$= 4(18+9) = 108$

VOLUME INTEGRAL

Let $\vec{A}(x, y, z)$ be a vector point function which is defined and continuous in a closed region R. Subdivide the region R into n subregions ΔR_k of volume ΔV_k, $k = 1, 2, \dots, n$. Let $P_k(x_k, y_k, z_k)$ be any point on each subregion ΔR_k as shown in figure (5.19).

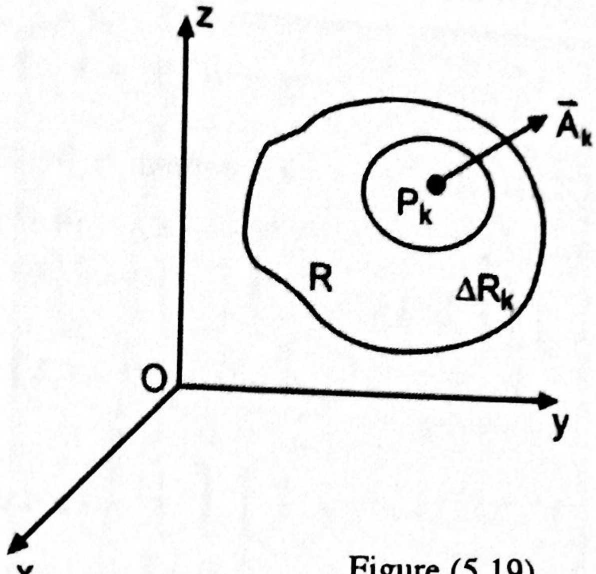

Figure (5.19)

Define $\vec{A}(x_k, y_k, z_k) = \vec{A}_k$. We multiply the value of $\vec{A}$ at the selected point (i.e. $\vec{A}_k$) with the volume of the corresponding subregion and form the sum

$$\sum_{k=1}^{n} \vec{A}_k \, \Delta V_k \qquad (1)$$

Now take the limit of sum (1) as $n \to \infty$ in such a way that each $\Delta V_k \to 0$. This limit, if it exists, is called the volume integral of $\vec{A}(x, y, z)$ over R and is denoted by $\iiint_R \vec{A} \, dV$

i.e.

$$\iiint_R \vec{A}(x, y, z) \, dV = \underset{n \to \infty}{\mathrm{Lt}} \sum_{k=1}^{n} A_k \, \Delta V_k \qquad (2)$$

If $\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$, then the volume integral (2) can be written as

$$\iiint_R \vec{A} \, dV = \hat{i} \iiint_R A_1 \, dV + \hat{j} \iiint_R A_2 \, dV + \hat{k} \iiint_R A_3 \, dV \qquad (3)$$

If we have a scalar point function $\phi(x, y, z)$ ... the volume

integral becomes $\iiint_R \phi \, dV$ (3)

In rectangular coordinate system, $dV = dx\,dy\,dz$ so the volume integral (3) can be written as

$$\iiint_R \phi(x, y, z)\, dx\,dy\,dz$$

which is the ordinary triple integral of $\phi(x, y, z)$ over the region R. If $\phi(x, y, z) = 1$, the volume V of the region R is given by $V = \iiint_R dV$.

EXAMPLE (13): **Evaluate $\iiint_R \vec{r}\, dV$ where R is the region bounded by the surfaces $x = 0$, $y = 0$, $y = 6$, $z = x^2$, $z = 4$.**

SOLUTION: The region R bounded by the given surfaces is shown in figure (5.20). Then

$$\iiint_R \vec{r}\, dV = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} (x\hat{i} + y\hat{j} + z\hat{k})\, dz\, dy\, dx$$

$$= \hat{i}\int_0^2\int_0^6\int_{x^2}^4 x\,dz\,dy\,dx$$

$$+ \hat{j}\int_0^2\int_0^6\int_{x^2}^4 y\,dz\,dy\,dx$$

$$+ \hat{k}\int_0^2\int_0^6\int_{x^2}^4 z\,dz\,dy\,dx$$

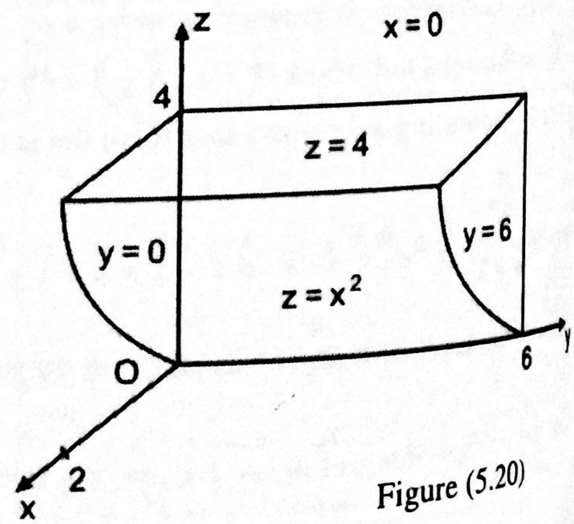

Figure (5.20)

$$= \hat{i}\int_0^2\int_0^6 x\, z\Big|_{x^2}^{4} dy\,dx + \hat{j}\int_0^2\int_0^6 y\, z\Big|_{x^2}^{4} dy\,dx + \hat{k}\int_0^2\int_0^6 \left|\frac{1}{2}z^2\right|_{x^2}^{4} dy\,dx$$

$$= \hat{i}\int_0^2\int_0^6 (4x - x^3)\,dy\,dx + \hat{j}\int_0^2\int_0^6 (4 - x^2)\,y\,dy\,dx + \hat{k}\int_0^2\int_0^6 \left(8 - \frac{1}{2}x^4\right) dy\,dx$$

$$= 6\hat{i}\int_0^2 (4x - x^3)\,dx + \hat{j}\int_0^2 (4 - x^2)\left|\frac{1}{2}y^2\right|_0^6 dx + 6\hat{k}\int_0^2\left(8 - \frac{1}{2}x^4\right)dx$$

$$= 6\hat{i}\left|2x^2 - \frac{1}{4}x^4\right|_0^2 + 18\hat{j}\left|4x - \frac{1}{3}x^3\right|_0^2 + 6\hat{k}\left|8x - \frac{1}{10}x^5\right|_0^2$$

$$= 6\hat{i}(4) + 18\hat{j}\left(\frac{16}{3}\right) + 6\hat{k}\left(\frac{64}{5}\right)$$

$$= 24\hat{i} + 96\hat{j} + \frac{384}{5}\hat{k}$$

19 SIMPLY AND MULTIPLY CONNECTED REGIONS

A simple closed curve is a closed curve which does not intersect itself anywhere . For example , the curve in figure [5.21 (a)] is a simple closed curve while the curve in figure [5.21 (b)] is not .

A region R is said to be **simply connected** if any simple closed curve lying in R can be continuously shrunk to a point . For example , the interior of a rectangle as shown in figure [5.21 (c)] is an example of a simply connected region .

A region R which is not simply connected is called **multiply connected**. For example , the region R exterior to C_2 and interior to C_1 is not simply connected because a circle drawn within R and enclosing C_2 cannot be shrunk to a point without crossing C_2 as shown in figure [5.21 (d)] . In other words , the regions which have holes are called multiply connected .

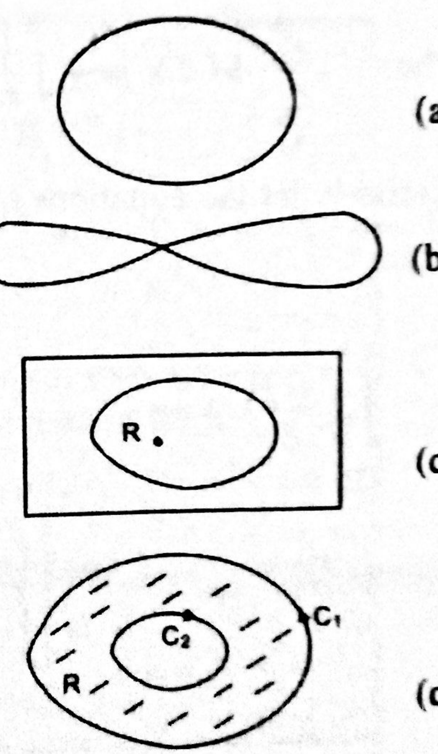

Figure (5.21)

$= 2\hat{i} + \hat{j} + 2\hat{k}$

PROBLEM (17): **If $\vec{A} = y\,\hat{i} - x\,\hat{j}$, evaluate $\iiint_R \nabla \times \vec{A}\, dV$ where R is the region enclosing the volume V.**

SOLUTION: We have

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\hat{k}$$

and so

$$\iiint_R \nabla \times \vec{A}\, dV = \iiint_R -2\hat{k}\, dV = -2\hat{k} \iiint_R dV = -2V\hat{k}$$

PROBLEM (18): **Evaluate $\iiint_R \phi\, dV$, where $\phi = 45x^2y$ and R is the region enclosed by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0$, $z = 0$.**

SOLUTION: The region R bounded by the given planes is shown in figure (5.45). Then

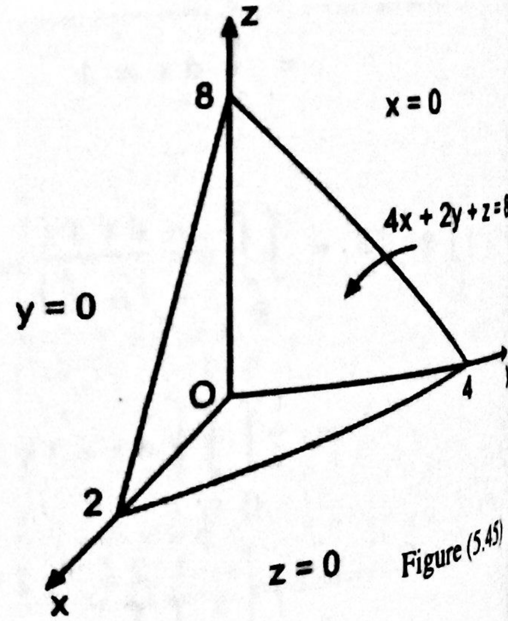

Figure (5.45)

$$\iiint_R \phi\, dV = \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} 45x^2y\, dz\, dy\, dx$$

$$= 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} x^2 y (8 - 4x - 2y)\, dy\, dx$$

$$= 45 \int_0^2 x^2 \left| (8-4x)\frac{y^2}{2} - \frac{2}{3}y^3 \right|_0^{4-2x} dx$$

$$= 45 \int_0^2 x^2 \left[(4-2x)^3 - \frac{2}{3}(4-2x)^3 \right] dx$$

$$= 45 \int_0^2 \frac{1}{3} x^2 (4-2x)^3\, dx$$

$$= 15 \int_0^2 x^2 (64 - 96x + 48x^2 - 8x^3)\, dx$$

$$= 15\int_0^2 (64x^2 - 96x^3 + 48x^4 - 8x^5)\,dx$$

$$= 15\left| 64\frac{x^3}{3} - 24x^4 + \frac{48}{5}x^5 - \frac{4}{3}x^6 \right|_0^2$$

$$= 15\left(\frac{512}{3} - 384 + \frac{1536}{5} - \frac{256}{3}\right) = 15\left[\frac{2560 - 5760 + 4608 - 1280}{15}\right]$$

$$= 128$$

...BLEM (19): **Find the volume of the region common to the intersecting cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.**

...UTION: The volume bounded by the given cylinders in the first octant is shown in ...(5.46). Then the required volume required is given by

$$\text{Volume } V = \iiint_R dV$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy\,dx$$

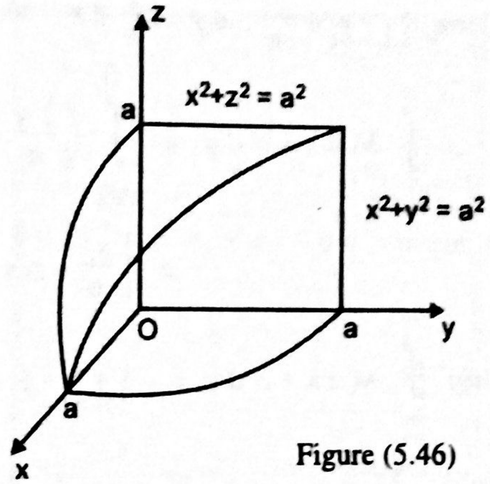

Figure (5.46)

$$= 8\int_0^a (a^2 - x^2)\,dx = 8\left| a^2x - \frac{x^3}{3} \right|_0^a$$

$$= 8\left(a^3 - \frac{a^3}{3}\right) = \frac{16a^3}{3}$$

...BLEM (20): **Verify Green's theorem in the plane for $M = y - \sin x$ and $N = \cos x$ where C is the triangle consisting of the line segment C_1 from $(0,0)$ to $\left(\frac{\pi}{2}, 0\right)$, then the line segment C_2 from $\left(\frac{\pi}{2}, 0\right)$ to $\left(\frac{\pi}{2}, 1\right)$, and then the line segment from $\left(\frac{\pi}{2}, 1\right)$ to $(0,0)$.**

...UTION: The given triangle is shown in figure (5.47).

...e show that

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \qquad (1)$$

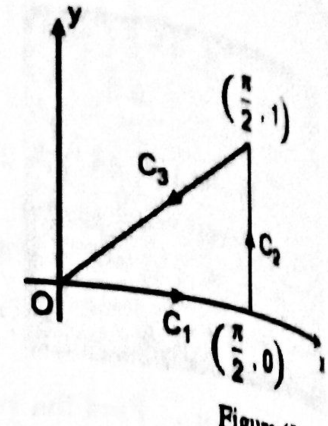

Figure (5.47)

Now $\oint_C M\,dx + N\,dy = \oint_C (y - \sin x)\,dx + \cos x\,dy$ (1)

Along C_1, $y = 0$, $dy = 0$, $0 \le x \le \pi/2$, and the integral (1) equals

$$\int_{C_1} M\,dx + N\,dy = \int_0^{\pi/2} -\sin x\,dx = \Big| \cos x \Big|_0^{\pi/2} = -1$$

Along C_2, $x = \pi/2$, $dx = 0$, $0 \le y \le 1$ and the integral (1) equals

$$\int_{C_2} M\,dx + N\,dy = \int_0^1 0\,dy = 0$$

Along C_3, $y = \frac{2x}{\pi}$, $dy = \frac{2}{\pi}\,dx$, where x varies from $\frac{\pi}{2}$ to 0 and the integral (1) equals

$$\int_{C_3} M\,dx + N\,dy = \int_{\pi/2}^{0} \left(\frac{2x}{\pi} - \sin x\right) dx + \frac{2}{\pi}\cos x\,dx$$

$$= \left| \frac{x^2}{\pi} + \cos x + \frac{2}{\pi}\sin x \right|_{\pi/2}^{0} = 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

Then $\oint_C M\,dx + N\,dy = -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\frac{\pi}{4} - \frac{2}{\pi}$ (2)

Since $M = y - \sin x$, $N = \cos x$, therefore $\frac{\partial N}{\partial x} = -\sin x$, $\frac{\partial M}{\partial y} = 1$.

and so $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (-\sin x - 1)\,dy\,dx$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{2x/\pi} (-\sin x - 1)\,dy\,dx$$

$$= \int_{x=0}^{\pi/2} \left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right) dx$$

$$= \left| -\frac{2}{\pi}(-x\cos x + \sin x) - \frac{x^2}{\pi} \right|_0^{\pi/2}$$

$$= -\frac{2}{\pi} - \frac{\pi}{4}$$ (3)

From equations (2) and (3), we see that Green's theorem is verified.

PROBLEM (21): Verify Green's theorem in the plane for $M = 2x - y^3$ and $N = -xy$, where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

SOLUTION: The boundary of R consists of the circle C_1: $x = 3\cos\theta$, $y = 3\sin\theta$ traversed counter – clockwise, and the circle C_2: $x = \cos\theta$, $y = \sin\theta$ traversed clockwise.

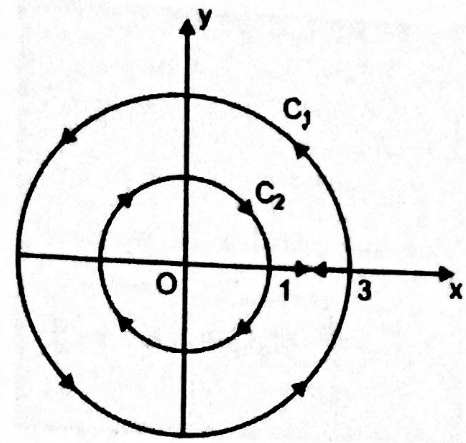

Figure (5.48)

We must show

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Now

$$\oint_C M\,dx + N\,dy = \oint_C (2x - y^3)\,dx - xy\,dy$$

$$= \oint_{C_1} (2x - y^3)\,dx - xy\,dy + \oint_{C_2} (2x - y^3)\,dx - xy\,dy \qquad (1)$$

For the circle C_1: $x = 3\cos\theta$, $y = 3\sin\theta$, $dx = -3\sin\theta\,d\theta$, $dy = 3\cos\theta\,d\theta$, where θ varies from 0 to 2π.

Then

$$\oint_{C_1} (2x - y^3)\,dx - xy\,dy = \int_0^{2\pi} (6\cos\theta - 27\sin^3\theta)(-3\sin\theta\,d\theta) - (9\cos\theta\sin\theta)(3\cos\theta\,d\theta)$$

$$= \int_0^{2\pi} (-18\cos\theta\sin\theta + 81\sin^4\theta - 27\cos^2\theta\sin\theta)\,d\theta$$

$$= \left| \frac{9}{2}\cos 2\theta + \left(\frac{243}{8}\theta - \frac{81}{4}\sin 2\theta + \frac{81}{32}\sin 4\theta\right) + 9\cos^3\theta \right|_0^{2\pi}$$

$$= \left[\frac{9}{2} + \frac{243}{8}(2\pi) + 9\right] - \left[\frac{9}{2} + 9\right]$$

$$= \frac{9}{2} + \frac{243}{4}\pi + 9 - \frac{9}{2} - 9 = \frac{243}{4}\pi$$

For the circle C_2: $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$, where θ varies from 2π to 0.

Then

$$\oint_{C_2} (2x - y^3)\,dx - xy\,dy = \int_{2\pi}^{0} (2\cos\theta - \sin^3\theta)(-\sin\theta\,d\theta) - \cos\theta\sin\theta(\cos\theta\,d\theta)$$

$$= \int_{2\pi}^{0} (-2\cos\theta\sin\theta + \sin^4\theta - \cos^2\theta\sin\theta)\,d\theta$$

$$= \left| \frac{\cos 2\theta}{2} + \left(\frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right) + \frac{\cos^3\theta}{3} \right|_{2\pi}^{0}$$

$$= \left(\frac{1}{2} + \frac{1}{3}\right) - \left(\frac{1}{2} + \frac{3}{4}\pi + \frac{1}{3}\right) = -\frac{3}{4}\pi$$

Thus from equation (1), we get

$$\oint_C M\,dx + N\,dy = \oint_C (2x - y^3)\,dx - xy\,dy = \frac{243}{4}\pi - \frac{3}{4}\pi = 60\pi$$

Next, $\frac{\partial M}{\partial y} = -3y^2$, $\frac{\partial N}{\partial x} = -y$, therefore

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (-y + 3y^2)\,dx\,dy$$

Changing to polar coordinates,

$$= \int_{\theta=0}^{2\pi} \int_{r=1}^{3} (-r\sin\theta + 3r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_{0}^{2\pi} \int_{1}^{3} (-r^2\sin\theta + 3r^3\sin^2\theta)\,dr\,d\theta$$

$$= \int_{0}^{2\pi} \left| -\frac{r^3}{3}\sin\theta + \frac{3}{4}r^4\sin^2\theta \right|_1^3 d\theta$$

$$= \int_{0}^{2\pi} \left[\left(-9\sin\theta + \frac{243}{4}\sin^2\theta\right) - \left(-\frac{1}{3}\sin\theta + \frac{3}{4}\sin^2\theta\right)\right] d\theta$$

$$= \int_{0}^{2\pi} \left(-\frac{26}{3}\sin\theta + 60\sin^2\theta\right) d\theta = \int_{0}^{2\pi} \left[-\frac{26}{3}\sin\theta + 30(1 - \cos 2\theta)\right] d\theta$$

$$= \left| \frac{26}{3}\cos\theta + 30\theta - 15\sin 2\theta \right|_0^{2\pi} = \frac{26}{3} + 60\pi - \frac{26}{3} = 60\pi$$

and thus the Green's theorem is verified.