5.9 SIMPLY AND MULTIPLY CONNECTED REGIONS

A simple closed curve is a closed curve which does not intersect itself anywhere. For example, the curve in figure [5.21 (a)] is a simple closed curve while the curve in figure [5.21 (b)] is not.

A region R is said to be **simply connected** if any simple closed curve lying in R can be continuously shrunk to a point. For example, the interior of a rectangle as shown in figure [5.21 (c)] is an example of a simply connected region.

A region R which is not simply connected is called **multiply connected**. For example, the region R exterior to C_2 and interior to C_1 is not simply connected because a circle drawn within R and enclosing C_2 cannot be shrunk to a point without crossing C_2 as shown in figure [5.21 (d)]. In other words, the regions which have holes are called multiply connected.

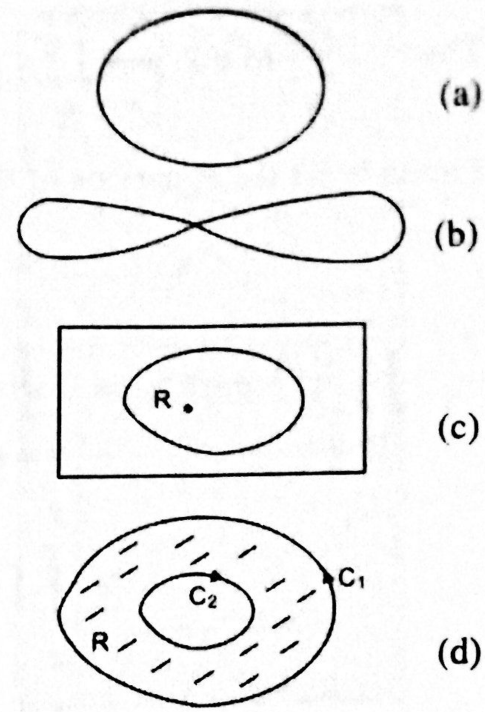

Figure (5.21)

5.10 GREEN'S THEOREM IN THE PLANE

We will consider vector functions of just x and y and derive a relationship between a line integral around a closed curve and a double integral over the part of the plane enclosed by the curve.

THOREM (5.6): **If R is a simply–connected region of the xy–plane bounded by a closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then**

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is described in the positive (counterclockwise) direction.

PROOF: We prove the theorem for a closed curve C which has the property that any straight line parallel to the coordinate axes cuts C in at most two points as shown in figure (5.22). Let the equations of the curves AEB and AFB be $y = f_1(x)$ and $y = f_2(x)$ respectively. If R is the region bounded by C, we have

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy\right]dx = \int_a^b \left| M(x,y) \right|_{y=f_1(x)}^{f_2(x)} dx$$

$$= \int_a^b \left[M(x,f_2) - M(x,f_1) \right] dx$$

$$= -\int_a^b M(x,f_1)\,dx - \int_b^a M(x,f_2)\,dx = -\oint_C M\,dx$$

Then $$\oint_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy \qquad (1)$$

Similarly let the equations of the curves EAF and EBF be $x = g_1(y)$ and $x = g_2(y)$ respectively. Then

$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_{y=e}^{f}\left[\int_{x=g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x}\,dx\right]dy = \int_e^f \left| N(x,y) \right|_{g_1(y)}^{g_2(y)} dy$$

$$= \int_e^f \left[N(g_2,y) - N(g_1,y) \right] dy$$

$$= \int_f^e N(g_1,y)\,dy + \int_e^f N(g_2,y)\,dy$$

$$= \oint_C N\,dy$$

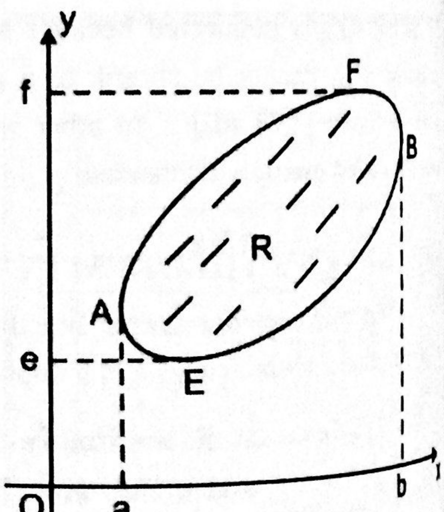

Figure (5.22)

Then $$\oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy \qquad (2)$$

Adding equations (1) and (2), we get

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

NOTE: (i) The proof can be extended to the curves C for which lines parallel to the coordinate axes may cut C in more than two points as shown in figure (5.23) (a).

The theorem also holds for a multiply–connected region R such as shown in figure (5.23) (b) .

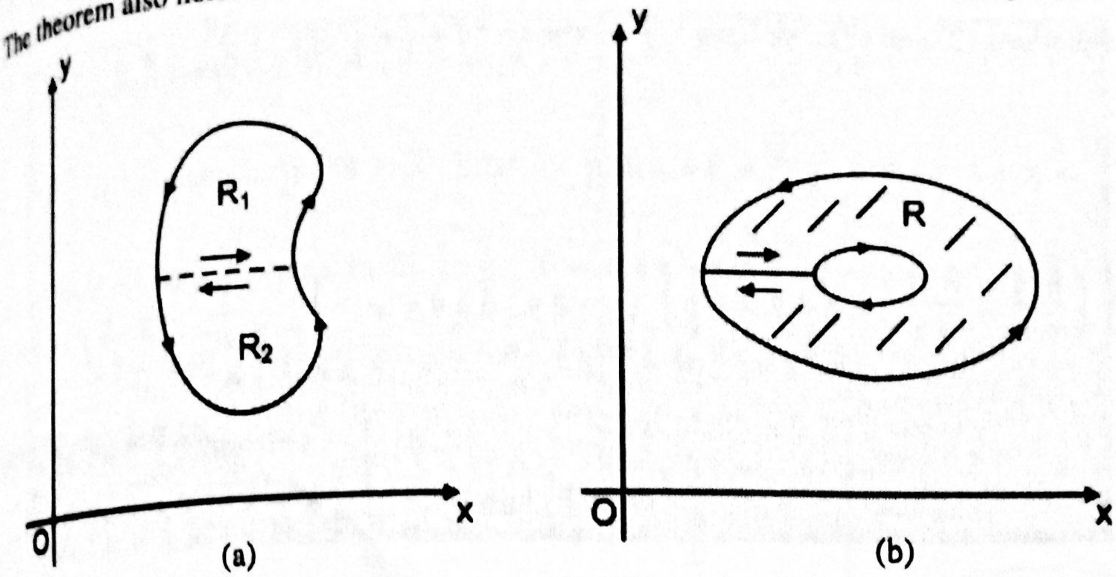

Figure (5.23)

AMPLE (14): **Verify Green's theorem in the plane for $M = xy + y^2$ and $N = x^2$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.**

UTION: The plane curves $y = x$ and $y = x^2$ intersect at $(0, 0)$ and $(1, 1)$. Let C_1 e curve $y = x^2$ and C_2 the curve $y = x$ and let the closed curve C be formed from C_1 and C_2 . positive direction in traversing C is shown in figure (5.24) .

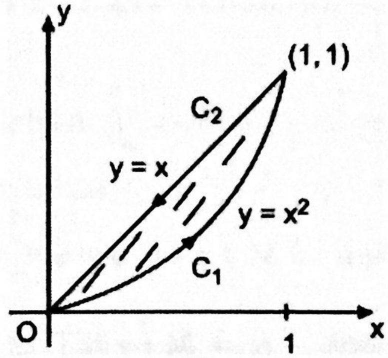

Figure (5.24)

we must show that

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\oint_C M\,dx + N\,dy = \oint_C (xy + y^2)\,dx + x^2\,dy \qquad (1)$$

g the curve C_1: $y = x^2$, $dy = 2x\,dx$, while x varies from 0 to 1 . The line integral (1) equals

$$\int_{C_1} M\,dx + N\,dy = \int_0^1 (x^3 + x^4)\,dx + 2x^3\,dx$$

$$= \int_0^1 (3x^3 + x^4)\,dx = \left| \frac{3}{4}x^4 + \frac{x^5}{5} \right|_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20} \qquad (2)$$

the curve C_2: $y = x$, $dy = dx$, while x varies from 1 to 0 . The line integral (1) equals

$$\int_{C_2} M\,dx + N\,dy = \int_1^0 2x^2\,dx + x^2\,dx = \int_1^0 3x^2\,dx = |x^3|_1^0 = -1 \qquad (3)$$

Then from equations (2) and (3), we have $\oint_C (xy+y^2)\,dx + x^2\,dy = \frac{19}{20} - 1 = -\frac{1}{20}$

Since $\frac{\partial M}{\partial y} = x + 2y$, and $\frac{\partial N}{\partial x} = 2x$, then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (x-2y)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x} (x-2y)\,dy\,dx$$

$$= \int_0^1 \left| xy - y^2 \right|_{x^2}^{x} dx = \int_0^1 (x^4 - x^3)\,dx$$

$$= \left| \frac{x^5}{5} - \frac{x^4}{4} \right|_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

so that the theorem is verified.

5.11 GREEN'S THEOREM IN THE PLANE IN VECTOR NOTATION

FIRST VECTOR FORM (OR TANGENTIAL FORM) OF GREEN'S THEOREM

We have $$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \qquad (1)$$

Now $M\,dx + N\,dy = (M\hat{i} + N\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j}) = \vec{A} \cdot d\vec{r}$

where $\vec{A} = M\hat{i} + N\hat{j}$ and $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$. Also, if $\vec{A} = M\hat{i} + N\hat{j}$ then

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = -\frac{\partial N}{\partial z}\hat{i} + \frac{\partial M}{\partial z}\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

so that $(\nabla \times \vec{A}) \cdot \hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

Then from equation (1) Green's theorem in the plane can be written

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_R (\nabla \times \vec{A}) \cdot \hat{k}\,dR \quad \text{where } dR = dx\,dy$$

A generalization of this to surface S in space having C as boundary leads quite naturally to Stokes' theorem. This form of Green's theorem is sometimes called Stokes' theorem in the plane. Thus Green's theorem in the plane is a special case of Stokes' theorem.

SECOND VECTOR FORM (OR NORMAL FORM) OF GREEN'S THEOREM

As above, $M\,dx + N\,dy = \vec{A}\cdot d\vec{r} = \vec{A}\cdot\hat{T}\,ds$

here $\frac{d\vec{r}}{ds} = \hat{T}$ = Unit tangent vector to C [see the figure (5.25)].

$\hat{n}$ is the outward drawn unit normal to C, then $\hat{T} = \hat{k}\times\hat{n}$ so that

$$M\,dx + N\,dy = \vec{A}\cdot\hat{T}\,ds = \vec{A}\cdot(\hat{k}\times\hat{n})\,ds = (\vec{A}\times\hat{k})\cdot\hat{n}\,ds$$

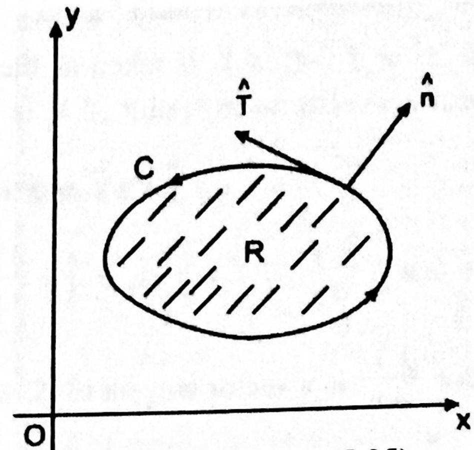

Figure (5.25)

nce $\vec{A} = M\hat{i} + N\hat{j}$, therefore

$$\vec{B} = \vec{A}\times\hat{k} = (M\hat{i} + N\hat{j})\times\hat{k} = N\hat{i} - M\hat{j}$$

d $\nabla\cdot\vec{B} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

en equation (1) becomes $\oint_C \vec{B}\cdot\hat{n}\,ds = \iint_R \nabla\cdot\vec{B}\,dR$

here $dR = dx\,dy$

12 STOKES' THEOREM

HEOREM (5.7): It states that if S is an open, two–sided surface bounded by a simple closed curve C, then if $\vec{A}$ has continuous first partial derivatives

$$\oint_C \vec{A}\cdot d\vec{r} = \iint_S (\nabla\times\vec{A})\cdot\hat{n}\,dS$$

where C is traversed in the positive direction.

n words the line integral of the tangential component of a vector function $\vec{A}$ taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of $\vec{A}$ taken over any surface S having C as its boundary.

PROOF:

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, then Stokes' theorem can be written as

$$\iint_S [\nabla\times(A_1\hat{i} + A_2\hat{j} + A_3\hat{k})]\cdot\hat{n}\,dS = \oint_C A_1\,dx + A_2\,dy + A_3\,dz$$

We prove this theorem for a surface S which has the property that its projections on the x y, y z, and z x planes are regions bounded by simple closed curves as shown in figure (5.26). Assume S to have representation $z = f(x, y)$ or $x = g(y, z)$ or $y = h(z, x)$, where f, g, h are continuous and differentiable functions.

Consider first $\iint\limits_S [\nabla \times (A_1 \hat{i})] . \hat{n}\, dS$

Since, $\nabla \times (A_1 \hat{i}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix} = \frac{\partial A_1}{\partial z}\hat{j} - \frac{\partial A_1}{\partial y}\hat{k}$

therefore, $[\nabla \times (A_1 \hat{i})] . \hat{n}\, dS = \left(\frac{\partial A_1}{\partial z}\, \hat{n} . \hat{j} - \frac{\partial A_1}{\partial y} \hat{n} . \hat{k} \right) dS$ (1)

If $z = f(x, y)$ is taken as the equation of S, then the position vector to any point of S is

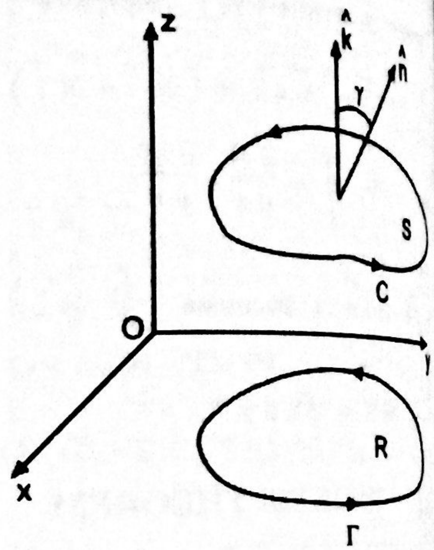

Figure (5.26)

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = x\hat{i} + y\hat{j} + f(x, y)\hat{k}$$

so that $\frac{\partial \vec{r}}{\partial y} = \hat{j} + \frac{\partial z}{\partial y}\hat{k} = \hat{j} + \frac{\partial f}{\partial y}\hat{k}$

But $\frac{\partial \vec{r}}{\partial y}$ is a vector tangent to S and thus perpendicular to $\hat{n}$, so that

$$\hat{n} . \frac{\partial \vec{r}}{\partial y} = \hat{n} . \hat{j} + \frac{\partial z}{\partial y}\hat{n} . \hat{k} = 0 \quad \text{or} \quad \hat{n} . \hat{j} = -\frac{\partial z}{\partial y}\, \hat{n} . \hat{k}$$

Substituting in equation (1) we get

$$[\nabla \times (A_1 \hat{i})] . \hat{n}\, dS = \left(-\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}\hat{n} . \hat{k} - \frac{\partial A_1}{\partial y}\hat{n} . \hat{k} \right) dS$$

$$= -\left(\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y} \right) \hat{n} . \hat{k}\, dS \qquad (2)$$

Now on S, $A_1(x, y, z) = A_1(x, y, f(x, y)) = F(x, y)$ (3)

hence $\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial F}{\partial y}$ and equation (2) becomes

$$[\nabla \times (A_1 \hat{i})] . \hat{n}\, dS = -\frac{\partial F}{\partial y}\, \hat{n} . \hat{k}\, dS = -\frac{\partial F}{\partial y}\, dx\, dy$$

Then $\iint\limits_S [\nabla \times (A_1 \hat{i})] . \hat{n}\, dS = \iint\limits_R -\frac{\partial F}{\partial y}\, dx\, dy$

where R is the projection of S on the xy–plane. By Green's theorem in the plane, the last integral equals $\oint\limits_\Gamma F\, dx$ where Γ is the boundary of R. From equation (3), since at each point (x, y) of Γ the value of F is the same as the value of A_1 at each point (x, y, z) of C, and since dx is the same for both curves , we must have

$$\oint_{\Gamma} F\,dx = \oint_{C} A_1\,dx$$

or

$$\iint_{S} [\nabla \times (A_1 \hat{i})] \cdot \hat{n}\,dS = \oint_{C} A_1\,dx \tag{4}$$

Similarly , by projections on the other coordinate planes , we have

$$\iint_{S} [\nabla \times (A_2 \hat{j})] \cdot \hat{n}\,dS = \oint_{C} A_2\,dy \tag{5}$$

$$\iint_{S} [\nabla \times (A_3 \hat{k})] \cdot \hat{n}\,dS = \oint_{C} A_3\,dz \tag{6}$$

Addition of equations (4) , (5) , and (6) completes the proof of the theorem .

RECTANGULAR FORM OF STOKES' THEOREM

Let $\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$ and $\hat{n} = n_1 \hat{i} + n_2 \hat{j} + n_3 \hat{k}$ be the outward drawn unit normal to the surface S . If α , β , and γ are the angles which the unit normal $\hat{n}$ makes with the positive directions of x , y , and z axes respectively , then

$$n_1 = \hat{n} \cdot \hat{i} = \cos\alpha$$

$$n_2 = \hat{n} \cdot \hat{j} = \cos\beta$$

and

$$n_3 = \hat{n} \cdot \hat{k} = \cos\gamma$$

The quantities $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the directions cosines of $\hat{n}$. Then

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

Thus

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

and

$$(\nabla \times \vec{A}) \cdot \hat{n} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos\gamma$$

Also

$$\vec{A} \cdot d\vec{r} = (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) = A_1\,dx + A_2\,dy + A_3\,dz$$

and Stokes' theorem becomes

$$\oint_C A_1\,dx + A_2\,dy + A_3\,dz$$

$$= \iint_S \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos\gamma\right] dS$$

EXAMPLE (15): **Verify Stokes' theorem for $\bar{A} = (2x - y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.**

SOLUTION: The surface S and its projection R on the xy–plane is shown in figure (5.27). The boundary C of S is a circle in the xy–plane of radius 1 and centre at the origin. Let $x = \cos\theta$, $y = \sin\theta$, $z = 0$, $0 \le \theta \le 2\pi$ be the parametric equations of C.

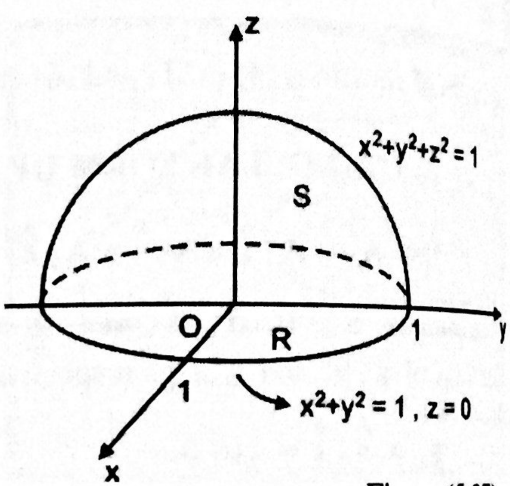

Figure (5.27)

Then $\displaystyle\oint_C \bar{A}\cdot d\bar{r} = \oint_C (2x - y)\,dx - yz^2\,dy - y^2 z\,dz$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta$$

$$= \int_0^{2\pi} (-2\sin\theta\cos\theta + \sin^2\theta)\,d\theta$$

$$= \int_0^{2\pi} \left[-\sin 2\theta + \left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta$$

$$= \left|\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right|_0^{2\pi} = \frac{1}{2} + \pi - \frac{1}{2} = \pi$$

Also, $\nabla \times \bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \hat{k}$

Then $\displaystyle\iint_S (\nabla \times \bar{A})\cdot\hat{n}\,dS = \iint_S \hat{k}\cdot\hat{n}\,dS = \iint_R dx\,dy$ (since $\hat{n}\cdot\hat{k}\,dS = dx\,dy$)

$$= 4\int_0^1 \int_0^{\sqrt{1 - x^2}} dy\,dx = 4\int_0^1 \sqrt{1 - x^2}\,dx$$

$x_1 = \sin\theta$, $dx = \cos\theta\, d\theta$, $0 \le \theta \le \pi/2$. Then

$$\iint_S (\nabla \times \vec{A}).\hat{n}\, dS = 4\int_0^{\pi/2} \cos^2\theta\, d\theta = 4\left(\frac{1}{2}\right)\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= 2\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{\pi/2} = 2\left(\frac{\pi}{2}\right) = \pi$$

Stokes' theorem is verified.

PROBLEM (25): Verify Stokes' theorem for $\vec{A} = (y - z + 2)\hat{i} + (yz + 4)\hat{j} - xz\hat{k}$, where S is the surface of the cube $x = 0$, $y = 0$, $z = 0$, $x = 2$, $y = 2$, $z = 2$ above the xy – plane.

SOLUTION: The surface S of the cube is shown in figure (5.52).

By Stokes' theorem $$\oint_C \vec{A}.d\vec{r} = \iint_S (\nabla \times \vec{A}).\hat{n}\, dS$$

Now $$\oint_C \vec{A}.d\vec{r} = \oint_{OABCO} (y - z + 2)\,dx + (yz + 4)\,dy - xz\,dz \qquad (1)$$

For OA, $y = 0$, $z = 0$, therefore $dy = dz = 0$, and integral (1) becomes

$$\int_{OA} \vec{A}.d\vec{r} = \int_{OA} 2\,dx = \int_0^2 2\,dx = 4$$

For AB, $x = 2$, $z = 0$, therefore $dx = dz = 0$ and integral (1) becomes

$$\int_{AB} \vec{A}.d\vec{r} = \int_{AB} 4\,dy = \int_0^2 4\,dy = 8$$

For BC, $y = 2, z = 0$, therefore $dy = dz = 0$ and integral (1)

becomes $$\int_{BC} \vec{A}.d\vec{r} = \int_{BC} 4\,dx = \int_2^0 4\,dx = -8$$

For CO, $x = 0$, $z = 0$ therefore $dx = dz = 0$ and integral (1)

becomes $$\int_{CO} \vec{A}.d\vec{r} = \int_{CO} 4\,dy = \int_2^0 4\,dy = -8$$

Thus from equation (1), we get

$$\oint_C \vec{A}.d\vec{r} = 4 + 8 - 8 - 8 = -4 \qquad (2)$$

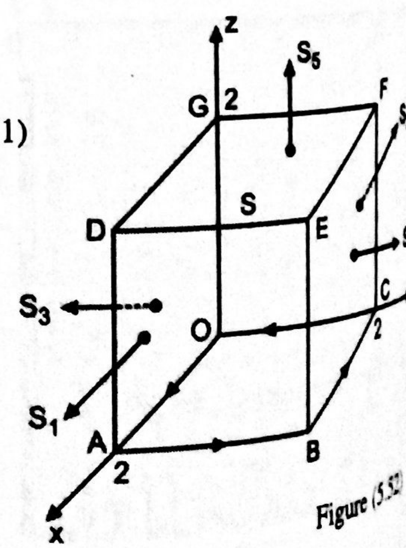

Figure (5.52)

Now $$\iint_S (\nabla \times \vec{A}).\hat{n}\,dS = \iint_S [-y\hat{i} + (-1 + z)\hat{j} - \hat{k}].\hat{n}\,dS \qquad (3)$$

where $S = S_1 + S_2 + S_3 + S_4 + S_5$.

S_1 (DABE), $x = 2$, $\hat{n} = \hat{i}$, and the integral (3) becomes

$$\iint_{S_1} (\nabla \times \bar{A}).\hat{n}\, dS_1 = \iint_{S_1} -y\, dS_1 = \int_0^2\int_0^2 -y\, dy\, dz = -2\int_0^2 dz = -4$$

S_2 (OGCF), $x = 0$, $\hat{n} = -\hat{i}$, and the integral (3) becomes

$$\iint_{S_2} (\nabla \times \bar{A}).\hat{n}\, dS_2 = \iint_{S_2} y\, dS_2 = \int_0^2\int_0^2 y\, dy\, dz = 2\int_0^2 dz = 4$$

S_3 (DAOG), $y = 0$, $\hat{n} = -\hat{j}$, and the integral (3) becomes

$$\iint_{S_3} (\nabla \times \bar{A}).\hat{n}\, dS_3 = \iint_{S_3} (1 - z)\, dS_3 = \int_0^2\int_0^2 (1 - z)\, dx\, dz = 2\int_0^2 (1 - z)\, dz = 0$$

S_4 (EBCF), $y = 2$, $\hat{n} = \hat{j}$, and the integral (3) becomes

$$\iint_{S_4} (\nabla \times \bar{A}).\hat{n}\, dS_4 = \iint_{S_4} (-1 + z)\, dS_4 = \int_0^2\int_0^2 (-1 + z)\, dx\, dz$$

$$= 2\int_0^2 (-1 + z)\, dz = 0$$

S_5 (DEFG), $z = 2$, $\hat{n} = \hat{k}$, and the integral (3) becomes

$$\iint_{S_5} (\nabla \times \bar{A}).\hat{n}\, dS_5 = \iint_{S_5} -1\, dS_5 = \int_0^2\int_0^2 -1\, dx\, dy = -2\int_0^2 dy = -4$$

From equation (3), we get

$$\iint_S (\nabla \times \bar{A}).\hat{n}\, dS = \iint_{S_1} (\ )\, dS_1 + \iint_{S_2} (\ )\, dS_2 + \iint_{S_3} (\ )\, dS_3 + \iint_{S_4} (\ )\, dS_4 + \iint_{S_5} (\ )\, dS_5$$

$$= -4 + 4 + 0 + 0 - 4 = -4 \qquad (4)$$

From equations (2) and (4) we see that Stokes' theorem is verified.

PROBLEM (26): Using Stokes' theorem, evaluate $\iint_S (\nabla \times \bar{A}).\hat{n}\, dS$,

if $\bar{A} = 4y\hat{i} + x\hat{j} + 2z\hat{k}$, and S is the surface of the hemisphere

$x^2 + y^2 + z^2$ ...

SOLUTION: The surface S of the hemisphere ... theorem, we have

$$\iint_S (\nabla \times \bar{A}).\hat{n}\, dS = \oint_C \bar{A}.d\bar{r} = \oint_C 4y\,dx + x\,dy + 2z\,dz \qquad (1)$$

where C is the circle $x^2 + y^2 = a^2$ in the xy-plane (i.e. $z = 0$) described in the counterclockwise direction. The parametric equations of this circle are $x = a\cos\theta$, $y = a\sin\theta$, $z = 0$ where $0 \le \theta \le 2\pi$. Then $dx = -a\sin\theta\, d\theta$, $dy = a\cos\theta\, d\theta$, $dz = 0$. Hence equation (1) becomes

$$\iint_S (\nabla \times \bar{A}).\hat{n}\, dS = \int_{\theta=0}^{2\pi} 4a\sin\theta(-a\sin\theta\, d\theta) + a\cos\theta(a\cos\theta\, d\theta) + 0$$

$$= \int_0^{2\pi} (-4a^2\sin^2\theta + a^2\cos^2\theta)\, d\theta$$

$$= a^2 \int_0^{2\pi} [-4\sin^2\theta + (1 - \sin^2\theta)]\, d\theta$$

$$= a^2 \int_0^{2\pi} (1 - 5\sin^2\theta)\, d\theta = a^2 \int_0^{2\pi} \left[1 - \frac{5}{2}(1 - \cos 2\theta)\right] d\theta$$

$$= a^2 \int_0^{2\pi} \left(-\frac{3}{2} + \frac{5}{2}\cos 2\theta\right) d\theta = a^2 \left| -\frac{3}{2}\theta + \frac{5}{4}\sin 2\theta \right|_0^{2\pi}$$

$$= a^2\left(-\frac{3}{2}\right)(2\pi) = -3a^2\pi .$$

PROBLEM (27): If $\bar{A} = 2yz\,\hat{i} - (x + 3y - 2)\,\hat{j} + (x^2 + z)\,\hat{k}$, then using Stokes' theorem evaluate $\iint_S (\nabla \times \bar{A}).\hat{n}\, dS$ over the surface of intersection of the cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$ which is included in the first octant.

SOLUTION: By Stokes' theorem, we have

$$\iint_S (\nabla \times \bar{A}).\hat{n}\, dS = \oint_C \bar{A}.d\bar{r}$$

$$= \oint_{ABCDA} 2yz\,dx - (x + 3y - 2)\,dy + (x^2 + z)\,dz \qquad (1)$$

…AB, $z=0$ therefore $dz = 0$ and integral (1) over this part of the curve becomes

$$\int_{AB} -(x+3y-2)\,dy = \int_0^a -\left(\sqrt{a^2-y^2}+3y-2\right)dy$$

$y = a\sin\theta$, $dy = a\cos\theta\, d\theta$, $0 \le \theta \le \pi/2$, then

$$\int_{AB} -(x+3y-2)\,dy = \int_0^{\pi/2} -(a\cos\theta + 3a\sin\theta - 2)\,a\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} -\left[\frac{a^2}{2}(1+\cos 2\theta) + 3a^2\sin\theta\cos\theta - 2a\cos\theta\right]d\theta$$

$$= -\left|\frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + \frac{3}{2}a^2\sin^2\theta - 2a\sin\theta\right|_0^{\pi/2}$$

$$= -\left[\frac{a^2}{2}\left(\frac{\pi}{2}\right) + \frac{3}{2}a^2 - 2a\right] = -\frac{a^2\pi}{4} - \frac{3}{2}a^2 + 2a$$

…BC, $x=0$, $y=a$ therefore $dx = dy = 0$ and integral (1) over this part of the curve becomes

$$\int_{BC} z\,dz = \int_0^a z\,dz = \frac{a^2}{2}$$

…CD, $x=0$, $z=a$ therefore $dx = dz = 0$

…integral (1) over this part of the curve becomes

$$\int_{CD} -(3y-2)\,dy = \int_a^0 -(3y-2)\,dy = \frac{3}{2}a^2 - 2a.$$

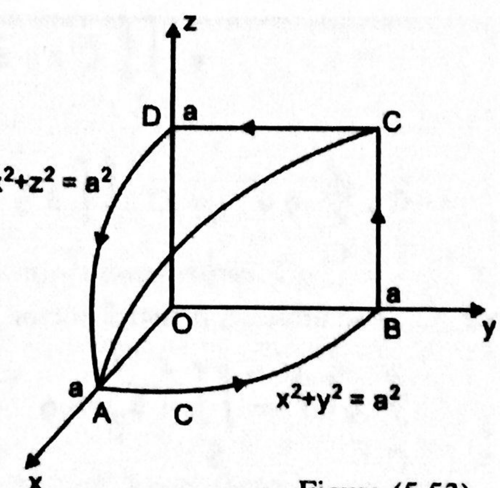

Figure (5.53)

…DA, $y=0$ therefore $dy = 0$ and the integral (1) over …part of the curve becomes

$$\int_{DA} (x^2+z)\,dz = \int_a^0 (a^2 - z^2 + z)\,dz$$

$$= \left|a^2 z - \frac{1}{3}z^3 + \frac{1}{2}z^2\right|_a^0 = -\frac{2}{3}a^3 - \frac{a^2}{2}$$

…from equation (1), we get

$$\iint_S (\nabla \times \bar{A})\cdot\hat{n}\,dS = -\frac{a^2\pi}{4} - \frac{3}{2}a^2 + 2a + \frac{a^2}{2} + \frac{3}{2}a^2 - 2a - \frac{2}{3}a^3 - \frac{a^2}{2}$$

$$= -\frac{a^2}{12}(3\pi + 8a)$$