin^2\theta\frac{\partial A_z}{\partial r}\hat{e}_\theta - \frac{\sin\theta\cos\theta}{r}\frac{\partial A_z}{\partial\theta}\hat{e}_\theta + \sin^2\theta\frac{\partial A_r}{\partial z}\hat{e}_\theta + \sin\theta\cos\theta\frac{\partial A_\theta}{\partial z}\hat{e}_\theta \tag{12}$$
$$\frac{\partial A_1}{\partial z} = \frac{\partial}{\partial z}(A_r\cos\theta - A_\theta\sin\theta) = \cos\theta\frac{\partial A_r}{\partial z} - \sin\theta\frac{\partial A_\theta}{\partial z}$$
$$\frac{\partial A_3}{\partial x} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)A_z = \cos\theta\frac{\partial A_z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial A_z}{\partial\theta}$$

$$\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\hat{j} = \left(\cos\theta\frac{\partial A_r}{\partial z}-\sin\theta\frac{\partial A_\theta}{\partial z}-\cos\theta\frac{\partial A_z}{\partial r}+\frac{\sin\theta}{r}\frac{\partial A_z}{\partial\theta}\right)\left(\sin\theta\,\hat{e}_r+\cos\theta\,\hat{e}_\theta\right)$$

$$= \sin\theta\cos\theta\frac{\partial A_r}{\partial z}\hat{e}_r-\sin^2\theta\frac{\partial A_\theta}{\partial z}\hat{e}_r-\sin\theta\cos\theta\frac{\partial A_z}{\partial r}\hat{e}_r+\frac{\sin^2\theta}{r}\frac{\partial A_z}{\partial\theta}\hat{e}_r$$

$$+\cos^2\theta\frac{\partial A_r}{\partial z}\hat{e}_\theta-\sin\theta\cos\theta\frac{\partial A_\theta}{\partial z}\hat{e}_\theta-\cos^2\theta\frac{\partial A_z}{\partial r}\hat{e}_\theta+\frac{\sin\theta\cos\theta}{r}\frac{\partial A_z}{\partial\theta}\hat{e}_\theta \qquad (13)$$

$$\frac{\partial A_y}{\partial x} = \left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(A_r\sin\theta+A_\theta\cos\theta\right)$$

$$= \sin\theta\cos\theta\frac{\partial A_r}{\partial r}-\frac{\sin^2\theta}{r}\frac{\partial A_r}{\partial\theta}-\frac{\sin\theta\cos\theta}{r}A_r$$

$$+\cos^2\theta\frac{\partial A_\theta}{\partial r}-\frac{\sin\theta\cos\theta}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{\sin^2\theta}{r}A_\theta$$

$$\frac{\partial A_x}{\partial y} = \left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(A_r\cos\theta-A_\theta\sin\theta\right)$$

$$= \sin\theta\cos\theta\frac{\partial A_r}{\partial r}+\frac{\cos^2\theta}{r}\frac{\partial A_r}{\partial\theta}-\frac{\sin\theta\cos\theta}{r}A_r$$

$$-\sin^2\theta\frac{\partial A_\theta}{\partial r}-\frac{\sin\theta\cos\theta}{r}\frac{\partial A_\theta}{\partial\theta}-\frac{\cos^2\theta}{r}A_\theta$$

$$\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{k} = \left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{e}_z$$

$$= \sin\theta\cos\theta\frac{\partial A_r}{\partial r}\hat{e}_z-\frac{\sin^2\theta}{r}\frac{\partial A_r}{\partial\theta}\hat{e}_z-\frac{\sin\theta\cos\theta}{r}A_r\hat{e}_z+\cos^2\theta\frac{\partial A_\theta}{\partial r}\hat{e}_z$$

$$-\frac{\sin\theta\cos\theta}{r}\frac{\partial A_\theta}{\partial\theta}\hat{e}_z+\frac{\sin^2\theta}{r}A_\theta\hat{e}_z-\sin\theta\cos\theta\frac{\partial A_r}{\partial r}\hat{e}_z-\frac{\cos^2\theta}{r}\frac{\partial A_r}{\partial\theta}\hat{e}_z$$

$$+\frac{\sin\theta\cos\theta}{r}A_r\hat{e}_z+\sin^2\theta\frac{\partial A_\theta}{\partial r}\hat{e}_z+\frac{\sin\theta\cos\theta}{r}\frac{\partial A_\theta}{\partial\theta}\hat{e}_z+\frac{\cos^2\theta}{r}A_\theta\hat{e}_z \qquad (14)$$

Substituting equations (12), (13), and (14) in equation (11), we get after simplification

$$\nabla\times\vec{A} = \frac{\sin^2\theta}{r}\frac{\partial A_z}{\partial\theta}\hat{e}_r-\cos^2\theta\frac{\partial A_\theta}{\partial z}\hat{e}_r-\sin^2\theta\frac{\partial A_z}{\partial r}\hat{e}_\theta+\sin^2\theta\frac{\partial A_r}{\partial z}\hat{e}_\theta$$

$$-\sin^2\theta\frac{\partial A_\theta}{\partial z}\hat{e}_r+\frac{\cos^2\theta}{r}\frac{\partial A_z}{\partial\theta}\hat{e}_r+\cos^2\theta\frac{\partial A_r}{\partial z}\hat{e}_\theta-\cos^2\theta\frac{\partial A_z}{\partial r}\hat{e}_\theta$$

$$-\frac{\sin^2\theta}{r}\frac{\partial A_r}{\partial\theta}\hat{e}_z+\cos^2\theta\frac{\partial A_\theta}{\partial r}\hat{e}_z+\frac{\sin^2\theta}{r}A_\theta\hat{e}_z-\frac{\cos^2\theta}{r}\frac{\partial A_r}{\partial\theta}\hat{e}_z$$

$$+\sin^2\theta\frac{\partial A_\theta}{\partial r}\hat{e}_z+\frac{\cos^2\theta}{r}A_\theta\hat{e}_z$$

$$= \frac{1}{r}\left(\cos^2\theta+\sin^2\theta\right)\frac{\partial A_z}{\partial\theta}\hat{e}_r-\left(\cos^2\theta+\sin^2\theta\right)\frac{\partial A_\theta}{\partial z}\hat{e}_r$$

$$+\left(\sin^2\theta+\cos^2\theta\right)\frac{\partial A_r}{\partial z}\hat{e}_\theta-\left(\sin^2\theta+\cos^2\theta\right)\frac{\partial A_z}{\partial r}\hat{e}_\theta$$

$$-\frac{1}{r}\left(\sin^2\theta+\cos^2\theta\right)\frac{\partial A_r}{\partial\theta}\hat{e}_z+\left(\cos^2\theta+\sin^2\theta\right)\frac{\partial A_\theta}{\partial r}\hat{e}_z+\frac{1}{r}\left(\sin^2\theta+\cos^2\theta\right)A_\theta\hat{e}_z$$

$$= \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e}_\theta + \left(\frac{\partial A_\theta}{\partial r} + \frac{1}{r}A_\theta - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)\hat{e}_z$$

$$= \frac{1}{r}\left[\left(\frac{\partial A_z}{\partial \theta} - r\frac{\partial A_\theta}{\partial z}\right)\hat{e}_r + \left(r\frac{\partial A_r}{\partial z} - r\frac{\partial A_z}{\partial r}\right)\hat{e}_\theta + \left(r\frac{\partial A_\theta}{\partial r} + A_\theta - \frac{\partial A_r}{\partial \theta}\right)\hat{e}_z\right]$$

$$= \frac{1}{r}\left[\left(\frac{\partial A_z}{\partial \theta} - \frac{\partial}{\partial z}(rA_\theta)\right)\hat{e}_r + \left(r\frac{\partial A_r}{\partial z} - r\frac{\partial A_z}{\partial r}\right)\hat{e}_\theta + \left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{e}_z\right]$$

$$= \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix} \qquad (15)$$

EXPRESSION FOR LAPLACIAN

We know that

$$\nabla\psi = \frac{\partial \psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{e}_\theta + \frac{\partial \psi}{\partial z}\hat{e}_z \qquad (16)$$

$$\nabla\cdot\vec{A} = \frac{\partial A_r}{\partial r} + \frac{1}{r}A_r + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \qquad (17)$$

From equations (16) and (17), we have

$$\nabla^2\psi = \nabla\cdot\nabla\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}$$

where $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$ (18)

is the Laplacian operator.

6.26 SPHERICAL POLAR COORDINATES

Let $P(x, y, z)$ be any point whose projection on the xy – plane is $Q(x, y)$. Then the spherical polar coordinates of P are (r, θ, ϕ) in which $r = OP$, $\theta = \angle ZOP$ and $\phi = \angle XOQ$.

From the figure (6.13), we have

$OQ = r\cos(90 - \theta) = r\sin\theta$ (since $\angle QOP = 90 - \theta$)

Therefore $x = OQ\cos\phi = r\sin\theta\cos\phi$

$y = OQ\sin\phi = r\sin\theta\sin\phi$

$z = OP\sin(90 - \theta) = r\cos\theta$

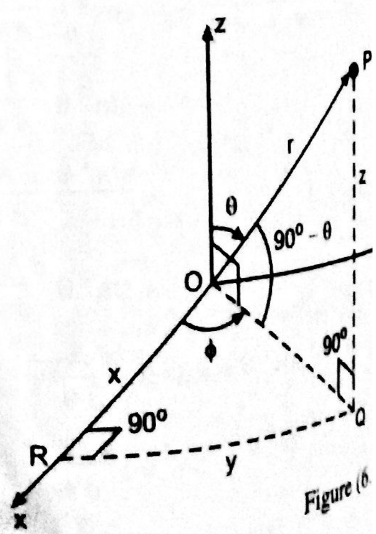

Figure (6

Hence, the transformation equations expressing the rectangular Cartesian coordinates in terms of spherical polar coordinates are :

THEOREM (6.5): In spherical polar coordinates, show that

$$\frac{\partial \hat{e}_r}{\partial r} = 0, \quad \frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta, \quad \frac{\partial \hat{e}_r}{\partial \phi} = \sin\theta\, \hat{e}_\phi$$

$$\frac{\partial \hat{e}_\theta}{\partial r} = 0, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r, \quad \frac{\partial \hat{e}_\theta}{\partial \phi} = \cos\theta\, \hat{e}_\phi$$

$$\frac{\partial \hat{e}_\phi}{\partial r} = 0, \quad \frac{\partial \hat{e}_\phi}{\partial \theta} = 0, \quad \frac{\partial \hat{e}_\phi}{\partial \phi} = -\sin\theta\, \hat{e}_r - \cos\theta\, \hat{e}_\theta$$

PROOF: We know that

$$\hat{e}_r = \sin\theta\cos\phi\, \hat{i} + \sin\theta\sin\phi\, \hat{j} + \cos\theta\, \hat{k}$$

$$\hat{e}_\theta = \cos\theta\cos\phi\, \hat{i} + \cos\theta\sin\phi\, \hat{j} - \sin\theta\, \hat{k}$$

$$\hat{e}_\phi = -\sin\phi\, \hat{i} + \cos\phi\, \hat{j}$$

$$\frac{\partial \hat{e}_r}{\partial r} = 0$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \cos\theta\cos\phi\, \hat{i} + \cos\theta\sin\phi\, \hat{j} - \sin\theta\, \hat{k} = \hat{e}_\theta$$

$$\frac{\partial \hat{e}_r}{\partial \phi} = -\sin\theta\sin\phi\, \hat{i} + \sin\theta\cos\phi\, \hat{j} = \sin\theta\left(-\sin\phi\, \hat{i} + \cos\phi\, \hat{j}\right) = \sin\theta\, \hat{e}_\phi$$

$$\frac{\partial \hat{e}_\theta}{\partial r} = 0$$

$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\sin\theta\cos\phi\, \hat{i} - \sin\theta\sin\phi\, \hat{j} - \cos\theta\, \hat{k} = -\left(\sin\theta\cos\phi\, \hat{i} + \sin\theta\sin\phi\, \hat{j} + \cos\theta\, \hat{k}\right)$$

$$= -\hat{e}_r$$

$$\frac{\partial \hat{e}_\theta}{\partial \phi} = -\cos\theta\sin\phi\, \hat{i} + \cos\theta\cos\phi\, \hat{j} = \cos\theta\left(-\sin\phi\, \hat{i} + \cos\phi\, \hat{j}\right) = \cos\theta\, \hat{e}_\phi$$

$$\frac{\partial \hat{e}_\phi}{\partial r} = 0$$

$$\frac{\partial \hat{e}_\phi}{\partial \theta} = 0$$

$$\frac{\partial \hat{e}_\phi}{\partial \phi} = -\cos\phi\, \hat{i} - \sin\phi\, \hat{j}$$

$$= -\cos\phi\left(\sin^2\theta + \cos^2\theta\right)\hat{i} - \sin\phi\left(\sin^2\theta + \cos^2\theta\right)\hat{j} - \sin\theta\cos\theta\, \hat{k} + \sin\theta\cos\theta\, \hat{k}$$

$$= -\cos\phi\sin^2\theta\, \hat{i} - \cos\phi\cos^2\theta\, \hat{i} - \sin\phi\sin^2\theta\, \hat{j} - \sin\phi\cos^2\theta\, \hat{j}$$
$$\qquad - \sin\theta\cos\theta\, \hat{k} + \sin\theta\cos\theta\, \hat{k}$$

$$= -\sin\theta\left(\sin\theta\cos\phi\, \hat{i} + \sin\theta\sin\phi\, \hat{j} + \cos\theta\, \hat{k}\right)$$
$$\qquad - \cos\theta\left(\cos\theta\cos\phi\, \hat{i} + \cos\theta\sin\phi\, \hat{j} - \sin\theta\, \hat{k}\right)$$

$$= -\sin\theta\, \hat{e}_r - \cos\theta\, \hat{e}_\theta$$

THEOREM (6.6): **Prove that in spherical polar coordinates ,**

$$\frac{d}{dt}\hat{e}_r = \dot{\theta}\,\hat{e}_\theta + \sin\theta\,\dot{\phi}\,\hat{e}_\phi$$

$$\frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\,\hat{e}_r + \cos\theta\,\dot{\phi}\,\hat{e}_\phi$$

$$\frac{d}{dt}\hat{e}_\phi = -\sin\theta\,\dot{\phi}\,\hat{e}_r - \cos\theta\,\dot{\phi}\,\hat{e}_\theta$$

PROOF: We know that $\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$, $\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

Then $\frac{d}{dt}\hat{e}_r = -\sin\theta\sin\phi\,\dot{\phi}\,\hat{i} + \cos\theta\,\dot{\theta}\cos\phi\,\hat{i} + \sin\theta\cos\phi\,\dot{\phi}\,\hat{j} + \cos\theta\,\dot{\theta}\sin\phi\,\hat{j} - \sin\theta\,\dot{\theta}\,\hat{k}$

$$= \dot{\theta}\left(\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}\right) + \sin\theta\,\dot{\phi}\left(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right)$$

$$= \dot{\theta}\,\hat{e}_\theta + \sin\theta\,\dot{\phi}\,\hat{e}_\phi \qquad (1)$$

$$\frac{d}{dt}\hat{e}_\theta = -\cos\theta\sin\phi\,\dot{\phi}\,\hat{i} - \sin\theta\,\dot{\theta}\cos\phi\,\hat{i} + \cos\theta\cos\phi\;\dot{\phi}\,\hat{j} - \sin\theta\;\dot{\theta}\sin\phi\,\hat{j} - \cos\theta\,\dot{\theta}\,\hat{k}$$

$$= -\dot{\theta}\left(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right) + \cos\theta\,\dot{\phi}\left(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right)$$

$$= -\dot{\theta}\,\hat{e}_r + \cos\theta\,\dot{\phi}\,\hat{e}_\phi \qquad (2)$$

$$\frac{d}{dt}\hat{e}_\phi = -\cos\phi\,\dot{\phi}\,\hat{i} - \sin\phi\,\dot{\phi}\,\hat{j}$$

$$= -\dot{\phi}\cos\phi\,\hat{i} - \dot{\phi}\sin\phi\,\hat{j}$$

$$= -\dot{\phi}\cos\phi\left(\cos^2\theta + \sin^2\theta\right)\hat{i} - \dot{\phi}\sin\phi\left(\cos^2\theta + \sin^2\theta\right)\hat{j} + \dot{\phi}\sin\theta\cos\theta\,\hat{k} - \dot{\phi}\sin\theta\cos\theta\,\hat{k}$$

$$= -\dot{\phi}\cos^2\theta\cos\phi\,\hat{i} - \dot{\phi}\sin^2\theta\cos\phi\,\hat{i} - \dot{\phi}\cos^2\theta\sin\phi\,\hat{j} - \dot{\phi}\sin^2\theta\sin\phi\,\hat{j}$$

$$+ \dot{\phi}\sin\theta\cos\theta\,\hat{k} - \dot{\phi}\sin\theta\cos\theta\,\hat{k}$$

$$= \left(-\dot{\phi}\sin^2\theta\cos\phi\,\hat{i} - \dot{\phi}\sin^2\theta\sin\phi\,\hat{j} - \dot{\phi}\sin\theta\cos\theta\,\hat{k}\right)$$

$$+ \left(-\dot{\phi}\cos^2\theta\cos\phi\,\hat{i} - \dot{\phi}\cos^2\theta\sin\phi\,\hat{j} + \dot{\phi}\sin\theta\cos\theta\,\hat{k}\right)$$

$$= -\dot{\phi}\sin\theta\left(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right)$$

$$-\dot{\phi}\cos\theta\left(\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}\right)$$

$$= -\sin\theta\,\dot{\phi}\,\hat{e}_r - \cos\theta\,\dot{\phi}\,\hat{e}_\theta$$

ORTHOGONALITY OF SPHERICAL COORDINATE SYSTEM

We know that the unit vectors in spherical polar coordinates are

$$\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\begin{aligned}\hat{e}_r\cdot\hat{e}_\theta &= (\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k})\cdot(\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k})\\ &= \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta\\ &= \sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) - \sin\theta\cos\theta\\ &= \sin\theta\cos\theta - \sin\theta\cos\theta\\ &= 0\end{aligned}$$

$$\begin{aligned}\hat{e}_\theta\cdot\hat{e}_\phi &= (\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k})\cdot(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})\\ &= -\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi\\ &= 0\end{aligned}$$

$$\begin{aligned}\hat{e}_r\cdot\hat{e}_\phi &= (\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k})\cdot(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})\\ &= -\sin\theta\cos\phi\sin\phi + \sin\theta\sin\phi\cos\phi\\ &= 0\end{aligned}$$

and so $\hat{e}_r, \hat{e}_\theta$, and $\hat{e}_\phi$ are mutually perpendicular and the coordinate system is orthogonal.

RELATIONSHIPS AMONG UNIT VECTORS IN SPHERICAL SYSTEM

THEOREM (6.7): Show that for spherical coordinate system.

$$\hat{e}_r\cdot\hat{e}_r = \hat{e}_\theta\cdot\hat{e}_\theta = \hat{e}_\phi\cdot\hat{e}_\phi = 1$$

$$\hat{e}_r\times\hat{e}_r = \hat{e}_\theta\times\hat{e}_\theta = \hat{e}_\phi\times\hat{e}_\phi = \vec{0}$$

$$\hat{e}_r\times\hat{e}_\theta = \hat{e}_\phi,\quad \hat{e}_\theta\times\hat{e}_\phi = \hat{e}_r,\quad \hat{e}_\phi\times\hat{e}_r = \hat{e}_\theta$$

PROOF: We know that the unit vectors in spherical polar coordinates are

$$\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

and

$$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

Then

$$\begin{aligned}\hat{e}_r\cdot\hat{e}_r &= (\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k})\cdot(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k})\\ &= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\\ &= \sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta = \cos^2\theta + \sin^2\theta = 1\end{aligned}$$

$$\begin{aligned}\hat{e}_\theta\cdot\hat{e}_\theta &= (\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k})\cdot(\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k})\\ &= \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta\\ &= \cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta = \cos^2\theta + \sin^2\theta = 1\end{aligned}$$

$$\hat{e}_\phi\cdot\hat{e}_\phi = (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})\cdot(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) = \sin^2\phi + \cos^2\phi = 1$$

$$\hat{e}_r \times \hat{e}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix}$$

$$= (\sin\theta\cos\theta\sin\phi - \sin\theta\cos\theta\sin\phi)\,\hat{i} + (\sin\theta\cos\theta\cos\phi - \sin\theta\cos\theta\cos\phi)\,\hat{j}$$

$$+ (\sin^2\theta\sin\phi\cos\phi - \sin^2\theta\sin\phi\cos\phi)\,\hat{k} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = \overrightarrow{0}$$

$$\hat{e}_\theta \times \hat{e}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix}$$

$$= (-\sin\theta\cos\theta\sin\phi + \sin\theta\cos\theta\sin\phi)\,\hat{i} + (-\sin\theta\cos\theta\cos\phi + \sin\theta\cos\theta\cos\phi)\,\hat{j}$$

$$+ (\cos^2\theta\sin\phi\cos\phi + \cos^2\theta\sin\phi\cos\phi)\,\hat{k} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = \overrightarrow{0}$$

$$\hat{e}_\phi \times \hat{e}_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\phi & \cos\phi & 0 \\ -\sin\phi & \cos\phi & 0 \end{vmatrix}$$

$$= 0\,\hat{i} + 0\,\hat{j} + (-\sin\phi\cos\phi + \sin\phi\cos\phi)\,\hat{k} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = \overrightarrow{0}$$

$$\hat{e}_r \times \hat{e}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix}$$

$$= (-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi)\,\hat{i} + (\cos^2\theta\cos\phi + \sin^2\theta\cos\phi)\,\hat{j}$$

$$+ (\sin\theta\cos\theta\sin\phi\cos\phi - \sin\theta\cos\theta\sin\phi\cos\phi)\,\hat{k}$$

$$= -\sin\phi(\sin^2\theta + \cos^2\theta)\,\hat{i} + \cos\phi(\cos^2\theta + \sin^2\theta)\,\hat{j} + 0\,\hat{k}$$

$$= -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} = \hat{e}_\phi$$

$$\hat{e}_\theta \times \hat{e}_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + (\cos\theta\cos^2\phi + \cos\theta\sin^2\phi)\,\hat{k}$$

$$= \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k} = \hat{e}_r$$

$$\hat{e}_\phi \times \hat{e}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix}$$

$$= \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} + (-\sin\theta\sin^2\phi - \sin\theta\cos^2\phi)\,\hat{k}$$

$$= \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k} = \hat{e}_\theta$$