and Stokes' theorem is verified .

5.13 GAUSS' DIVERGENCE THEOREM

Gauss' divergence theorem has wide applications in mathematics , physics and engineering and is used to derive equations governing the flow of fluids , heat conduction , wave propagation , and electrical fields .

THEOREM (5.8): **It states that if R is the region bounded by a closed surface S and $\vec{A}$ is a vector point function with continuous first partial derivatives , then**

$$\iint_S \vec{A} \cdot \hat{n} \, dS = \iiint_R \nabla \cdot \vec{A} \, dV$$

where $\hat{n}$ is the outward drawn unit normal to S .

In words the surface integral of the normal component of a vector function $\vec{A}$ taken over a closed surface S is equal to the integral of the divergence of $\vec{A}$ taken over the region R enclosed by the surface .

PROOF: If $\vec{A}$ is expressed in terms of components as $\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$, then the divergence theorem can be written as

$$\iint_S (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) \cdot \hat{n} \, dS = \iiint_R \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV$$

To establish this , we prove that the respective integrals on each side are equal .

We prove this for a closed surface S , which has the property that any line parallel to the coordinate axes cuts S in atmost two points . Under this assumption, it follows that S is a double valued surface over its projection on each of the coordinate planes . Let R′ be the projection of S on the x y – plane .

Divide the surface S into lower and upper parts S_1 and S_2 and assume the equations of S_1 and S_2 to be $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively .

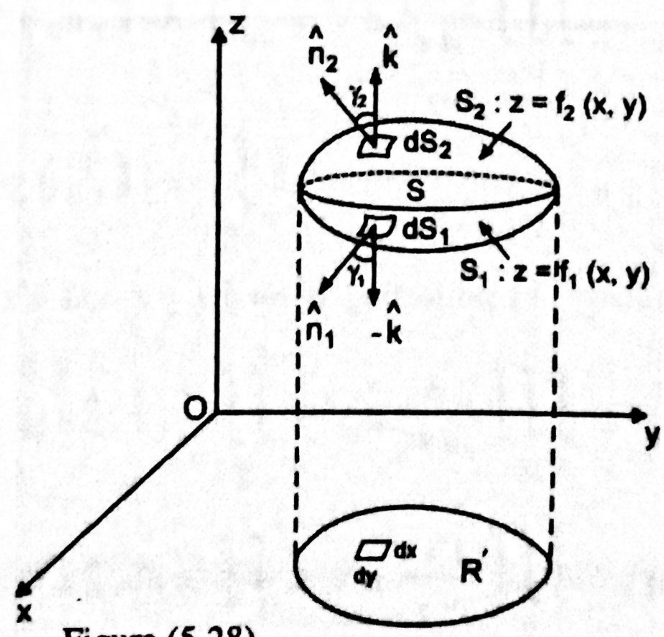

Figure (5.28)

Consider

$$\iiint_R \frac{\partial A_3}{\partial z} dV = \iiint_R \frac{\partial A_3}{\partial z} dz\,dy\,dx$$

$$= \iint_{R'} \left[\int_{z=f_1(x,y)}^{f_2(x,y)} \frac{\partial A_3}{\partial z} dz \right] dy\,dz$$

$$= \iint_{R'} A_3(x,y,z) \Big|_{f_1(x,y)}^{f_2(x,y)} dy\,dx$$

$$= \iint_{R'} \{ A_3[x,y,f_2(x,y)] - A_3[x,y,f_1(x,y)] \} dy\,dx \quad (1)$$

For the upper part S_2, $dy\,dx = \cos\gamma_2\, dS_2 = \hat{k}\cdot\hat{n}_2\, dS_2$, since the normal $\hat{n}_2$ to S_2 makes an acute angle with $\hat{k}$. For the lower part S_1, $dy\,dx = \cos\gamma_1\, dS_1 = -\hat{k}\cdot\hat{n}_1\, dS_1$, since the normal $\hat{n}_1$ to S_1 makes an angle γ_1 with $-\hat{k}$.

Then $$\iint_{R'} A_3[x,y,f_2(x,y)]\,dy\,dx = \iint_{S_2} A_3\hat{k}\cdot\hat{n}_2\, dS_2$$

and $$\iint_{R'} A_3[x,y,f_1(x,y)]\,dy\,dx = -\iint_{S_1} A_3\hat{k}\cdot\hat{n}_1\, dS_1$$

and therefore equation (1) becomes

$$\iiint_R \frac{\partial A_3}{\partial z} dV = \iint_{S_2} A_3\hat{k}\cdot\hat{n}_2\, dS_2 + \iint_{S_1} A_3\hat{k}\cdot\hat{n}_1\, dS_1$$

$$= \iint_S A_3\hat{k}\cdot\hat{n}\, dS \quad (2)$$

Similarly, by projecting S on the yz and zx coordinate planes, we obtain respectively,

$$\iiint_R \frac{\partial A_1}{\partial x} dV = \iint_S A_1\hat{i}\cdot\hat{n}\, dS \quad (3)$$

and $$\iiint_R \frac{\partial A_2}{\partial y} dV = \iint_S A_2\hat{j}\cdot\hat{n}\, dS \quad (4)$$

addition of equations (2), (3), and (4) completes the proof of the theorem.

Notice that Gauss' divergence theorem is a generalization of Green's theorem in the plane where the (plane) region R and its boundary (curve) C are replaced by a (space) region R and its closed boundary (surface) S. For this reason the divergence theorem is often called Green's theorem in space.

RECTANGULAR FORM OF GAUSS'S DIVERGENCE THEOREM

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, and $\hat{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$

then $\nabla.\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$

and $\vec{A}.\hat{n} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}).(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})$

$= A_1\cos\alpha + A_2\cos\beta + A_3\cos\gamma$

and the Gauss' divergence theorem can be written as

$$\iiint_R \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dx\,dy\,dz = \iint_S (A_1\cos\alpha + A_2\cos\beta + A_3\cos\gamma)\,dS$$

EXAMPLE (16) **Verify the divergence theorem for $\vec{A} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ where S is the surface of the cube bounded by**

$x = 0,\quad x = 1,\quad y = 0,\quad y = 1,\quad z = 0,\quad z = 1.$

SOLUTION: The given cube is shown in figure (5.29). By the divergence theorem, we have

$$\iint_S \vec{A}.\hat{n}\,dS = \iiint_R \nabla.\vec{A}\,dV$$

Now $$\iiint_R \nabla.\vec{A}\,dV = \iiint_R \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_R (4z - y)\,dV = \int_0^1\int_0^1\int_0^1 (4z - y)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^1 \left|2z^2 - yz\right|_0^1 dy\,dx = \int_0^1\int_0^1 (2 - y)\,dy\,dx$$

$$= \int_0^1 \left|2y - \frac{y^2}{2}\right|_0^1 dx = \frac{3}{2}\int_0^1 dx = \frac{3}{2}$$

Now $$\iint_S \vec{A}.\hat{n}\,dS = \iint_{S_1} \vec{A}.\hat{n}\,dS_1 + \iint_{S_2} \vec{A}.\hat{n}\,dS_2$$

$$+\iint_{S_3} \vec{A}\cdot\hat{n}\,dS_3 + \iint_{S_4} \vec{A}\cdot\hat{n}\,dS_4$$

$$+\iint_{S_5} \vec{A}\cdot\hat{n}\,dS_5 + \iint_{S_6} \vec{A}\cdot\hat{n}\,dS_6$$

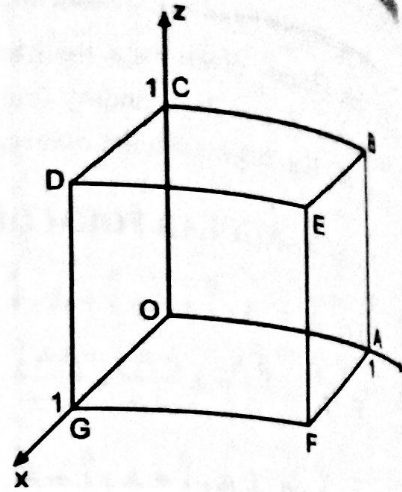

Figure (5.29)

For S_1 (DEFG), $\hat{n} = \hat{i}$, $x = 1$. Then

$$\iint_{DEFG} \vec{A}\cdot\hat{n}\,dS = \iint_{DEFG} \vec{A}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

$$= \int_0^1\int_0^1 (4z\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{i}\,dy\,dz = \int_0^1\int_0^1 4z\,dy\,dz = 2. \qquad (1)$$

For S_2 (ABCO), $\hat{n} = -\hat{i}$, $x = 0$. Then

$$\iint_{ABCO} \vec{A}\cdot\hat{n}\,dS = \int_0^1\int_0^1 (-y^2\hat{j} + yz\hat{k})\cdot(-\hat{i})\,dy\,dz = 0 \qquad (2)$$

For S_3 (ABEF), $\hat{n} = \hat{j}$, $y = 1$. Then

$$\iint_{ABEF} \vec{A}\cdot\hat{n}\,dS = \iint_{ABEF} \vec{A}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

$$= \int_0^1\int_0^1 (4xz\hat{i} - \hat{j} + z\hat{k})\cdot\hat{j}\,dx\,dz = \int_0^1\int_0^1 -dx\,dz = -1 \qquad (3)$$

For S_4 (OGDC), $\hat{n} = -\hat{j}$, $y = 0$. Then

$$\iint_{OGDC} \vec{A}\cdot\hat{n}\,dS = \int_0^1\int_0^1 (4xz\hat{i})\cdot(-\hat{j})\,dx\,dz = 0 \qquad (4)$$

For S_5 (BCDE), $\hat{n} = \hat{k}$, $z = 1$. Then

$$\iint_{BCDE} \vec{A}\cdot\hat{n}\,dS = \iint_{BCDE} \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \int_0^1\int_0^1 (4x\hat{i} - y^2\hat{j} + y\hat{k})\cdot\hat{k}\,dx\,dy = \int_0^1\int_0^1 y\,dx\,dy = \frac{1}{2} \qquad (5)$$

For S_6 (AFGO), $\hat{n} = -\hat{k}$, $z = 0$. Then

$$\iint_{AFGO} \vec{A}.\hat{n}\, dS = \int_0^1\int_0^1 (-y^2\hat{j}).(-\hat{k})\, dx\, dy = 0 \qquad (6)$$

Adding, equations (1) – (6), we get

$$\iint_S \vec{A}.\hat{n}\, dS = 2+0+(-1)+0+\frac{1}{2}+0 = \frac{3}{2}$$

and the theorem is verified.

5.14 THE GRADIENT THEOREM

THEOREM (5.9): If ϕ is a continuous scalar function in a region R bounded by a closed surface S, then prove that $\iiint_R \nabla\phi\, dV = \iint_S \phi\hat{n}\, dS$

PROOF: In the divergence theorem, let $\vec{A} = \phi\vec{C}$ where $\vec{C}$ is a constant vector.

Then $$\iiint_R \nabla.(\phi\vec{C})\, dV = \iint_S \phi\vec{C}.\hat{n}\, dS \qquad (1)$$

Since $\nabla.(\phi\vec{C}) = \nabla\phi.\vec{C} + \phi\nabla.\vec{C} = \nabla\phi.\vec{C} = \vec{C}.\nabla\phi$ (since $\nabla.\vec{C} = 0$)

and $\phi\vec{C}.\hat{n} = \vec{C}.(\phi\hat{n})$, equation (1) becomes

$$\iiint_R \vec{C}.\nabla\phi\, dV = \iint_S \vec{C}.(\phi\hat{n})\, dS$$

Taking $\vec{C}$ outside the integrals, we get $\vec{C}.\iiint_R \nabla\phi\, dV = \vec{C}.\iint_S \phi\hat{n}\, dS$

and since $\vec{C}$ is an arbitrary constant vector, we have $\iiint_R \nabla\phi\, dV = \iint_S \phi\hat{n}\, dS$

5.15 THE CURL THEOREM

THEOREM (5.10): If $\vec{B}$ is a continuous vector function in a region R bounded by a closed surface S, then prove that $\iiint_R \nabla\times\vec{B}\, dV = \iint_S \hat{n}\times\vec{B}\, dS$.

PROOF: In the divergence theorem, let $\vec{A} = \vec{B} \times \vec{C}$ where $\vec{C}$ is a constant vector

Then $$\iiint_R \nabla \cdot (\vec{B} \times \vec{C})\, dV = \iint_S (\vec{B} \times \vec{C}) \cdot \hat{n}\, dS$$

Since $\nabla \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\nabla \times \vec{B}) - \vec{B} \cdot (\nabla \times \vec{C}) = \vec{C} \cdot (\nabla \times \vec{B})$

and $(\vec{B} \times \vec{C}) \cdot \hat{n} = \vec{B} \cdot (\vec{C} \times \hat{n}) = (\vec{C} \times \hat{n}) \cdot \vec{B} = \vec{C} \cdot (\hat{n} \times \vec{B})$

$$\iiint_R \vec{C} \cdot (\nabla \times \vec{B})\, dV = \iint_S \vec{C} \cdot (\hat{n} \times \vec{B})\, dS$$

Taking $\vec{C}$ outside the integrals, we get

$$\vec{C} \cdot \iiint_R (\nabla \times \vec{B})\, dV = \vec{C} \cdot \iint_S \hat{n} \times \vec{B}\, dS$$

and since $\vec{C}$ is an arbitrary constant vector, we have

$$\iiint_R (\nabla \times \vec{B})\, dV = \iint_S \hat{n} \times \vec{B}\, dS$$

ONE CONDENSED NOTATIONAL FORM

The divergence theorem, the gradient theorem, and the curl theorem can be stated in one condensed notational form :

$$\iiint_R \nabla * \alpha\, dV = \iint_S d\vec{S} * \alpha$$

where α is any scalar or vector quantity, and the asterisk ($*$) represents any acceptable form of multiplication i.e. the dot, cross, or simple product.

5.16 GREEN'S IDENTITIES

We now prove the Green's identities which are also called the Green's theorems.

GREEN'S FIRST IDENTITY

THEOREM (5.11): **If ϕ and ψ are scalar point functions with continuous second order derivatives in a region R bounded by a closed surface S, then**

$$\iiint_R [\phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi)]\, dV = \iint_S (\phi \nabla \psi) \cdot d\vec{S}$$

PROOF: Let $\vec{A} = \phi \nabla \psi$ in the divergence theorem

$$\iiint_R \nabla.(\phi \nabla \psi) dV = \iint_S (\phi \nabla \psi).\hat{n} dS = \iint_S (\phi \nabla \psi).d\vec{S} \qquad (1)$$

But $\nabla.(\phi \nabla \psi) = (\nabla \phi).(\nabla \psi) + \phi(\nabla.\nabla \psi) = \phi \nabla^2 \psi + (\nabla \phi).(\nabla \psi)$

Thus equation (1) becomes

$$\iiint_R [\phi \nabla^2 \psi + (\nabla \phi).(\nabla \psi)] dV = \iint_S (\phi \nabla \psi).d\vec{S}$$

GREEN'S SECOND IDENTITY

THEOREM (5.12): If ϕ and ψ are scalar point functions with continuous second order derivatives in a region R bounded by a closed surface S, then

$$\iiint_R (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dV = \iint_S (\phi \nabla \psi - \psi \nabla \phi).d\vec{S}$$

PROOF: We have from Green's first identity

$$\iiint_R [\phi \nabla^2 \psi + (\nabla \phi).(\nabla \psi)] dV = \iint_S (\phi \nabla \psi).d\vec{S} \qquad (1)$$

Interchanging ϕ and ψ in equation (1), we get

$$\iiint_R [\psi \nabla^2 \phi + (\nabla \psi).(\nabla \phi)] dV = \iint_S (\psi \nabla \phi).d\vec{S} \qquad (2)$$

Subtracting equation (2) from equation (1), we have

$$\iiint_R (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dV = \iint_S (\phi \nabla \psi - \psi \nabla \phi).d\vec{S}$$

which is called **Green's second identity or symmetrical theorem**.

ALTERNATIVE FORMS OF GREEN'S IDENTITIES

We know that $\nabla \psi.\hat{n} = \frac{\partial \psi}{\partial n}$ and $\nabla \phi.\hat{n} = \frac{\partial \phi}{\partial n}$.

Thus $\nabla \psi.d\vec{S} = \nabla \psi.\hat{n} dS = \frac{\partial \psi}{\partial n} dS$

and $\nabla \phi.d\vec{S} = \nabla \phi.\hat{n} dS = \frac{\partial \phi}{\partial n} dS$

Hence Green's first and second identities can be written respectively, as

$$\iiint_R [\phi\nabla^2\psi - (\nabla\phi).(\nabla\psi)]\,dV = \iint_S \phi\frac{\partial\psi}{\partial n}\,dS \qquad (1)$$

and

$$\iiint_R (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV = \iint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS \qquad (2)$$

GREEN'S THIRD IDENTITY OR GREEN'S FORMULA

THEOREM (5.13): Let ϕ be a scalar point function with continuous second order derivatives in a region R bounded by a closed surface S. Let $\vec{r}$ be the position vector of any point $P(x,y,z)$ on S relative to an origin O. Prove that

$$\iint_S \left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]dS - \iiint_R \frac{1}{r}\nabla^2\phi\,dV = \begin{cases} 0 & \text{if origin } O \text{ lies outside } R \\ 4\pi\phi_0 & \text{if origin } O \text{ lies inside } R \end{cases}$$

where ϕ_0 is the value of ϕ at O.

PROOF: We know that Green's second identity is :

$$\iint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS = \iiint_R (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV \qquad (1)$$

If in equation (1) we take for ψ the harmonic function $\frac{1}{r}$, where r is the distance from a fixed point O to a variable point P within R. Then

$$\iint_S \left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial n}\right]dS = \iiint_R \left[\phi\nabla^2\left(\frac{1}{r}\right) - \frac{1}{r}\nabla^2\phi\right]dV \qquad (2)$$

CASE (1): When O lies outside the region R

If the origin O lies outside the region R bounded by S as shown in figure (5.30), then $r \neq 0$. Hence, the function $\frac{1}{r}$ and its derivative are finite at all points of the region. Furthermore, since $\frac{1}{r}$ is harmonic, therefore $\nabla^2\left(\frac{1}{r}\right) = 0$. Thus equation (2) reduces to

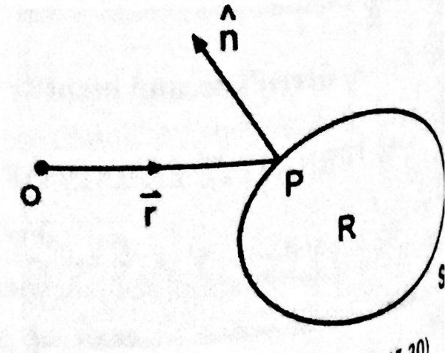

Figure (5.30)

$$\iint_S \left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]dS - \iiint_R \frac{1}{r}\nabla^2\phi\,dV = 0 \qquad (3)$$