Chapter 6

CURVILINEAR COORDINATES

6.1 INTRODUCTION

So far we have restricted ourselves completely to a rectangular Cartesian coordinate system, which has the advantage that all the three unit vectors $\hat{i}, \hat{j}, \hat{k}$ are constant unit vectors. In applications, it is often useful to use other coordinate systems, for example, when a problem involves cylinderical or spherical symmetry. In this chapter, we shall discuss the general orthogonal curvilinear coordinate system and show how the gradient, divergence, curl, and Laplacian can be transformed into this system. In particular, we shall discuss the two most important coordinate systems for space, i.e. the cylinderical coordinate system and the spherical coordinate system. We shall see that the cylinderical coordinates simplify the equations of cylinders, while spherical coordinates simplify the equations of spheres and cones. We shall also derive the formulas for the gradient, divergence, curl, and Laplacian in cylinderical and spherical coordinate systems.

6.2 TRANSFORMATION OF COORDINATES

Let the rectangular coordinates (x, y, z) of any point be expressed as functions of u_1, u_2, u_3, so that

$$\left.\begin{aligned} x &= x(u_1, u_2, u_3) \\ y &= y(u_1, u_2, u_3) \\ z &= z(u_1, u_2, u_3) \end{aligned}\right] \qquad (1)$$

A theorem from elementary calculus shows that if the functions in equations (1) are single–valued and have continuous partial derivatives, then equations (1) can be solved uniquely for u_1, u_2, u_3 in terms of x, y, and z, i.e.

$$\left.\begin{aligned} u_1 &= u_1(x, y, z) \\ u_2 &= u_2(x, y, z) \\ u_3 &= u_3(x, y, z) \end{aligned}\right] \qquad (2)$$

Given a point P with rectangular coordinates (x, y, z), we can from equations (2) associate a unique set of coordinates (u_1, u_2, u_3) called the **curvilinear coordinates** of the point P. Hence any point P can then be defined in space not only by rectangular coordinates (x, y, z) but also by curvilinear coordinates (u_1, u_2, u_3). The sets of equations (1) and (2) define the transformations of coordinates.

6.3 COORDINATE SURFACES AND COORDINATE CURVES

The coordinate surfaces (or level surfaces) are families of surfaces obtained by setting the coordinate equations equal to a constant. Thus if C_1, C_2, C_3 are constants, then the surfaces $u_1 = C_1$, $u_2 = C_2$, $u_3 = C_3$ are called **coordinate surfaces**. The coordinate surfaces are generally curved and each pair of these surfaces intersect in curves called **coordinate curves** in space. Thus u_1 – coordinate curve is that along which only u_1 varies while u_2 and u_3 are constants. Similarly, along u_2 – coordinate curve only u_2 varies while u_1 and u_3 are constants, and along u_3 – coordinate curve only u_3 varies while u_1 and u_2 are constants as shown in figure (6.1).

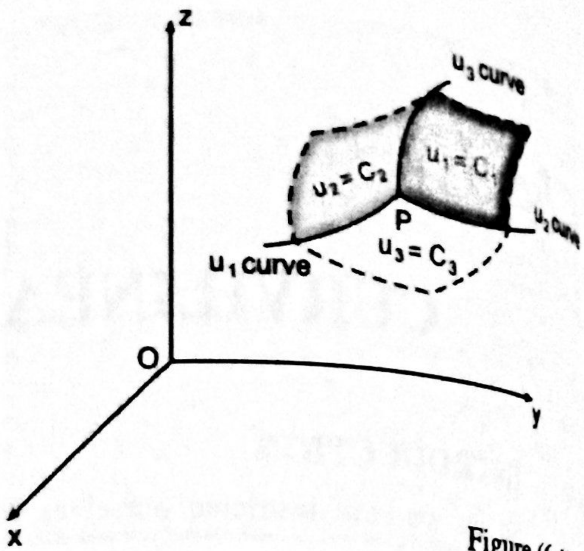

Figure (6.1)

6.4 UNIT VECTORS IN CURVILINEAR COORDINATE SYSTEM

Since the three coordinate curves are generally not straight lines, as in the rectangular coordinate system, such a coordinate system is called the **curvilinear** coordinate system.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a point P. Then the set of equations

$$x = x(u_1, u_2, u_3)$$

$$y = y(u_1, u_2, u_3)$$

$$z = z(u_1, u_2, u_3)$$

can be written $\vec{r} = \vec{r}\,(u_1, u_2, u_3)$

The vector $\dfrac{\partial \vec{r}}{\partial u_1}$ is tangent to the u_1 – coordinate curve at P. Then if $\hat{e}_1$ is the unit tangent vector at P in this direction we can write $\hat{e}_1 = \dfrac{\dfrac{\partial \vec{r}}{\partial u_1}}{\left|\dfrac{\partial \vec{r}}{\partial u_1}\right|}$

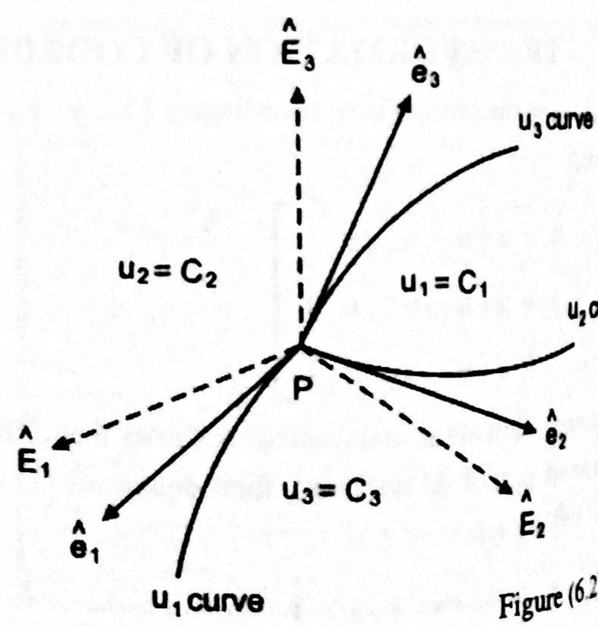

Figure (6.2)

so that $\dfrac{\partial \vec{r}}{\partial u_1} = h_1 \hat{e}_1$, where $h_1 = \left|\dfrac{\partial \vec{r}}{\partial u_1}\right|$

Similarly if $\hat{e}_2$ and $\hat{e}_3$ are unit tangent vectors to the u_2 and u_3 – curves at P respectively, then

$$\frac{\partial \vec{r}}{\partial u_2} = h_2 \hat{e}_2 \quad \text{and} \quad \frac{\partial \vec{r}}{\partial u_3} = h_3 \hat{e}_3, \quad \text{where} \quad h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|, \text{ and } h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right|$$

The quantities h_1, h_2, and h_3 are called, the **scale factors**. The unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are in the directions of increasing u_1, u_2, u_3 respectively. In general, h_1, h_2, h_3 are functions of u_1, u_2, u_3; and $h_1 \neq 0, h_2 \neq 0, h_3 \neq 0$. Hence $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are also functions of u_1, u_2, u_3.

Since ∇u_1 is a vector at P normal to the surface $u_1 = C_1$, a unit vector in this direction is given by

$$\hat{E}_1 = \frac{\nabla u_1}{|\nabla u_1|}$$

Similarly, the unit vectors $\hat{E}_2 = \frac{\nabla u_2}{|\nabla u_2|}$ and $\hat{E}_3 = \frac{\nabla u_3}{|\nabla u_3|}$

at P are unit normal to the surfaces $u_2 = C_2$ and $u_3 = C_3$ respectively.

Thus at each point P of a curvilinear coordinate system there exist, in general, two sets of unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ tangent to the coordinate curves and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ normal to the coordinate surfaces. These two sets of unit vectors generally vary in direction from point to point because the coordinate curves are curved. However, the two sets become identical if and only if the curvilinear coordinate system is orthogonal [see figure (6.3) below].

6.5 ORTHOGONAL CURVILINEAR COORDINATE SYSTEM

If the coordinate curves intersect at right angles, the curvilinear coordinate system is called **orthogonal**. The u_1, u_2, and u_3 coordinate curves of an orthogonal curvilinear system are similar to the x, y, and z coordinate lines of a rectangular Cartesian system. For this system, the two sets of unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ are the same. [see theorem (6.1) below]. In an orthogonal curvilinear coordinate system, the unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are mutually orthogonal at every point,

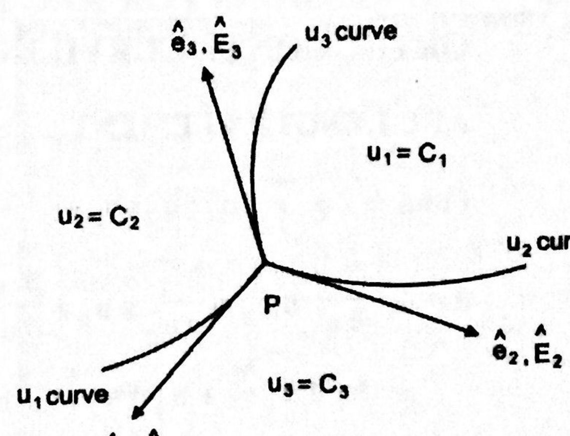

Figure (6.3)

i.e. $\hat{e}_1 . \hat{e}_2 = \hat{e}_2 . \hat{e}_3 = \hat{e}_3 . \hat{e}_1 = 0$.

and $\hat{e}_1 . \hat{e}_1 = \hat{e}_2 . \hat{e}_2 = \hat{e}_3 . \hat{e}_3 = 1$

Furthermore, if this system is right-handed, then

$$\hat{e}_1 \times \hat{e}_2 = -\hat{e}_2 \times \hat{e}_1 = \hat{e}_3$$

$$\hat{e}_2 \times \hat{e}_3 = -\hat{e}_3 \times \hat{e}_2 = \hat{e}_1$$

$$\hat{e}_3 \times \hat{e}_1 = -\hat{e}_1 \times \hat{e}_3 = \hat{e}_2$$

The vector $\bar{A}$ in an orthogonal curvilinear coordinate system can be expressed in terms of unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ as $\bar{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$

where A_1, A_2, A_3 are the components of $\bar{A}$.

THEOREM (6.1): Prove that if u_1, u_2, u_3 are orthogonal curvilinear coordinates, then

(i) $|\nabla u_j| = h_j^{-1}$ (ii) $\hat{e}_j = \hat{E}_j,\ j = 1, 2, 3$

SOLUTION: (i) Since ∇u_1 is a vector normal to the surface $u_1 = C_1$, therefore it is parallel to $\hat{e}_1$.

Thus $\hat{e}_1 = h_1 \nabla u_1$ where h_1 is a scalar factor of proportionality between $\hat{e}_1$ and ∇u_1.

or $\nabla u_1 = \dfrac{\hat{e}_1}{h_1}$ and so $|\nabla u_1| = \dfrac{|\hat{e}_1|}{h_1} = \dfrac{1}{h_1}$ (since $|\hat{e}_1| = 1$)

or $|\nabla u_1| = h_1^{-1}$.

Similarly $|\nabla u_2| = h_2^{-1}$ and $|\nabla u_3| = h_3^{-1}$.

Combining the three equations, we can write $|\nabla u_j| = h_j^{-1},\ j = 1, 2, 3$.

(ii) By definition,

$$\hat{E}_j = \frac{\nabla u_j}{|\nabla u_j|} = h_j \nabla u_j = \hat{e}_j,\quad j = 1, 2, 3$$ and the result is proved.

6.6 EXPRESSIONS FOR ARC LENGTH, AREA, AND VOLUME ELEMENTS IN ORTHOGONAL CURVILINEAR COORDINATES

ARC LENGTH ELEMENT

From $\vec{r} = \vec{r}(u_1, u_2, u_3)$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

$$= h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

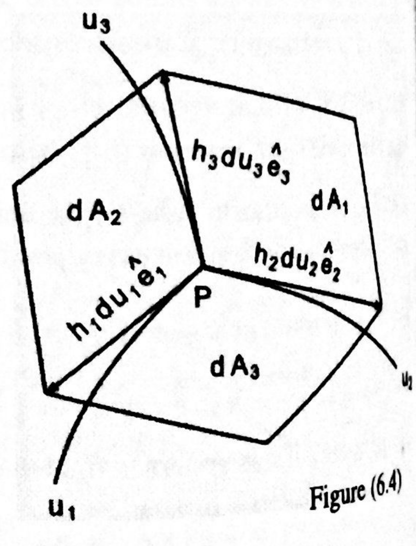

Figure (6.4)

Then the differential of arc length ds is determined from

$$(ds)^2 = d\vec{r} \cdot d\vec{r} \quad (1)$$

For an orthogonal curvilinear coordinate system, we have

$$\hat{e}_1 \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3 = 1$$

$$\hat{e}_1 \cdot \hat{e}_2 = \hat{e}_2 \cdot \hat{e}_3 = \hat{e}_3 \cdot \hat{e}_1 = 0$$

Thus equation (1) gives

$$(ds)^2 = d\vec{r} \cdot d\vec{r} = h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2 \quad (2)$$

and the element of arc length ds is obtained by taking the square root of equation (2).

Along u_1– curve, u_2 and u_3 are constants so that $d\vec{r} = h_1 du_1 \hat{e}_1$. Then the differential of arc length ds_1 along u_1 at P is $h_1 du_1$. Similarly, the differential of arc lengths along u_2 and u_3– curves at P are $ds_2 = h_2 du_2$ and $ds_3 = h_3 du_3$, respectively.

AREA ELEMENT

From the figure (6.4) , the area elements are given by

$$dA_1 = |(h_2\,du_2\,\hat{e}_2) \times (h_3\,du_3\,\hat{e}_3)|$$

$$= h_2 h_3\,|\hat{e}_2 \times \hat{e}_3|\,du_2\,du_3 = h_2 h_3\,du_2\,du_3 \quad (\text{since } |\hat{e}_2 \times \hat{e}_3| = |\hat{e}_1| = 1)$$

...ilarly, $dA_2 = |(h_1\,du_1\,\hat{e}_1) \times (h_3\,du_3\,\hat{e}_3)| = h_1 h_3\,du_1\,du_3$

$$dA_3 = |(h_1\,du_1\,\hat{e}_1) \times (h_2\,du_2\,\hat{e}_2)| = h_1 h_2\,du_1\,du_2$$

VOLUME ELEMENT

We know that the absolute value of the scalar triple product gives the volume of the ...llelopiped. Thus

$$dV = |(h_1\,du_1\,\hat{e}_1) \cdot (h_2\,du_2\,\hat{e}_2) \times (h_3\,du_3\,\hat{e}_3)|$$

$$= h_1 h_2 h_3\,du_1\,du_2\,du_3\,|\hat{e}_1 \cdot \hat{e}_2 \times \hat{e}_3| = h_1 h_2 h_3\,du_1\,du_2\,du_3$$

$$(\text{since } |\hat{e}_1 \cdot \hat{e}_2 \times \hat{e}_3| = 1)$$

EXPRESSI... ...HOGONAL CURVILINEAR

6.8 GRADIENT, DIVERGENCE, CURL, AND LAPLACIAN IN ORTHOGONAL CURVILINEAR COORDINATES

EXPRESSION FOR GRADIENT

Let $\nabla\psi = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$ (1)

where f_1, f_2, and f_3 are to be determined.

Since $$d\vec{r} = \frac{\partial\vec{r}}{\partial u_1}du_1 + \frac{\partial\vec{r}}{\partial u_2}du_2 + \frac{\partial\vec{r}}{\partial u_3}du_3$$

$$= h_1 du_1\hat{e}_1 + h_2 du_2\hat{e}_2 + h_3 du_3\hat{e}_3$$

we have

$$d\psi = \nabla\psi\cdot d\vec{r} = h_1 f_1 du_1 + h_2 f_2 du_2 + h_3 f_3 du_3 \tag{2}$$

But $$d\psi = \frac{\partial\psi}{\partial u_1}du_1 + \frac{\partial\psi}{\partial u_2}du_2 + \frac{\partial\psi}{\partial u_3}du_3 \tag{3}$$

From equations (2) and (3) equating the coefficients of du_1, du_2, and du_3, we get

$$f_1 = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1},\quad f_2 = \frac{1}{h_2}\frac{\partial\psi}{\partial u_2},\quad f_3 = \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}$$

Then from equation (1), we have

$$\nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\hat{e}_3 \tag{4}$$

This indicates the operator equivalence

$$\nabla = \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3} \tag{5}$$

which is the expression for the del operator in orthogonal curvilinear coordinates.

ALTERNATIVE FORM

Note that by taking $\psi = u_1$ in equation (4), we get

$$\nabla u_1 = \frac{1}{h_1}\hat{e}_1 \quad \text{i.e.} \quad \hat{e}_1 = h_1\nabla u_1$$

Similarly, by taking $\psi = u_2$ and $\psi = u_3$ we get

$$\hat{e}_2 = h_2\nabla u_2 \quad \text{and} \quad \hat{e}_3 = h_3\nabla u_3$$

Thus equation (4) takes an alternative form

$$\nabla\psi = \frac{\partial\psi}{\partial u_1}\nabla u_1 + \frac{\partial\psi}{\partial u_2}\nabla u_2 + \frac{\partial\psi}{\partial u_3}\nabla u_3 \tag{6}$$

EXPRESSION FOR DIVERGENCE

We have

$$\nabla\cdot\vec{A} = \nabla\cdot(A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3)$$

$$= \nabla\cdot(A_1\hat{e}_1) + \nabla\cdot(A_2\hat{e}_2) + \nabla\cdot(A_3\hat{e}_3) \tag{7}$$

$\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$ form a right-handed system, therefore

$$\hat{e}_1 = \hat{e}_2 \times \hat{e}_3 = h_2 h_3 \nabla u_2 \times \nabla u_3$$

$$\hat{e}_2 = \hat{e}_3 \times \hat{e}_1 = h_3 h_1 \nabla u_3 \times \nabla u_1$$

$$\hat{e}_3 = \hat{e}_1 \times \hat{e}_2 = h_1 h_2 \nabla u_1 \times \nabla u_2$$

$$\nabla.(A_1 \hat{e}_1) = \nabla.(A_1 h_2 h_3 \nabla u_2 \times \nabla u_3)$$

$$= \nabla(A_1 h_2 h_3).\nabla u_2 \times \nabla u_3 + A_1 h_2 h_3 \nabla.(\nabla u_2 \times \nabla u_3) \quad (8)$$

Using the formulas, $\nabla.(\vec{A} \times \vec{B}) = \vec{B}.(\nabla \times \vec{A}) - \vec{A}.(\nabla \times \vec{B})$ and $\nabla \times \nabla \phi = \vec{0}$, we get

$$\nabla.(\nabla u_2 \times \nabla u_3) = \nabla u_3.(\nabla \times \nabla u_2) - \nabla u_2.(\nabla \times \nabla u_3) = 0$$

So equation (8) becomes

$$\nabla.(A_1 \hat{e}_1) = \nabla(A_1 h_2 h_3).\frac{\hat{e}_2}{h_2} \times \frac{\hat{e}_3}{h_3} + 0$$

$$= \nabla(A_1 h_2 h_3).\frac{\hat{e}_1}{h_2 h_3}$$

$$= \left[\frac{1}{h_1}\frac{\partial}{\partial u_1}(A_1 h_2 h_3)\hat{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial u_2}(A_1 h_2 h_3)\hat{e}_2 + \frac{1}{h_3}\frac{\partial}{\partial u_3}(A_1 h_2 h_3)\hat{e}_3\right].\frac{\hat{e}_1}{h_2 h_3}$$

$$= \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}(A_1 h_2 h_3)$$

Similarly $\nabla.(A_2 \hat{e}_2) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_2}(A_2 h_3 h_1)$

$$\nabla.(A_3 \hat{e}_3) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_3}(A_3 h_1 h_2)$$

Thus from equation (7), we get

$$\nabla.\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right] \quad (9)$$

EXPRESSION FOR CURL

We have $\nabla \times \vec{A} = \nabla \times (A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3)$

$$= \nabla \times (A_1 \hat{e}_1) + \nabla \times (A_2 \hat{e}_2) + \nabla \times (A_3 \hat{e}_3) \quad (10)$$

$$\nabla \times (A_1 \hat{e}_1) = \nabla \times (A_1 h_1 \nabla u_1) \quad (\text{since } \hat{e}_1 = h_1 \nabla u_1)$$

$$= \nabla(A_1 h_1) \times \nabla u_1 + A_1 h_1 \nabla \times \nabla u_1$$

$$= \nabla(A_1 h_1) \times \frac{\hat{e}_1}{h_1} + \vec{0} \quad (\text{since } \nabla \times \nabla u_1 = \vec{0})$$

$$= \left[\frac{1}{h_1}\frac{\partial}{\partial u_1}(A_1 h_1)\hat{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial u_2}(A_1 h_1)\hat{e}_2 + \frac{1}{h_3}\frac{\partial}{\partial u_3}(A_1 h_1)\hat{e}_3\right] \times \frac{\hat{e}_1}{h_1}$$

$$= \frac{1}{h_3 h_1}\frac{\partial}{\partial u_3}(A_1 h_1)\hat{e}_2 - \frac{1}{h_1 h_2}\frac{\partial}{\partial u_2}(A_1 h_1)\hat{e}_3$$

Similarly $\nabla \times (A_2 \hat{e}_2) = \frac{1}{h_1 h_2} \frac{\partial}{\partial u_1}(A_2 h_2)\hat{e}_3 - \frac{1}{h_2 h_3} \frac{\partial}{\partial u_3}(A_2 h_2)\hat{e}_1$

and $\quad \nabla \times (A_3 \hat{e}_3) = \frac{1}{h_2 h_3} \frac{\partial}{\partial u_2}(A_3 h_3)\hat{e}_1 - \frac{1}{h_3 h_1} \frac{\partial}{\partial u_1}(A_3 h_3)\hat{e}_2$

Thus equation (10) becomes

$$\nabla \times \vec{A} = \frac{\hat{e}_1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(A_3 h_3) - \frac{\partial}{\partial u_3}(A_2 h_2)\right] + \frac{\hat{e}_2}{h_3 h_1}\left[\frac{\partial}{\partial u_3}(A_1 h_1) - \frac{\partial}{\partial u_1}(A_3 h_3)\right] + \frac{\hat{e}_3}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(A_2 h_2) - \frac{\partial}{\partial u_2}(A_1 h_1)\right]$$

This can be written as

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \tag{11}$$

EXPRESSION FOR LAPLACIAN

We know that

$$\nabla \psi = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial \psi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial \psi}{\partial u_3}\hat{e}_3$$

$$\nabla . \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right]$$

If $\quad \vec{A} = \nabla \psi$, then $A_1 = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}$, $\quad A_2 = \frac{1}{h_2}\frac{\partial \psi}{\partial u_2}$, $\quad A_3 = \frac{1}{h_3}\frac{\partial \psi}{\partial u_3}$ and thus

$$\nabla . \vec{A} = \nabla . \nabla \psi = \nabla^2 \psi$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial u_3}\right)\right] \tag{12}$$

EXAMPLE (1): **Consider the curvilinear coordinate system defined for $z \geq 0$ by**

$x = u_1 - u_2, \quad y = u_1 + u_2, \quad z = u_3^2$

(i) **Find the unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ and show that the system is orthogonal and right–handed. Also find the scale factors h_1, h_2, h_3.**

(ii) **Find the expressions for $(ds)^2$ and dV.**

(iii) **Find $\nabla \psi$ in this system for $\psi(u_1, u_2, u_3) = u_1 u_2 + u_3^2$.**

(iv) **Find $\nabla . \vec{A}$ and $\nabla \times \vec{A}$ for the vector field**

$\vec{A} = u_3 u_1 \hat{e}_1 + u_3 u_2 \hat{e}_2 + u_1 u_2 \hat{e}_3$.

(v) **Find $\nabla^2 \psi$ if $\psi = u_1^3 + u_2^3 + u_3^3$.**

RECTANGULAR CARTESIAN COORDINATES

Let $P(x, y, z)$ be any point whose projection on the ... is $Q(x, y)$. Then the rectangular Cartesian ...ates (x, y, z) of P are defined as $x = OR$, $y = RQ$, $z = QP$ as shown in figure (6.5). In rectangular Cartesian ...te system, the unit vectors are denoted by $\hat{i}, \hat{j}$, and $\hat{k}$

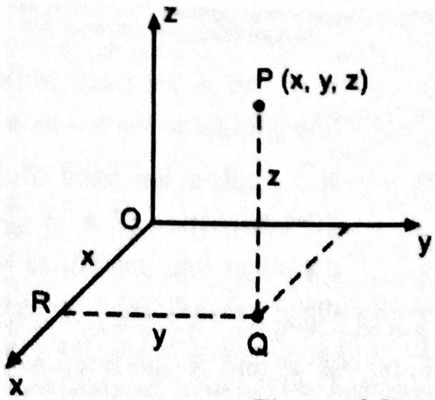

Figure (6.5)

...vector $\vec{A}$ can be represented in terms of these unit ...s:

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

...ition vector $\vec{r}$ in this system is given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

...scale factors are given by

$$h_1 = \left|\frac{\partial \vec{r}}{\partial x}\right| = |\hat{i}| = 1, \quad h_2 = \left|\frac{\partial \vec{r}}{\partial y}\right| = |\hat{j}| = 1, \quad h_3 = \left|\frac{\partial \vec{r}}{\partial z}\right| = |\hat{k}| = 1$$

...e rectangular Cartesian coordinate system is a particular case of an orthogonal curvilinear ...te system where $u_1 = x$, $u_2 = y$, $u_3 = z$ and $h_1 = 1$, $h_2 = 1$, $h_3 = 1$.

COORDINATE SURFACES

In rectangular Cartesian coordinate system, the coordinate surfaces are :

If x is held constant while y and z vary, then the equation $x = C_1$ represents a plane parallel to the yz–plane as shown in figure [6.6 (a)].

If y is held constant while x and z vary, then the equation $y = C_2$ represents a plane parallel to the zx–plane as shown in figure [6.6 (b)].

If z is held constant while x and y vary, then the equation $z = C_3$ represents a plane parallel to the xy–plane as shown in figure [6.6 (c)].

...e coordinate surfaces are mutually orthogonal in the sense that any two of them intersect at right ... Furthermore, each point in this system is the intersection of the three coordinate surfaces ..., $y = C_2$, and $z = C_3$.

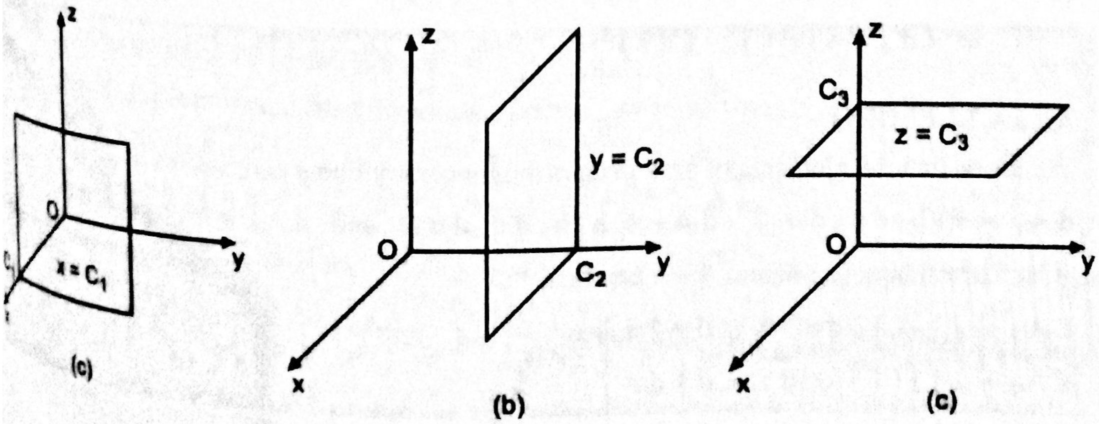

Figure (6.6)

EXPRESSION FOR CURL

We know that in orthogonal curvilinear coordinates, we have

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

In rectangular Cartesian coordinates, this becomes

$$\nabla \times \vec{A} = \frac{1}{(1)(1)(1)}\begin{vmatrix} (1)\hat{i} & (1)\hat{j} & (1)\hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (1)A_1 & (1)A_2 & (1)A_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k} \qquad (7)$$

EXPRESSION FOR LAPLACIAN

We know that in orthogonal curvilinear coordinates, we have

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial u_3}\right)\right]$$

In rectangular Cartesian coordinates, this becomes

$$\nabla^2 \psi = \frac{1}{(1)(1)(1)}\left[\frac{\partial}{\partial x}\left(\frac{(1)(1)}{(1)}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{(1)(1)}{(1)}\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{(1)(1)}{(1)}\frac{\partial \psi}{\partial z}\right)\right]$$

$$= \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \qquad (8)$$

6.13 CYLINDERICAL POLAR COORDINATES

Let $P(x, y, z)$ be any point whose projection on the xy–plane is $Q(x, y)$. Then the cylinderical coordinates of P are (r, θ, z) in which $r = OQ$, $\theta = \angle XOQ$ and $z = QP$. From the figure (6.9), the transformation equations expressing the rectangular Cartesian coordinates in terms of cylinderical polar coordinates are:

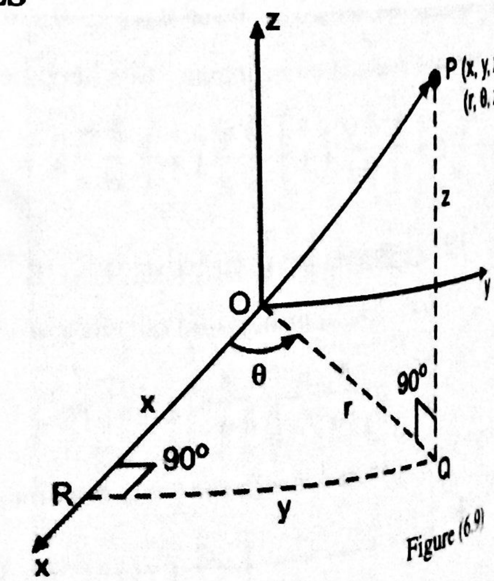

Figure (6.9)

$$x = r\cos\theta \qquad (1)$$

$$y = r\sin\theta \qquad (2)$$

$$z = z \qquad (3)$$

where $r \geq 0$, $0 \leq \theta < 2\pi$, and $-\infty < z < \infty$.

COORDINATE SURFACES

In cylinderical coordinate system, the coordinate surfaces are:

If r is held constant while θ and z vary, then the equation $r = C_1$ represents a right circular cylinder of radius C_1 and axis along z–axis (or z–axis if $C_1 = 0$) as shown in figure [6.10 (a)].

If θ is held constant while r and z vary, then the equation $\theta = C_2$ represents a half plane through the z–axis making an angle θ with the xz – plane as shown in figure [6.10 (b)].

If z is held constant, while r and θ vary, then the equation $z = C_3$ represents a plane perpendicular to z–axis as shown in figure [6.10 (c)].

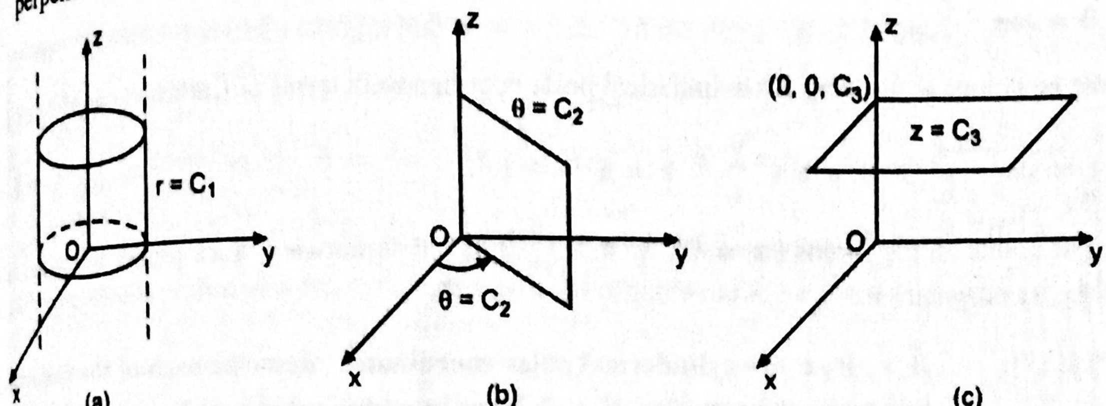

Figure (6.10)

COORDINATE CURVES

The coordinate curves for cylinderical polar coordinate system are :

If θ and z are fixed while r varies, then the intersection of $\theta = C_2$ and $z = C_3$ is a straight line called the r–coordinate curve or simply the r–curve .

) If r and z are fixed while θ varies, then the intersection of $r = C_1$ and $z = C_3$ is a circle (or point) called the θ–coordinate curve or simply the θ–curve .

i) If r and θ are fixed while z varies, then the intersection of $r = C_1$ and $\theta = C_2$ is a straight line called the z–coordinate curve or simply the z–curve .

Thus the r–curves are straight lines radiating from and normal to the z–axis, the θ–curves are circles centered on the z–axis and parallel to the xy–plane ; and the z–curves are the straight lines parallel to the z–axis as shown in figure (6.11).

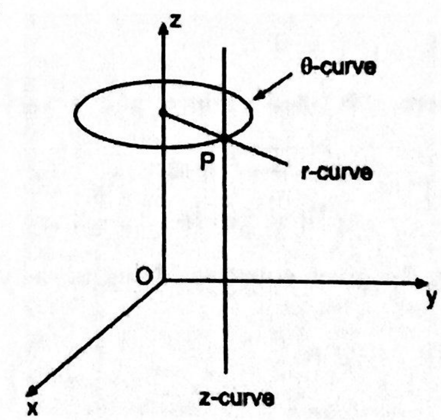

Figure (6.11)

6.14 CYLINDERICAL COORDINATES IN TERMS OF CARTESIAN COORDINATES

We know that the equations expressing the rectangular Cartesian coordinates in terms of cylinderical polar coordinates are:

$x = r\cos\theta$ (1)

$y = r\sin\theta$ (2)

$z = z$ (3)

Squaring equations (1) and (2) and adding , we get

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

or $\quad r = \sqrt{x^2 + y^2} \quad$ (since r is positive) (4)

Dividing equation (2) by equation (1) , we get

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

or $\quad \theta = \tan^{-1}\frac{y}{x}$ (5)

Hence , the equations expressing the cylinderical polar coordinates in terms of Cartesian coordinates are:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\frac{y}{x}, \quad z = z \quad (6)$$

NOTE: For points on the z–axis (x = 0 , y = 0) , θ is indeterminate . Such points are called singular points of the transformation .

EXAMPLE (2): **If r , θ , z are cylinderical polar coordinates , describe each of the following loci and write the equation of each locus in rectangular coordinates :**

(i) $r = 4$ **(ii)** $\theta = \frac{\pi}{2}$ **(iii)** $z = 3$

(iv) $\theta = \frac{\pi}{3}, z = 1$ **(v)** $r = 4, z = 0$ **(vi)** $r = 2, \theta = \frac{\pi}{6}$

SOLUTION: In cylinderical coordinates , $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\frac{y}{x}, \quad z = z$$

(i) $r = 4$

Here r is fixed while θ and z vary . We can write the given equation as

$$\sqrt{x^2 + y^2} = 4$$

or $\quad x^2 + y^2 = 16$

i.e. the given equation represents a cylinder with axis as the z–axis and radius 4 .

(ii) $\theta = \frac{\pi}{2}$

Here θ is fixed while r and z vary . We can write the given equation as

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{2}$$

or $\quad \frac{y}{x} = \tan\frac{\pi}{2} = \infty$

This implies x = 0 i.e. the given equation represents the yz–plane where $y \geq 0$.

(iii) $z = 3$

Here z is fixed while r and θ vary . The given equation represents a plane parallel to the x y – plane at a distance 3 units from the origin .

$\theta = \frac{\pi}{3}, \quad z = 1$

θ and z are fixed while only r varies . We can write $\theta = \frac{\pi}{3}$ as

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{3} \quad \text{or} \quad \frac{y}{x} = \tan\frac{\pi}{3} = \sqrt{3}$$

$$y = \sqrt{3}\,x , \quad z = 1$$

the given equations represent a straight line $y = \sqrt{3}\,x$, in the plane $z = 1$ where $x \geq 0$, $y \geq 0$.

$r = 4, \quad z = 0$

r and z are fixed while only θ varies . We can write $r = 4$ as

$$\sqrt{x^2+y^2} = 4 \quad \text{or} \quad x^2+y^2 = 16 , \quad z = 0$$

the given equations represent a circle with centre at the origin and radius 4 in the xy–plane .

$r = 2, \ \theta = \frac{\pi}{6}$

Here r and θ are fixed while only z varies . We can write $r = 2$ as

$$\sqrt{x^2+y^2} = 2 \quad \text{or} \quad x^2+y^2 = 4$$

and $\theta = \frac{\pi}{6}$ as $\tan^{-1}\frac{y}{x} = \frac{\pi}{6}$ or $\frac{y}{x} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ or $y = \frac{1}{\sqrt{3}}x$

the given equations represent a straight line parallel to the z–axis and passing through the point of intersection of the circle $x^2+y^2 = 4$ and the straight line $y = \frac{1}{\sqrt{3}}x$.

UNIT VECTORS IN CYLINDERICAL COORDINATE SYSTEM

The position vector of any point P in cylinderical polar coordinates is

$$\bar{R} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + z\hat{k}$$

The tangent vectors in the directions of r , θ , and z respectively , are given by

$$\frac{\partial \bar{R}}{\partial r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\frac{\partial \bar{R}}{\partial \theta} = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}$$

$$\frac{\partial \bar{R}}{\partial z} = \hat{k}$$

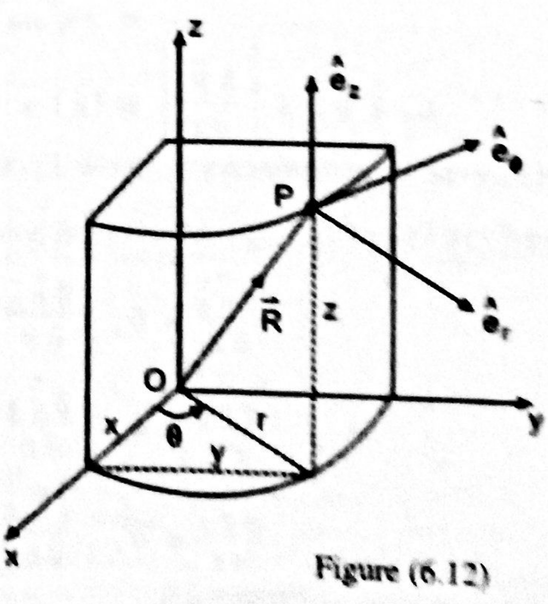

Figure (6.12)

Unit vectors in these directions of r , θ , and z are given by

$$\frac{\partial \hat{e}_r}{\partial r} = 0, \quad \frac{\partial \hat{e}_r}{\partial \theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{e}_\theta, \quad \frac{\partial \hat{e}_r}{\partial z} = 0$$

$$\frac{\partial \hat{e}_\theta}{\partial r} = 0, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} = -\hat{e}_r, \quad \frac{\partial \hat{e}_\theta}{\partial z} = 0$$

$$\frac{\partial \hat{e}_z}{\partial r} = 0, \quad \frac{\partial \hat{e}_z}{\partial \theta} = 0, \quad \frac{\partial \hat{e}_z}{\partial z} = 0$$

THEOREM (6.3): **Prove that in cylinderical polar coordinates**

$$\frac{d}{dt}\hat{e}_r = \dot{\theta}\,\hat{e}_\theta, \quad \frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\,\hat{e}_r, \quad \frac{d}{dz}\hat{e}_z = \vec{0}$$

where dots denote differentiation w.r.t. time t.

PROOF: We know that $\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$, $\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$, $\hat{e}_z = \hat{k}$

Then $$\frac{d}{dt}\hat{e}_r = -\sin\theta\frac{d\theta}{dt}\hat{i} + \cos\theta\frac{d\theta}{dt}\hat{j} = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\frac{d\theta}{dt} = \dot{\theta}\,\hat{e}_\theta$$

$$\frac{d}{dt}\hat{e}_\theta = -\cos\theta\frac{d\theta}{dt}\hat{i} - \sin\theta\frac{d\theta}{dt}\hat{j} = -(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\frac{d\theta}{dt} = -\dot{\theta}\,\hat{e}_r$$

and $$\frac{d}{dz}\hat{e}_z = \frac{d}{dz}\hat{k} = \vec{0}$$

6.16 ORTHOGONALITY OF CYLINDERICAL COORDINATE SYSTEM

We know that the unit vectors in cylinderical polar coordinates are

$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$, $\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$, and $\hat{e}_z = \hat{k}$

Then $$\hat{e}_r.\hat{e}_\theta = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j}).(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$= -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$$

$$\hat{e}_\theta.\hat{e}_z = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}).(\hat{k}) = 0$$

$$\hat{e}_r.\hat{e}_z = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j}).(\hat{k}) = 0$$

and so $\hat{e}_r, \hat{e}_\theta$, and $\hat{e}_z$ are mutually perpendicular and the coordinate system is orthogonal.

6.17 RELATIONSHIPS AMONG UNIT VECTORS IN CYLINDERICAL SYSTEM

THEOREM (6.4): **Prove that for cylinderical coordinate system.**

$$\hat{e}_r.\hat{e}_r = \hat{e}_\theta.\hat{e}_\theta = \hat{e}_z.\hat{e}_z = 1$$

$$\hat{e}_r \times \hat{e}_r = \hat{e}_\theta \times \hat{e}_\theta = \hat{e}_z \times \hat{e}_z = \vec{0}$$

$$\hat{e}_r \times \hat{e}_\theta = \hat{e}_z, \quad \hat{e}_\theta \times \hat{e}_z = \hat{e}_r, \quad \hat{e}_z \times \hat{e}_r = \hat{e}_\theta$$

PROOF: We know that $\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$, $\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$, $\hat{e}_z = \hat{k}$

Then $$\hat{e}_r.\hat{e}_r = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j}).(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) = \cos^2\theta + \sin^2\theta = 1$$

$$\hat{e}_\theta . \hat{e}_\theta = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}).(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) = \sin^2\theta + \cos^2\theta = 1$$

$$\hat{e}_z . \hat{e}_z = \hat{k}.\hat{k} = 1$$

$$\hat{e}_r \times \hat{e}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = \vec{0}$$

$$\hat{e}_\theta \times \hat{e}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = \vec{0}$$

$$\hat{e}_z \times \hat{e}_z = \hat{k} \times \hat{k} = \vec{0}$$

$$\hat{e}_r \times \hat{e}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = 0\,\hat{i} + 0\,\hat{j} + (\cos^2\theta + \sin^2\theta)\,\hat{k} = \hat{k} = \hat{e}_z$$

$$\hat{e}_\theta \times \hat{e}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + 0\,\hat{k} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} = \hat{e}_r$$

$$\hat{e}_z \times \hat{e}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} + 0\,\hat{k} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{e}_\theta$$

6.18 CARTESIAN UNIT VECTORS IN TERMS OF CYLINDERICAL UNIT VECTORS

We know that

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad (1)$$

$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \qquad (2)$$

$$\hat{e}_z = \hat{k} \qquad (3)$$

Multiplying equation (1) by $\cos\theta$ and equation (2) by $\sin\theta$ and then subtracting , we get

$$\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta = (\cos^2\theta + \sin^2\theta)\,\hat{i}$$

or $$\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \qquad (4)$$

Multiplying equation (1) by $\sin\theta$ and equation (2) by $\cos\theta$ and then adding , we get

$$\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta = (\sin^2\theta + \cos^2\theta)\,\hat{j}$$

or $$\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta \qquad (5)$$

Also $$\hat{k} = \hat{e}_z \qquad (6)$$

In matrix notation , equations (4) , (5) , and (6) can be written as

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_z \end{bmatrix} \qquad (7)$$

$$= \frac{1}{r}\left[\left(\frac{\partial A_z}{\partial \theta} - r\frac{\partial A_\theta}{\partial z}\right)\hat{e}_r + \left(r\frac{\partial A_r}{\partial z} - r\frac{\partial A_z}{\partial r}\right)\hat{e}_\theta + \left(r\frac{\partial A_\theta}{\partial r} + A_\theta - \frac{\partial A_r}{\partial \theta}\right)\hat{e}_z\right]$$

$$= \frac{1}{r}\left[\left(\frac{\partial A_z}{\partial \theta} - \frac{\partial}{\partial z}(r A_\theta)\right)\hat{e}_r + \left(r\frac{\partial A_r}{\partial z} - r\frac{\partial A_z}{\partial r}\right)\hat{e}_\theta + \left(\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{e}_z\right]$$

$$= \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & r A_\theta & A_z \end{vmatrix} \qquad (15)$$

EXPRESSION FOR LAPLACIAN

We know that

$$\nabla \psi = \frac{\partial \psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{e}_\theta + \frac{\partial \psi}{\partial z}\hat{e}_z \qquad (16)$$

$$\nabla \cdot \vec{A} = \frac{\partial A_r}{\partial r} + \frac{1}{r}A_r + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \qquad (17)$$

From equations (16) and (17), we have

$$\nabla^2 \psi = \nabla \cdot \nabla \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}$$

where $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$ (18)

is the Laplacian operator.

6.26 SPHERICAL POLAR COORDINATES

Let P (x , y , z) be any point whose projection on the xy – plane is Q (x , y). Then the spherical polar coordinates of P are (r , θ , φ) in which $r = OP$, $\theta = \angle ZOP$ and $\phi = \angle XOQ$.

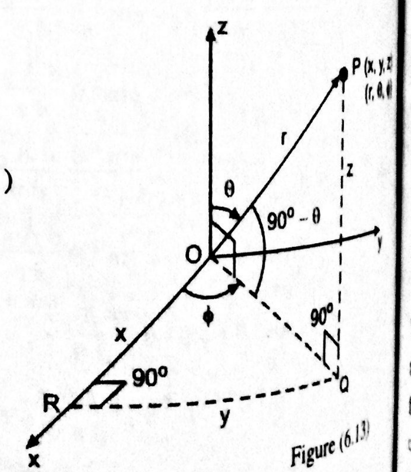

Figure (6.13)

From the figure (6.13), we have

$OQ = r\cos(90 - \theta) = r\sin\theta$ (since $\angle QOP = 90 - \theta$)

Therefore $x = OQ\cos\phi = r\sin\theta\cos\phi$

$y = OQ\sin\phi = r\sin\theta\sin\phi$

$z = OP\sin(90 - \theta) = r\cos\theta$

Hence, the transformation equations expressing the rectangular Cartesian coordinates in terms of spherical polar coordinates are :

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$ (2)

$z = r \cos\theta$ (3)

$r \geq 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi$

COORDINATE SURFACES

[illegible] spherical coordinate system, the coordinate surfaces are:

[illegible] r is held constant while θ and ϕ vary, then the equation $r = C_1$ represents a sphere with centre at the origin (or origin if $C_1 = 0$) as shown in figure [6.14 (a)].

[illegible] θ is held constant while r and ϕ vary, then the equation $\theta = C_2$ represents a cone with vertex at O, axis OZ and generating angle θ as shown in figure [6.14 (b)].

[illegible] ϕ is held constant while r and θ vary, then the equation $\phi = C_3$ represents a half plane through the z - axis as shown in figure [6.14 (c)].

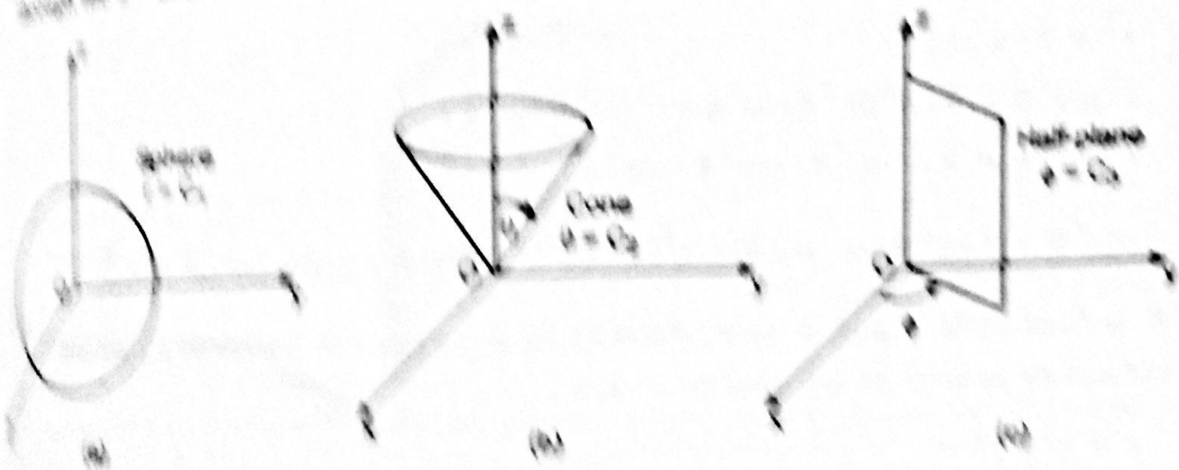

Figure (6.14)

COORDINATE CURVES

The coordinate curves for spherical polar coordinate system are :

[illegible] θ and ϕ are fixed while r varies, then the intersection of $\theta = C_2$ and $\phi = C_3$ is a straight line called the r - coordinate curve or simply the r - curve.

[illegible] r and ϕ are fixed while θ varies, then the intersection of $r = C_1$ and $\phi = C_3$ is a semi - circle ($C_1 \neq 0$) called the θ - coordinate curve or simply the θ - curve.

[illegible] r and θ are fixed while ϕ varies, then the intersection of $r = C_1$ and $\theta = C_2$ is a circle [illegible] called the ϕ - coordinate curve or simply the ϕ - curve.

[illegible] r - curves are the straight lines radiating from [illegible] the θ - curves are the semi circles originating [illegible] axis with centres on the origin ; the ϕ - [illegible] are the circles with centres on the z - axis and [illegible] xy - plane, as shown in figure (6.15).

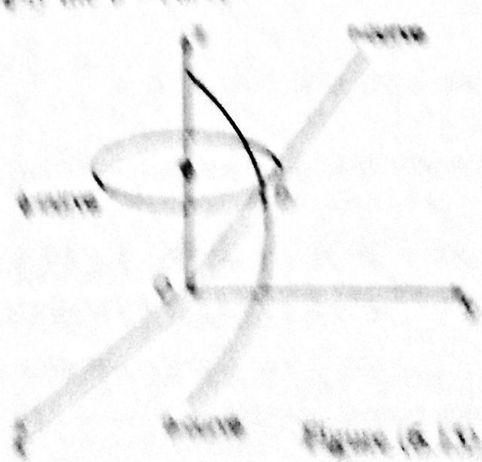

Figure (6.15)

EXAMPLE (3): **Express each of the following loci in spherical polar coordinates :**

(i) **the sphere** $x^2+y^2+z^2 = 9$ (ii) **the cone** $z^2 = 3(x^2+y^2)$

(iii) **the paraboloid** $z = x^2+y^2$ (iv) **the plane** $z = 0$

(v) **the plane** $y = x$.

SOLUTION: We know that in spherical coordinates

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta$$

(i) $x^2+y^2+z^2 = 9$

or $r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta = 9$

$r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) + r^2\cos^2\theta = 9$

$r^2\sin^2\theta + r^2\cos^2\theta = 9$ or $r^2(\sin^2\theta+\cos^2\theta) = 9$

or $r^2 = 9$ or $r = 3$ (since r is always positive)

Since r is fixed while θ and ϕ vary, therefore the given equation represents a sphere with centre at the origin and radius 3.

(ii) $z^2 = 3(x^2+y^2)$

$r^2\cos^2\theta = 3(r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi)$

$r^2\cos^2\theta = 3\,r^2\sin^2\theta(\cos^2\phi+\sin^2\phi)$

$\cos^2\theta = 3\sin^2\theta$ or $\cos\theta = \sqrt{3}\sin\theta$. This is possible only if $\theta = \frac{\pi}{6}$.

Since θ is fixed while r and ϕ vary, therefore the given equation represents a cone with vertex at the origin and making an angle of $\pi/6$ with the z – axis.

(iii) $z = x^2+y^2$

$r\cos\theta = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi = r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) = r^2\sin^2\theta$

or $\cos\theta = r\sin^2\theta$

(iv) $z = 0$ or $r\cos\theta = 0$

Since $r \neq 0$, therefore $\cos\theta = 0$ or $\theta = \frac{\pi}{2}$.

i.e. is the given equation represents the xy – plane.

(v) $y = x$ or $r\sin\theta\sin\phi = r\sin\theta\cos\phi$ or $\sin\phi = \cos\phi$

This is possible only if $\phi = \frac{\pi}{4}, \frac{5\pi}{4}$. Since ϕ is fixed while r and θ vary, therefore the plane $y = x$ is made up of two half planes through the z – axis $\phi = \frac{\pi}{4}$ and $\phi = \frac{5\pi}{4}$.

6.27 EQUATIONS EXPRESSING SPHERICAL COORDINATES IN TERMS OF CARTESIAN COORDINATES

We know that the equation expressing the rectangular Cartesian coordination in terms of spherical polar coordinates are :

$$x = r\sin\theta\cos\phi \quad (1)$$

$$y = r\sin\theta\sin\phi \quad (2)$$

$$z = r\cos\theta \quad (3)$$

ing and adding equations (1), (2), and (3), we get

$$x^2+y^2+z^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta$$

$$= r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) + r^2\cos^2\theta$$

$$= r^2\sin^2\theta + r^2\cos^2\theta = r^2(\sin^2\theta+\cos^2\theta) = r^2$$

$$r = \sqrt{x^2+y^2+z^2} \quad (4)$$

ing and adding equations (1) and (2), we get

$$x^2+y^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi$$

$$= r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) = r^2\sin^2\theta$$

$$r\sin\theta = \sqrt{x^2+y^2} \quad (5)$$

ding equation (5) by equation (3), we get

$$\frac{r\sin\theta}{r\cos\theta} = \frac{\sqrt{x^2+y^2}}{z} \quad \text{or} \quad \tan\theta = \frac{\sqrt{x^2+y^2}}{z}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} \quad (6)$$

ly, dividing equation (2) by equation (1), we get

$$\frac{r\sin\theta\sin\phi}{r\sin\theta\cos\phi} = \frac{y}{x} \quad \text{or} \quad \tan\phi = \frac{y}{x}$$

$$\phi = \tan^{-1}\frac{y}{x} \quad (7)$$

e, the equations expressing the spherical polar coordinates in terms of Cartesian coordinates are:

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \quad \phi = \tan^{-1}\frac{y}{x} \quad (8)$$

UNIT VECTORS IN SPHERICAL COORDINATES SYSTEM

The position vector of any point $P(x, y, z)$ in spherical coordinates is given by

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

gent vectors in the directions of r, θ, and ϕ ly, are given by

$$\frac{\partial\bar{r}}{\partial r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\frac{\partial\bar{r}}{\partial\theta} = r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}$$

$$\frac{\partial\bar{r}}{\partial\phi} = -r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}$$

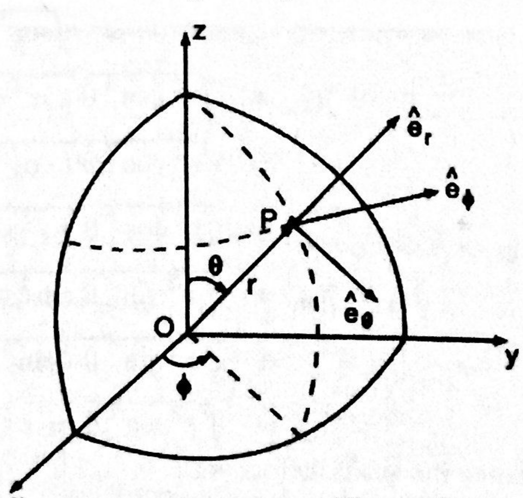

Figure (6.16)

The unit vectors in the directions of r , θ , and φ are

$$\hat{e}_r = \frac{\dfrac{\partial \vec{r}}{\partial r}}{\left|\dfrac{\partial \vec{r}}{\partial r}\right|} = \frac{\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}}$$

$$= \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k} \qquad (1)$$

$$\hat{e}_\theta = \frac{\dfrac{\partial \vec{r}}{\partial \theta}}{\left|\dfrac{\partial \vec{r}}{\partial \theta}\right|} = \frac{r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}}{\sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}}$$

$$= \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k} \qquad (2)$$

$$\hat{e}_\phi = \frac{\dfrac{\partial \vec{r}}{\partial \phi}}{\left|\dfrac{\partial \vec{r}}{\partial \phi}\right|} = \frac{-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}}{\sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}}$$

$$= -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} \qquad (3)$$

In matrix notation , equations (1) , (2) , and (3) can be written as

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \qquad (4)$$

SCALE FACTORS

The scale factors for the spherical coordinate system are given by

$$h_1 = h_r = \left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta + \cos^2\theta} = 1 \qquad (5)$$

$$h_2 = h_\theta = \sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}$$

$$= \sqrt{r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta}$$

$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r \qquad (6)$$

$$h_3 = h_\phi = \sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}$$

$$= \sqrt{r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)}$$

$$= \sqrt{r^2\sin^2\theta} = r\sin\theta \qquad (7)$$

Hence the scale factors are :

$$h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_\phi = r\sin\theta$$