5.6 NORMAL SURFACE INTEGRAL

Let S be a surface (open or closed) and $\vec{A}(x, y, z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ a vector function which is defined and continuous over S. Let one side of S be taken arbitrarily as the positive side (if S is a closed surface this is taken as the outer side).

Subdivide the surface S into n elements of surface area ΔS_k, $k = 1, 2, \ldots, n$. Let $P_k(x_k, y_k, z_k)$ be any point on each surface element ΔS_k. Define $\vec{A}(x_k, y_k, z_k) = \vec{A}_k$. Let $\hat{n}_k$ be the positive outward drawn unit normal to ΔS_k at P_k as shown in the figure (5.14).

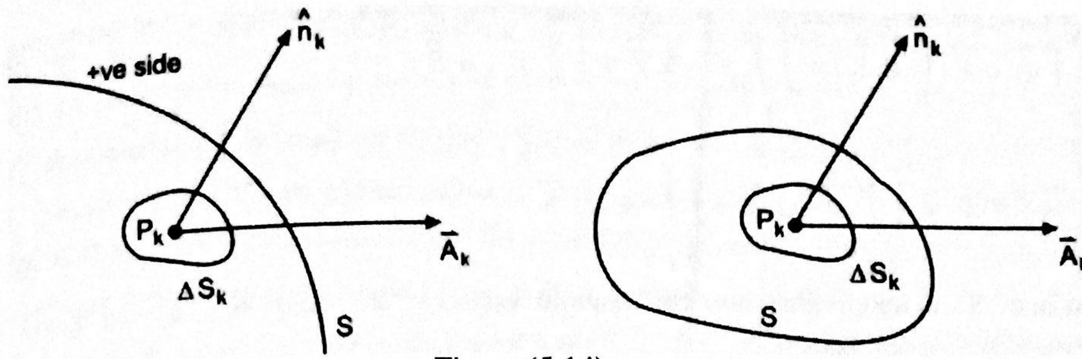

Figure (5.14)

We multiply the normal component of $\vec{A}_k$ at P_k with the surface area of the corresponding element and form the sum $\sum_{k=1}^{n} \vec{A}_k \cdot \hat{n}_k \, \Delta S_k$ (1)

Now take the limit of this sum as $n \to \infty$ in such a way that each $\Delta S_k \to 0$. This limit, if it exists is called the normal surface integral of $\vec{A}$ over S and is denoted by $\iint_S \vec{A} \cdot \hat{n} \, dS$

By defining a vector $d\vec{S} = \hat{n} \, dS$, where dS is the magnitude of this vector and whose direction is that of $\hat{n}$, the above integral can be written as

$$\iint_S \vec{A} \cdot \hat{n} \, dS = \iint_S \vec{A} \cdot d\vec{S} \qquad (2)$$

This surface integral of the vector field $\vec{A}$ is called the flux of $\vec{A}$ through S. The other forms of surface integrals are $\iint_S \phi \, d\vec{S}$, $\iint_S \vec{A} \times d\vec{S}$

where ϕ is a scalar point function. If S is a closed surface, the surface integrals are written as

$$\oiint_S \vec{A} \cdot d\vec{S}, \quad \oiint_S \phi \, d\vec{S} \quad \text{or} \quad \oiint_S \vec{A} \times d\vec{S}$$

Integrals that involve the differential surface element $d\vec{S}$ are called surface integral.

NOTE: If $\vec{A}$ represents the fluid velocity, then integral (2) represents the net amount of fluid crossing the surface in unit time. The same formula represents the current flow, magnetic flux, flow of heat, etc.

PROPERTIES OF SURFACE INTEGRALS

Like the double integrals, surface integrals have the following properties:

(i) $\iint_S K\vec{A} \cdot d\vec{S} = K \iint_S \vec{A} \cdot d\vec{S}$ (K any real constant)

(ii) $$\iint_S [\vec{A} + \vec{B}] \cdot d\vec{S} = \iint_S \vec{A} \cdot d\vec{S} + \iint_S \vec{B} \cdot d\vec{S}$$

(iii) $$\iint_S \vec{A} \cdot d\vec{S} = \iint_{S_1} \vec{A} \cdot d\vec{S} + \iint_{S_2} \vec{A} \cdot d\vec{S}$$

where the surface S is subdivided into two smooth surfaces S_1 and S_2 having atmost a curve in common.

(iv) If the surface S is partitioned by smooth curves into a finite number of non–overlaping smooth patches $S_1, S_2, \ldots\ldots, S_n$ (i.e. if S is piecewise smooth), then the normal surface integral of $\vec{A}$ over S is the sum of the normal surface integrals of $\vec{A}$ over all the smooth patches , i.e.

$$\iint_S \vec{A} \cdot d\vec{S} = \iint_{S_1} \vec{A} \cdot d\vec{S} + \iint_{S_2} \vec{A} \cdot d\vec{S} + \ldots\ldots\ldots + \iint_{S_n} \vec{A} \cdot d\vec{S}.$$

5.7 EVALUATION OF THE SURFACE INTEGRALS

To evaluate surface integrals , it is convenient to express them as double integrals taken over the projected area of the surface S on one of the coordinate planes .

THEOREM (5.5): Let R be the projection of the surface S on the xy–plane , then prove that

$$\iint_S \vec{A} \cdot \hat{n}\, dS = \iint_R \vec{A} \cdot \hat{n} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

PROOF: Let the surface S and its projection R on the xy–plane be as shown in figure (5.15) (a).

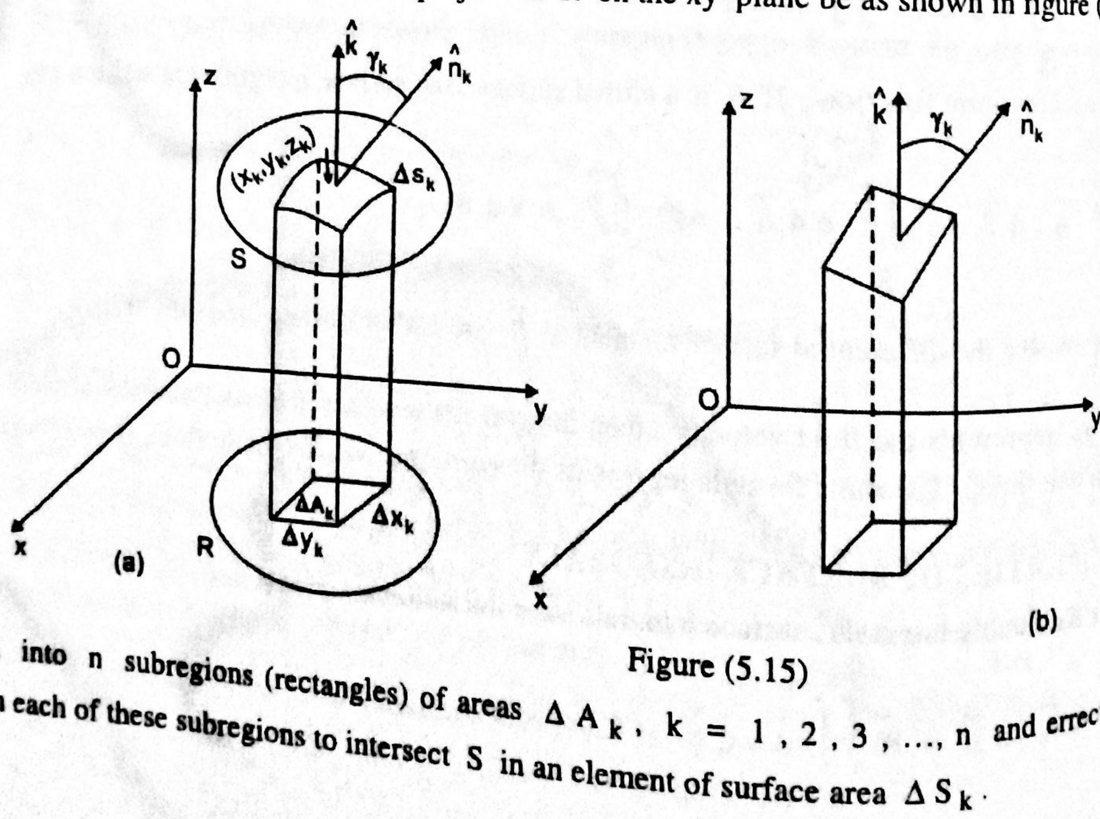

Figure (5.15)

Divide R into n subregions (rectangles) of areas ΔA_k, $k = 1, 2, 3, \ldots, n$ and erect a vertical column on each of these subregions to intersect S in an element of surface area ΔS_k.

Choose a point (x_k, y_k, z_k) on each surface element ΔS_k and draw the unit normal $\hat{n}_k$ to this element at this point. Let γ_k be the acute angle between this unit normal and the positive z – axis.

If this surface element is sufficiently small, it can be regarded as a plane as shown in figure (5.15) (b).

We know from geometry, that if two planes intersect at an acute angle, an area in one plane may be projected into the other by multiplying the cosine of the included angle as shown in figure (5.16).

Since the angle between two planes is the angle between their normals,

therefore $\Delta S_k \cos \gamma_k \approx \Delta A_k$

or $\Delta S_k \approx \sec \gamma_k \, \Delta A_k = \dfrac{\Delta x_k \Delta y_k}{|\hat{n}_k \cdot \hat{k}|}$

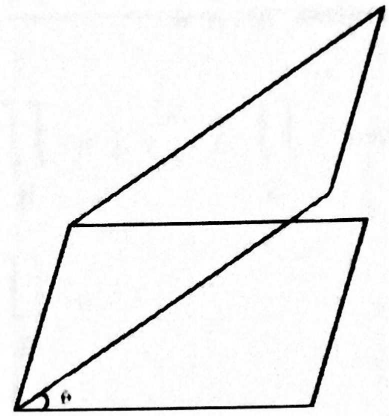
Figure (5.16)

Thus the sum (1) in the definition of the normal surface integral

$$\sum_{k=1}^{n} \vec{A}_k \cdot \hat{n}_k \, \Delta S_k \approx \sum_{k=1}^{n} \vec{A}_k \cdot \hat{n}_k \frac{\Delta x_k \Delta y_k}{|\hat{n}_k \cdot \hat{k}|}$$

and the limit of this sum can be written as

$$\iint_S \vec{A} \cdot \hat{n} \, dS = \iint_R \vec{A} \cdot \hat{n} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

NOTE: Similarly, we can prove that if R is the projection of the surface S on the yz–plane, then

$$\iint_S \vec{A} \cdot \hat{n} \, dS = \iint_R \vec{A} \cdot \hat{n} \frac{dy\,dz}{|\hat{n} \cdot \hat{i}|}$$

and if R is the projection of S on the zx–plane, then $\displaystyle\iint_S \vec{A} \cdot \hat{n} \, dS = \iint_R \vec{A} \cdot \hat{n} \frac{dz\,dx}{|\hat{n} \cdot \hat{j}|}$

EXAMPLE (11): **Evaluate $\displaystyle\iint_S \vec{A} \cdot \hat{n} \, dS$ where $\vec{A} = 18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}$ and S is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant.**

SOLUTION:

The surface S and its projection R on the xy–plane are shown in figure (5.17).

We know that $\displaystyle\iint_S \vec{A} \cdot \hat{n} \, dS = \iint_R \vec{A} \cdot \hat{n} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$

Also we know that a normal vector to the surface $2x + 3y + 6z = 12$ is given by

$\nabla(2x + 3y + 6z) = 2\hat{i} + 3\hat{j} + 6\hat{k}$.

Then a unit normal $\hat{n}$ to any point of S is $\hat{n} = \dfrac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7} = \dfrac{2}{7}\hat{i} + \dfrac{3}{7}\hat{j} + \dfrac{6}{7}\hat{k}$

Thus $\hat{n}\cdot\hat{k} = \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)\cdot\hat{k} = \frac{6}{7}$ and so $\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{7}{6}\,dx\,dy$

Also $\vec{A}\cdot\hat{n} = (18z\hat{i} - 12\hat{j} + 3y\hat{k})\cdot\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) = \frac{36z - 36 + 18y}{7}$

Now from the equation of the surface S, $z = \frac{12 - 2x - 3y}{6}$

therefore $\vec{A}\cdot\hat{n} = \frac{6(12 - 2x - 3y) - 36 + 18y}{7} = \frac{36 - 12x}{7}$

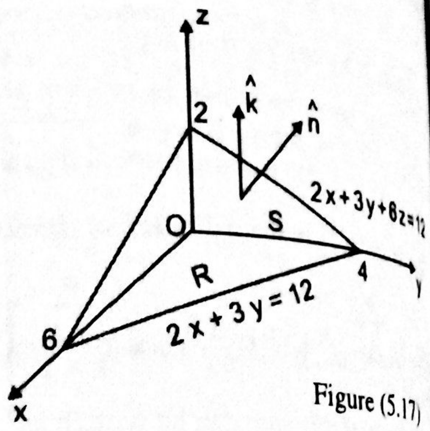

Figure (5.17)

Then

$$\iint_S \vec{A}\cdot\hat{n}\,dS = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint_R \frac{36 - 12x}{7}\,\frac{7}{6}\,dx\,dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{(12-2x)/3} (6 - 2x)\,dy\,dx$$

$$= \int_0^6 (6 - 2x)\left(\frac{12 - 2x}{3}\right)dx = \int_{x=0}^{6}\left(24 - 12x + \frac{4x^2}{3}\right)dx$$

$$= \left| 24x - 6x^2 + \frac{4}{9}x^3 \right|_0^6 = 144 - 216 + 96 = 240 - 216 = 24$$

NOTE: If we had chosen the positive unit normal $\hat{n}$ opposite to that in the figure above, we would have obtained the result -24.

EXAMPLE (12): **Evaluate $\iint_S \vec{A}\cdot\hat{n}\,dS$ for $\vec{A} = y\hat{i} + 2x\hat{j} - z\hat{k}$ where S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.**

SOLUTION: The surface S and its projection R in the yz–plane are shown in figure (5.18). A vector normal to S is given by

$$\nabla(2x + y) = 2\hat{i} + \hat{j}$$

therefore, $\hat{n} = \frac{2\hat{i} + \hat{j}}{\sqrt{5}}$. Also $\hat{n}\cdot\hat{i} = \frac{2}{\sqrt{5}}$

and $\vec{A}\cdot\hat{n} = \frac{2y}{\sqrt{5}} + \frac{2x}{\sqrt{5}} = \frac{2}{\sqrt{5}}(y + x)$

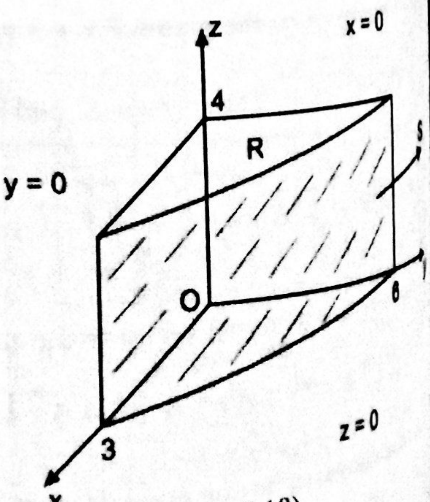

Figure (5.18)

and $$\iint_S \vec{A}.\hat{n}\,dS = \iint_R \vec{A}.\hat{n}\frac{dy\,dz}{|\hat{n}.\hat{i}|} = \int_{y=0}^{6}\int_{z=0}^{4}\frac{2}{\sqrt{5}}(y+x).\frac{\sqrt{5}}{2}\,dz\,dy$$

$$= \int_0^6\int_0^4\left[y+\left(\frac{6-y}{2}\right)\right]dz\,dy = \int_0^6\int_0^4\left(3+\frac{y}{2}\right)dz\,dy$$

$$= \int_0^6\left(3+\frac{y}{2}\right)|z|_0^4\,dy = 4\int_0^6\left(3+\frac{y}{2}\right)dy = 4\left|3y+\frac{y^2}{4}\right|_0^6$$

$$= 4(18+9) = 108$$

5.8 VOLUME INTEGRAL

Let $\vec{A}(x,y,z)$ be a vector point function which is defined and continuous in a closed region R. Subdivide the region R into n subregions ΔR_k of volume ΔV_k, $k=1,2,\ldots,n$. Let $P_k(x_k,y_k,z_k)$ be any point on each subregion ΔR_k as shown in figure (5.19).

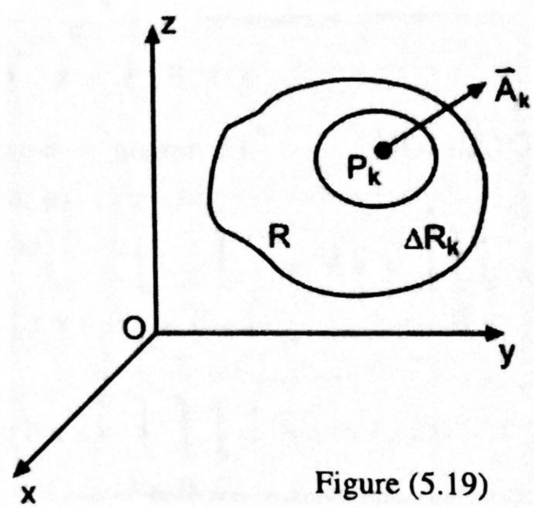

Figure (5.19)

Define $\vec{A}(x_k,y_k,z_k) = \vec{A}_k$. We multiply the value of $\vec{A}$ at the selected point (i.e. $\vec{A}_k$) with the volume of the corresponding subregion and form the sum

$$\sum_{k=1}^{n}\vec{A}_k\,\Delta V_k \qquad (1)$$

Now take the limit of sum (1) as $n\to\infty$ in such a way that each $\Delta V_k\to 0$. This limit, if it exists, is called the volume integral of $\vec{A}(x,y,z)$ over R and is denoted by $\iiint_R \vec{A}\,dV$

i.e. $$\iiint_R \vec{A}(x,y,z)\,dV = \underset{n\to\infty}{\text{Lt}}\sum_{k=1}^{n}A_k\Delta V_k \qquad (2)$$

If $\vec{A} = A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$, then the volume integral (2) can be written as

$$\iiint_R \vec{A}\,dV = \hat{i}\iiint_R A_1\,dV+\hat{j}\iiint_R A_2\,dV+\hat{k}\iiint_R A_3\,dV \qquad (3)$$

If $\vec{A} = yz\hat{i} + zx\hat{j} + xy\hat{k}$, evaluate $\iint_S \vec{A}\cdot\hat{n}\,dS$ where S is the surface of the sphere $x^2+y^2+z^2 = 1$ lying in the first octant.

The surface S of the sphere in the first octant and its projection R on the xy-plane are shown in Figure (5.37). We know that

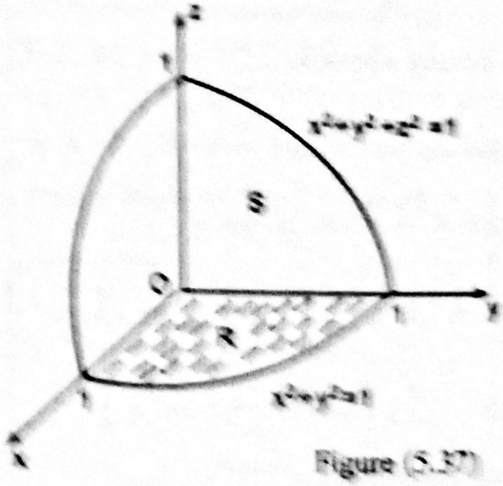

Figure (5.37)

$$\iint_S \vec{A}\cdot\hat{n}\,dS = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \qquad (1)$$

A vector normal to the surface S is given by

$$\nabla(x^2+y^2+z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Therefore, $\hat{n}$ at any point of S is

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$\vec{A}\cdot\hat{n} = (yz\hat{i}+zx\hat{j}+xy\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k}) = xyz+xyz+xyz = 3xyz$$

$$\hat{n}\cdot\hat{k} = (x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{k} = z \quad \text{so that} \quad \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{z}$$

Hence, from equation (1), we have

$$\iint_S \vec{A}\cdot\hat{n}\,dS = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} (3xyz)\,\frac{dy\,dx}{z} = 3\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dy\,dx$$

$$= 3\int_0^1 x\left|\frac{y^2}{2}\right|_0^{\sqrt{1-x^2}} dx = 3\int_0^1 \frac{x(1-x^2)}{2}\,dx$$

$$= \frac{3}{2}\int_0^1 (x-x^3)\,dx = \frac{3}{2}\left|\frac{x^2}{2}-\frac{x^4}{4}\right|_0^1 = \frac{3}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{3}{8}$$

Evaluate (i) $\iint_S \vec{A}\cdot\hat{n}\,dS$, (ii) $\iint_S \phi\,\hat{n}\,dS$

where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$, $\phi = \frac{3}{8}xyz$ and S is the surface of the cylinder $x^2+y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

SOLUTION: The surface S and its projection R on the xz–plane are shown in figure (5.38). Note that the projection of S on the xy–plane cannot be used here. Then

(i) $$\iint_S \vec{A}\cdot\hat{n}\,dS = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|} \qquad (1)$$

A normal vector to $x^2+y^2 = 16$ is $\nabla(x^2+y^2) = 2x\hat{i}+2y\hat{j}$

Thus the unit normal $\hat{n}$ to S, is $\hat{n} = \dfrac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \dfrac{x\hat{i}+y\hat{j}}{4}$

since $x^2+y^2 = 16$ on S.

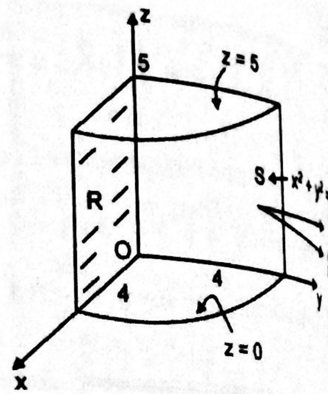

Figure (5.38)

$$\vec{A}\cdot\hat{n} = (z\hat{i}+x\hat{j}-3y^2z\hat{k})\cdot\frac{(x\hat{i}+y\hat{j})}{4} = \frac{1}{4}(xz+xy)$$

and $$\hat{n}\cdot\hat{j} = \frac{x\hat{i}+y\hat{j}}{4}\cdot\hat{j} = \frac{y}{4}$$

Then from equation (1), we have

$$\iint_S \vec{A}\cdot\hat{n}\,dS = \iint_R \frac{1}{4}(xz+xy)\frac{4}{y}\,dx\,dz$$

$$= \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{16-x^2}}+x\right)dx\,dz$$

$$= \int_0^5 \left| -z\sqrt{16-x^2}+\frac{x^2}{2}\right|_0^4 dz = \int_0^5 (4z+8)\,dz = \left|2z^2+8z\right|_0^5 = 90$$

(ii) $$\iint_S \phi\,\hat{n}\,dS = \iint_R \phi\,\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|} \qquad (2)$$

Thus equation (2) becomes

$$\iint_S \phi\,\hat{n}\,dS = \iint_R \frac{3}{8}xyz\frac{(x\hat{i}+y\hat{j})}{4}\frac{4}{y}\,dx\,dz = \frac{3}{8}\int_{z=0}^{5}\int_{x=0}^{4} xz(x\hat{i}+y\hat{j})\,dx\,dz$$

$$= \frac{3}{8}\int_0^5\int_0^4 (x^2z\hat{i}+xz\sqrt{16-x^2}\,\hat{j})\,dx\,dz$$

$$= \frac{3}{8}\int_0^5 \left[z\hat{i}\left|\frac{x^3}{3}\right|_0^4 + z\hat{j}\left| -\frac{1}{3}(16-x^2)^{3/2}\right|_0^4 \right] dz$$

$$= \frac{3}{8}\int_0^5 \left(\frac{64}{3} z\,\hat{i} + \frac{64}{3} z\,\hat{j}\right) dz = 8 \left| \hat{i}\frac{z^2}{2} + \hat{j}\frac{z^2}{2} \right|_0^5$$

$$= 8\left(\frac{25}{2}\hat{i} + \frac{25}{2}\hat{j}\right) = 100\,\hat{i} + 100\,\hat{j}$$

PROBLEM (11): **Evaluate $\iint_S \vec{A}\cdot\hat{n}\,dS$, if $\vec{A} = 2y\,\hat{i} - z\,\hat{j} + x^2\,\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$.**

SOLUTION: The surface S and its projection R on the yz-plane are shown in figure (5.39). Then

$$\iint_S \vec{A}\cdot\hat{n}\,dS = \iint_R \vec{A}\cdot\hat{n}\,\frac{dz\,dy}{|\hat{n}\cdot\hat{i}|} \qquad (1)$$

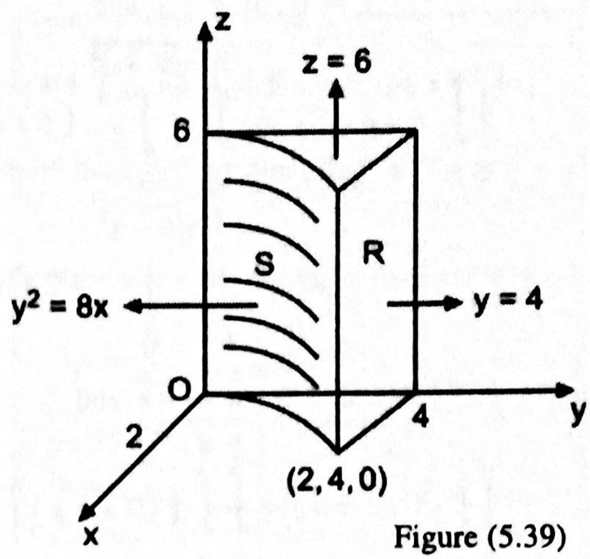

Figure (5.39)

A vector normal to S is $\nabla(8x - y^2) = 8\,\hat{i} - 2y\,\hat{j}$

therefore, $\hat{n} = \dfrac{8\,\hat{i} - 2y\,\hat{j}}{\sqrt{64 + 4y^2}} = \dfrac{4\,\hat{i} - y\,\hat{j}}{\sqrt{16 + y^2}}$

Also $\hat{n}\cdot\hat{i} = \dfrac{4}{\sqrt{16 + y^2}}$ and $\vec{A}\cdot\hat{n} = \dfrac{8y + zy}{\sqrt{16 + y^2}}$

Thus equation (1) becomes

$$\iint_S \vec{A}\cdot\hat{n}\,dS = \int_0^4\int_0^6 \frac{8y + zy}{\sqrt{16 + y^2}}\,\frac{\sqrt{16 + y^2}}{4}\,dz\,dy$$

$$= \frac{1}{4}\int_0^4\int_0^6 (8y + zy)\,dz\,dy = \frac{1}{4}\int_0^4 \left| 8yz + y\frac{z^2}{2} \right|_0^6 dy$$

$$= \frac{1}{4}\int_0^4 66\,y\,dy = \frac{33}{4}\,|y^2|_0^4 = \frac{33}{4}(16) = 132$$

PROBLEM (12): **If $\vec{A} = 6z\,\hat{i} + (2x + y)\,\hat{j} - x\,\hat{k}$, evaluate $\iint_S \vec{A}\cdot\hat{n}\,dS$ over the entire surface S of the region bounded by the cylinder $x^2 + z^2 = 9$, $x = 0$, $y = 0$, $z = 0$ and $y = 8$.**

SOLUTION: The surface S of the region bounded by the given curves is shown in figure (5.40).

Then
$$\iint_S \vec{A}\cdot\hat{n}\,dS = \iint_{S_1} \vec{A}\cdot\hat{n}\,dS_1 + \iint_{S_2} \vec{A}\cdot\hat{n}\,dS_2 + \iint_{S_3} \vec{A}\cdot\hat{n}\,dS_3 + \iint_{S_4} \vec{A}\cdot\hat{n}\,dS_4 + \iint_{S_5} \vec{A}\cdot\hat{n}\,dS_5$$

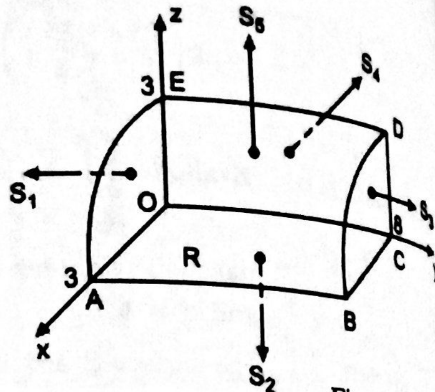

Figure (5.40)

For S_1 (OEA), $y = 0$, $\hat{n} = -\hat{j}$ and

$$\iint_{S_1} \vec{A}\cdot\hat{n}\,dS_1 = \int_0^3 \int_0^{\sqrt{9-x^2}} (6z\hat{i}+2x\hat{j}-x\hat{k})\cdot(-\hat{j})\,dz\,dx$$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} -2x\,dz\,dx = \int_0^3 \sqrt{9-x^2}\,(-2x)\,dx = -18$$

For S_2 (OABC), $z = 0$, $\hat{n} = -\hat{k}$ and

$$\iint_{S_2} \vec{A}\cdot\hat{n}\,dS_2 = \int_0^3 \int_0^8 [(2x+y)\hat{j}-x\hat{k}]\cdot(-\hat{k})\,dy\,dx = \int_0^3 \int_0^8 x\,dy\,dx = 36$$

For S_3 (CDB), $y = 8$, $\hat{n} = \hat{j}$ and

$$\iint_{S_3} \vec{A}\cdot\hat{n}\,dS_3 = \int_0^3 \int_0^{\sqrt{9-x^2}} [6z\hat{i}+(2x+8)\hat{j}-x\hat{k}]\cdot\hat{j}\,dz\,dx$$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} (2x+8)\,dz\,dx = \int_0^3 (2x+8)\sqrt{9-x^2}\,dx = 18+18\pi$$

For S_4 (OCDE), $x = 0$, $\hat{n} = -\hat{i}$ and

$$\iint_{S_4} \vec{A}\cdot\hat{n}\,dS_4 = \int_0^8 \int_0^3 (6z\hat{i}+y\hat{j})\cdot(-\hat{i})\,dz\,dy = \int_0^8 \int_0^3 -6z\,dz\,dy = -216$$

For S_5 (ABDE), a vector normal to S_5 is $\nabla(x^2+z^2) = 2x\hat{i}+2z\hat{k}$

therefore, $\hat{n} = \dfrac{2x\hat{i}+2z\hat{k}}{\sqrt{4x^2+4z^2}} = \dfrac{x}{3}\hat{i}+\dfrac{z}{3}\hat{k}$. Also $\hat{n}\cdot\hat{k} = \dfrac{z}{3}$.

... for the projection R on the xy-plane.

$$\iint_{S_5} \bar{A}\cdot\hat{n}\,dS_5 = \int_0^3\int_0^8 [6z\hat{i} + (2x+y)\hat{j} - x\hat{k}]\cdot\left(\frac{x}{3}\hat{i}+\frac{z}{3}\hat{k}\right)\frac{3}{z}\,dy\,dx$$

$$= \int_0^3\int_0^8 \left(2xz - \frac{xz}{3}\right)\frac{3}{z}\,dy\,dx = \int_0^3\int_0^8 5x\,dy\,dx = 180.$$

$$\therefore \iint_S \bar{A}\cdot\hat{n}\,dS = -18 + 36 + 18 + 18\pi - 216 + 180 = 18\pi$$

PROBLEM (13): If $\bar{A} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$, evaluate $\iint_S \bar{A}\cdot\hat{n}\,dS$ over the entire surface of the region above the xy-plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$.

SOLUTION: The surface S and its projection R on the xy-plane are shown in figure (5.41).

$$\therefore \iint_S \bar{A}\cdot\hat{n}\,dS = \iint_{S_1} \bar{A}\cdot\hat{n}\,dS_1 + \iint_{S_2} \bar{A}\cdot\hat{n}\,dS_2$$

On S_1, $z = 4$, $\hat{n} = \hat{k}$ and $\hat{n}\cdot\hat{k} = 1$

$$\therefore \iint_{S_1} \bar{A}\cdot\hat{n}\,dS_1 = \iint_{S_1} 12\,dS_1 = 12\iint_R dy\,dx$$

$$= 12\pi(4)^2 = 192\pi$$

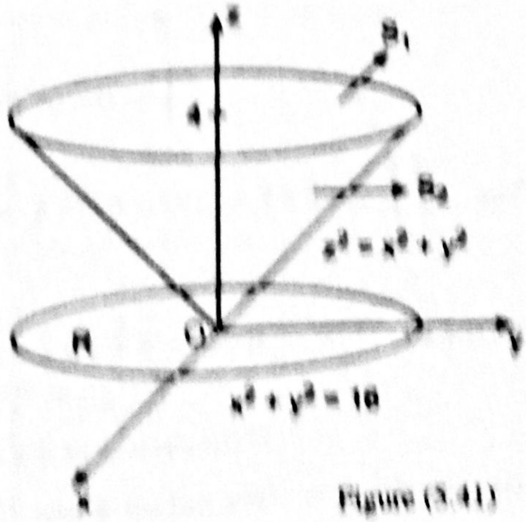

Figure (5.41)

On S_2, the normal vector is given by

$$\nabla(x^2+y^2-z^2) = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$$

Hence $\hat{n} = \dfrac{2x\hat{i}+2y\hat{j}-2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{x\hat{i}+y\hat{j}-z\hat{k}}{\sqrt{2}\,z}$

$\therefore$ $\bar{A}\cdot\hat{n} = \dfrac{1}{\sqrt{2}}(4x^2 + xy^2z - 3z)$ and $\hat{n}\cdot\hat{k} = -\dfrac{1}{\sqrt{2}}$

$$\text{Hence } \iint_{S_2} \bar{A}\cdot\hat{n}\,dS_2 = \iint_R \bar{A}\cdot\hat{n}\,\frac{dy\,dx}{|\hat{n}\cdot\hat{k}|}$$

$$= \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \frac{1}{\sqrt{2}}(4x^2 + xy^2z - 3z)\sqrt{2}\,dy\,dx$$

$$= \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left(4x^2 + x y^2 \sqrt{x^2+y^2} - 3\sqrt{x^2+y^2} \right) dy\, dx$$

Let $x = r\cos\theta$, $y = r\sin\theta$, therefore $0 \le r \le 4$ and $0 \le \theta \le 2\pi$

and
$$\iint_{S_2} \vec{A}\cdot\hat{n}\, dS_2 = \int_0^{2\pi}\int_0^4 \left(4r^2\cos^2\theta + r^4\cos\theta\sin^2\theta - 3r\right) r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left| r^4\cos^2\theta + \frac{r^6}{6}\cos\theta\sin^2\theta - r^3 \right|_0^4 d\theta$$

$$= \int_0^{2\pi} \left(256\cos^2\theta + \frac{(4)^6}{6}\cos\theta\sin^2\theta - 64 \right) d\theta$$

$$= \int_0^{2\pi} \left[128(1+\cos 2\theta) + \frac{2048}{3}\cos\theta\sin^2\theta - 64 \right] d\theta$$

$$= \left| 64\theta + 64\sin 2\theta + \frac{2048}{9}\sin^3\theta \right|_0^{2\pi} = 128\pi$$

Thus $\iint_S \vec{A}\cdot\hat{n}\, dS = 192\pi + 128\pi = 320\pi$

PROBLEM (14): **If $\vec{A} = x\hat{i} + y\hat{j} - 2z\hat{k}$, evaluate $\iint_S \vec{A}\cdot\hat{n}\, dS$, where S is the surface of the sphere $x^2+y^2+z^2 = a^2$ above the xy – plane.**

SOLUTION: The surface S and its projection R on the xy–plane is shown in figure (5.42). A normal to the surface of the sphere $x^2+y^2+z^2 = a^2$ is

$$\nabla(x^2+y^2+z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Then the unit normal $\hat{n}$ to any point of S is

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x}{a}\hat{i} + \frac{y}{a}\hat{j} + \frac{z}{a}\hat{k}$$

since $x^2+y^2+z^2 = a^2$. Also $\hat{n}\cdot\hat{k} = \frac{z}{a}$

Then
$$\iint_S \vec{A}\cdot\hat{n}\, dS = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\, dy}{|\hat{n}\cdot\hat{k}|}$$

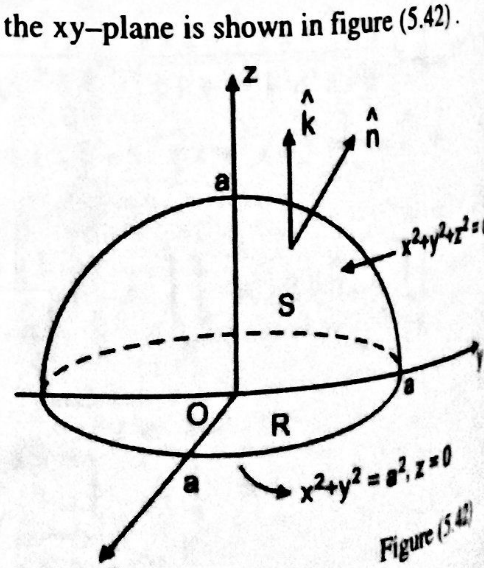

Figure (5.42)

$$= \iint_R (x\hat{i} + y\hat{j} - 2z\hat{k}) \cdot \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{a} \frac{a}{z}\, dx\, dy$$

$$= \iint_R \frac{(x^2 + y^2 - 2z^2)}{z}\, dx\, dy$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{3(x^2+y^2) - 2a^2}{\sqrt{a^2 - x^2 - y^2}}\, dy\, dx$$

(since $z^2 = a^2 - x^2 - y^2$ from the equation of the sphere S)

To evaluate this double integral, let $x = r\cos\theta$, $y = r\sin\theta$ and $dy\,dx = r\,dr\,d\theta$. Then the double integral becomes

$$\iint_S (\nabla \times \vec{A}) \cdot \hat{n}\, dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \frac{3r^2 - 2a^2}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{a} \frac{3(r^2 - a^2) + a^2}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{a} \left(-3r\sqrt{a^2 - r^2} + \frac{a^2 r}{\sqrt{a^2 - r^2}} \right) dr\, d\theta$$

$$= \int_0^{2\pi} \left| (a^2 - r^2)^{3/2} - a^2\sqrt{a^2 - r^2} \right|_0^a d\theta$$

$$= \int_0^{2\pi} (a^3 - a^3)\, d\theta = 0$$

PROBLEM (15): **Evaluate $\iint_S \vec{r} \cdot \hat{n}\, dS$ where S is the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$**

SOLUTION:

The surface S of the ellipsoid and its projection R on the xy–plane are shown in Figure (5.43). We know that

$$\iint_S \vec{r} \cdot \hat{n}\, dS = \iint_R \vec{r} \cdot \hat{n} \frac{dy\, dx}{|\hat{n} \cdot \hat{k}|} \qquad (1)$$

Also, we know that a normal vector to the surface S is given by

$$\nabla\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) = \frac{2x}{a^2}\hat{i}+\frac{2y}{b^2}\hat{j}+\frac{2z}{c^2}\hat{k}$$

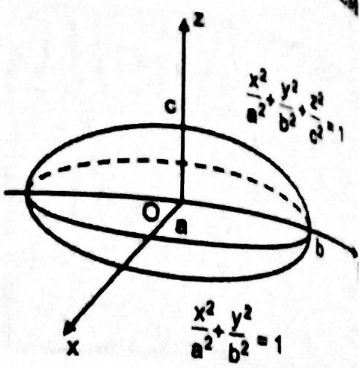

Figure (5.43)

Then a unit normal $\hat{n}$ to any point of S is

$$\hat{n} = \frac{(2x/a^2)\,\hat{i}+(2y/b^2)\,\hat{j}+(2z/c^2)\,\hat{k}}{\sqrt{(4x^2/a^4)+(4y^2/b^4)+(4z^2/c^4)}}$$

$$= \frac{(x/a^2)\,\hat{i}+(y/b^2)\,\hat{j}+(z/c^2)\,\hat{k}}{\sqrt{(x^2/a^4)+(y^2/b^4)+(z^2/c^4)}}$$

Also $\quad \bar{r}\cdot\hat{n} = \dfrac{(x^2/a^2)+(y^2/b^2)+(z^2/c^2)}{\sqrt{(x^2/a^4)+(y^2/b^4)+(z^2/c^4)}} = \dfrac{1}{\sqrt{(x^2/a^4)+(y^2/b^4)+(z^2/c^4)}}$

and $\quad \hat{n}\cdot\hat{k} = \dfrac{z/c^2}{\sqrt{(x^2/a^4)+(y^2/b^4)+(z^2/c^4)}}$

Thus for the projection R on the xy–plane, equation (1) becomes

$$\iint_S \bar{r}\cdot\hat{n}\,dS = 2\int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} \frac{1}{\sqrt{(x^2/a^4)+(y^2/b^4)+(z^2/c^4)}}\;\frac{\sqrt{(x^2/a^4)+(y^2/b^4)+(z^2/c^4)}}{z/c^2}\,dy\,dx$$

$$= 8\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} \frac{c^2}{z}\,dy\,dx$$

$$= 8\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} \frac{c^2}{c\sqrt{1-(x^2/a^2)-(y^2/b^2)}}\,dy\,dx$$

$$= 8c\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} \frac{1}{\sqrt{1-(x^2/a^2)-(y^2/b^2)}}\,dy\,dx$$

Let $\quad y = b\sqrt{1-x^2/a^2}\,\sin\theta$, then $dy = b\sqrt{1-x^2/a^2}\,\cos\theta\,d\theta$, and $0\le\theta\le\pi/2$.

Then $\quad \displaystyle\iint_S \bar{r}\cdot\hat{n}\,dS = 8c\int_0^a\int_0^{\pi/2} \frac{b\sqrt{1-x^2/a^2}\,\cos\theta}{\sqrt{1-x^2/a^2}\,\cos\theta}\,d\theta\,dx$

$$= 8c\int_0^a\int_0^{\pi/2} b\,d\theta\,dx = 8bc\int_0^a\int_0^{\pi/2} d\theta\,dx = 8bc\left(\frac{\pi}{2}\right)a = 4\pi abc$$