Chapter 5

LINE , SURFACE , AND VOLUME INTEGRALS AND RELATED INTEGRAL THEOREMS

5.1 INTRODUCTION

So far , we have dealt with derivative operations on vector fields . In this chapter , we shall define line integrals , surface integrals , and volume integrals and consider some important applications of these integrals . We shall see that a line integral is a natural generalization of the definite integral , the surface integral is a generalization of a double integral , and volume integral is a generalization of a triple integral of calculus .

Line integrals can be transformed into double integrals with the help of Green's theorem in the plane . With the help of Stokes' theorem , line integrals can be transformed into surface integrals , and conversely . Surface integrals can be transformed into triple integrals and conversely with the help of Gauss divergence theorem . These transformations are of great practical importance . The corresponding theorems of Green's , Stokes' , and Gauss serve as powerful tools in many practical as well as theoretical problems .

5.2 TANGENTIAL LINE INTEGRAL

Let $\vec{A}(x, y, z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be a vector point function which is defined and continuous along the arc AB of the space curve C . Subdivide the arc AB into n segments by means of the points $P_1, P_2, \ldots, P_{n-1}$ chosen arbitrarily and write $A = P_0$ and $B = P_n$ as shown in figure (5.1) . Consider one such segment $P_{k-1}P_k$ and let the arc length of this segment be Δs_k, $k = 1, 2, \ldots, n$. Let $Q_k(x_k, y_k, z_k)$ be any point on the segment $P_{k-1}P_k$ and define $\vec{A}(x_k, y_k, z_k) = \vec{A}_k$. Let $\hat{T}_k$ be unit tangent vector to C at Q_k.

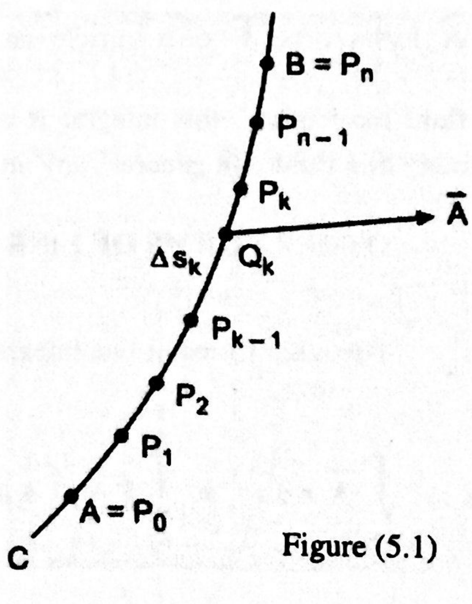

Figure (5.1)

We multiply the tangential component of $\vec{A}$ at Q_k with the arc length Δs_k of the corresponding segment $P_{k-1}P_k$ and form the sum $\sum_{k=1}^{n} \vec{A}_k \cdot \hat{T}_k \, \Delta s_k$.

Now take the limit of this sum as $n \to \infty$ in such a way that the arc length of each segment $\Delta s_k \to 0$. This limit, if it exists, is called the tangential line integral of $\vec{A}$ along C from A to B and is denoted

by $\int_A^B \vec{A}\cdot\hat{T}\,ds$ or $\int_C \vec{A}\cdot\hat{T}\,ds$

i.e. $\underset{n\to\infty}{\text{Lt}} \sum_{k=1}^{n} \vec{A}_k\cdot\hat{T}_k\,\Delta s_k = \int_A^B \vec{A}\cdot\hat{T}\,ds$

Since $\hat{T} = \frac{d\vec{r}}{ds}$ where $\vec{r}$ is the position vector of any point on C, it is usual to put $\hat{T}\,ds = d\vec{r}$, and

thus the line integral $\int_A^B \vec{A}\cdot\hat{T}\,ds = \int_A^B \vec{A}\cdot d\vec{r} = \int_C \vec{A}\cdot d\vec{r} = \int_C A_1 dx + A_2 dy + A_3 dz$

where $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ is called the **differential displacement vector**.

The line integral $\int_C \vec{A}\cdot d\vec{r}$ is sometimes called a scalar line integral of a vector field $\vec{A}$.

If C is a closed curve which we shall suppose a simple closed curve (i.e. a curve which does not intersect itself anywhere), the line integral around C is often denoted by

$$\oint_C \vec{A}\cdot d\vec{r} = \oint_C A_1 dx + A_2 dy + A_3 dz$$

If $\vec{A}$ is the force $\vec{F}$ on a particle moving along C, this line integral represents the work done by a force.

In fluid mechanics, this integral is called the circulation of $\vec{A}$ around C, where $\vec{A}$ represents the velocity of a fluid. In general, any integral which is to be evaluated along a curve is called a line integral.

OTHER FORMS OF LINE INTEGRALS

The other forms of line integrals are $\int_C \phi\,d\vec{r} = \hat{i}\int_C \phi\,dx + \hat{j}\int_C \phi\,dy + \hat{k}\int_C \phi\,dz$

and $\int_C \vec{A}\times d\vec{r} = \int_C (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\times(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$$= \hat{i}\int_C (A_2 dz - A_3 dy) + \hat{j}\int_C (A_3 dx - A_1 dz) + \hat{k}\int_C (A_1 dy - A_2 dx)$$

Thus integrals that involve differential displacement vector $d\vec{r}$ are called **line integrals**.

GENERAL PROPERTIES OF LINE INTEGRALS

The following are the properties of line integrals that are useful in computation and applications:

(i) $$\int_C K\vec{A}\cdot d\vec{r} = K\int_C \vec{A}\cdot d\vec{r} \quad (\text{K any real constant})$$

(ii) $$\int_C (\vec{A}+\vec{B})\cdot d\vec{r} = \int_C \vec{A}\cdot d\vec{r} + \int_C \vec{B}\cdot d\vec{r}$$

(iii) $$\int_C \vec{A}\cdot d\vec{r} = \int_{C_1} \vec{A}\cdot d\vec{r} + \int_{C_2} \vec{A}\cdot d\vec{r}$$

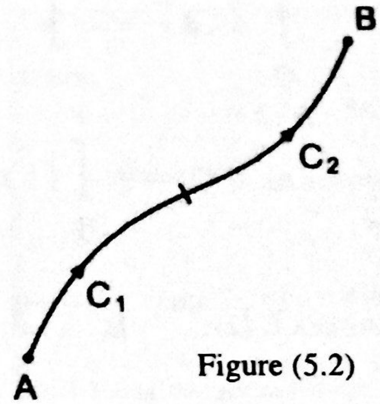

Figure (5.2)

where the path C is subdivided into two arcs C_1 and C_2 that have the same orientation as C as shown in figure (5.2). If the sense of orientation along C is reversed, the value of the integral is multiplied by -1.

(iv) If C is piecewise smooth, consisting of smooth curves $C_1, C_2, \ldots, C_n$ as shown in figure (5.3), the line integral of $\vec{A}$ over C is defined as the sum of the line integrals of $\vec{A}$ over each of the smooth curves making up C:

$$\int_C \vec{A}\cdot d\vec{r} = \int_{C_1} \vec{A}\cdot d\vec{r} + \int_{C_2} \vec{A}\cdot d\vec{r} + \ldots + \int_{C_n} \vec{A}\cdot d\vec{r}$$

In this sum, the orientation along C must be maintained over the curves $C_1, C_2, \ldots, C_n$. That is, the initial point of C_j is the terminal point of C_{j-1}. This requirement is indicated by the arrows as shown in figure (5.3).

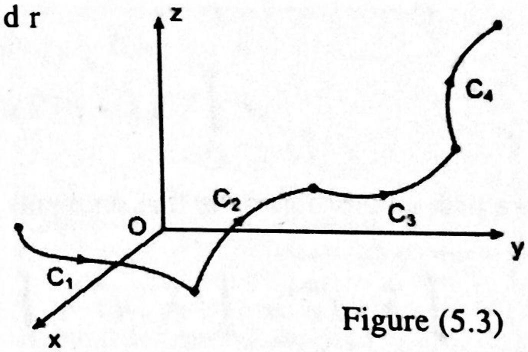

Figure (5.3)

EXAMPLE (1): **If $\vec{A} = 3xy\,\hat{i} - y^2\,\hat{j}$, evaluate $\int_C \vec{A}\cdot d\vec{r}$ where C is the curve in the xy-plane, $y = 2x^2$, from $(0,0)$ to $(1,2)$.**

SOLUTION: The curve C defined by $y = 2x^2$ in the xy-plane is shown in figure (5.4). Since the integration is performed in the xy-plane ($z = 0$), we can take

$\vec{r} = x\,\hat{i} + y\,\hat{j}$, therefore $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$.

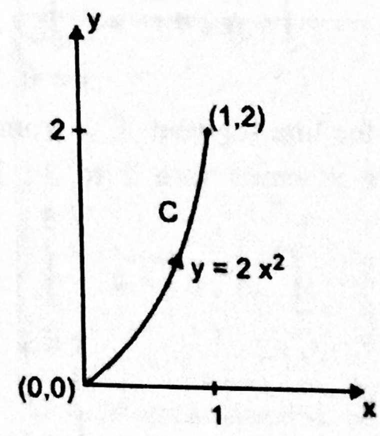

Figure (5.4)

Thus
$$\int_C \vec{A}\cdot d\vec{r} = \int_C (3xy\,\hat{i} - y^2\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C 3xy\,dx - y^2\,dy$$

Let $y = 2x^2$, then $dy = 4x\,dx$. Also x varies from 0 to 1.

$$\int_C \vec{A}\cdot d\vec{r} = \int_{x=0}^{1} 3x(2x^2)\,dx - 4x^4(4x)\,dx$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx = \left|\frac{3}{2}x^4 - \frac{8}{3}x^6\right|_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}$$

EXAMPLE (2): **If $\vec{A} = (2x+y)\,\hat{i} + (3y-x)\,\hat{j}$, evaluate $\int_C \vec{A}\cdot d\vec{r}$ where C is the curve in the xy–plane consisting of the line segment C_1 from $(0,0)$ to $(2,0)$ and then the line segment C_2 from $(2,0)$ to $(3,2)$.**

SOLUTION: The path C consisting of line segments C_1 and C_2 is shown in figure (5.5).

Since integration is performed in the xy–plane, therefore $\vec{r} = x\,\hat{i} + y\,\hat{j}$ and so $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$.

Then $$\int_C \vec{A}\cdot d\vec{r} = \int_C [(2x+y)\,\hat{i} + (3y-x)\,\hat{j}]\cdot[dx\,\hat{i} + dy\,\hat{j}]$$

$$= \int_C (2x+y)\,dx + (3y-x)\,dy \qquad (1)$$

For a path consisting of the line segments C_1 and C_2, we have

$$\int_C \vec{A}\cdot d\vec{r} = \int_{C_1} \vec{A}\cdot d\vec{r} + \int_{C_2} \vec{A}\cdot d\vec{r} \qquad (2)$$

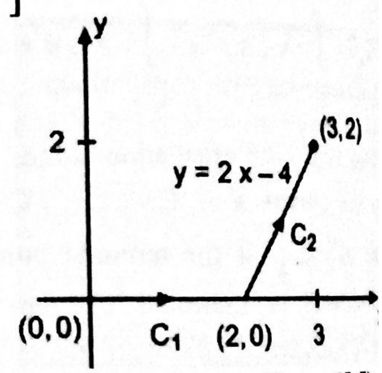

Figure (5.5)

On the line segment C_1 from $(0,0)$ to $(2,0)$, $y = 0$ and so $dy = 0$, while x varies from 0 to 2. The integral (1) over this part of the path is

$$\int_{C_1} \vec{A}\cdot d\vec{r} = \int_{x=0}^{2} 2x\,dx = \left|x^2\right|_0^2 = 4$$

For the line segment C_2 from $(2,0)$ to $(3,2)$, the equation is $y = 2x - 4$ and so $dy = 2dx$ while x varies from 2 to 3. The integral (1) over this part of the path is

$$\int_{C_2} \vec{A}\cdot d\vec{r} = \int_{x=2}^{3} [2x + (2x-4)]\,dx + [3(2x-4) - x]\,2\,dx$$

$$= \int_2^3 (14x - 28)\,dx = \left|7x^2 - 28x\right|_2^3 = (63 - 84) - (28 - 56) = 7$$

From equation (2) we get

$$\int_C \vec{A}\cdot d\vec{r} = 4+7 = 11$$

EXAMPLE (3): **If $\vec{A} = (x-3y)\hat{i} + (y-2x)\hat{j}$, evaluate $\oint_C \vec{A}\cdot d\vec{r}$ where C is an ellipse $\frac{x^2}{9}+\frac{y^2}{4} = 1$ in the xy–plane traversed in the positive (counterclockwise) direction.**

SOLUTION: The curve C which is an ellipse with semi–major axis as 3 and semi–minor axis as 2 is shown in figure (5.6). Since the integration is performed in the xy-plane, we take $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

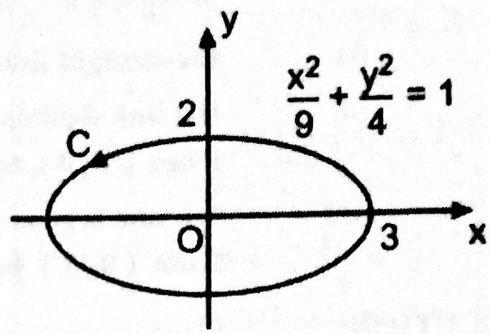

Figure (5.6)

Thus $$\oint_C \vec{A}\cdot d\vec{r} = \oint_C (x-3y)\,dx + (y-2x)\,dy \qquad (1)$$

The parametric equations of this ellipse are $x = 3\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$ therefore, $dx = -3\sin t\,dt$, $dy = 2\cos t\,dt$. Hence from equation (1), we get

$$\oint_C \vec{A}\cdot d\vec{r} = \int_{t=0}^{2\pi} (3\cos t - 6\sin t)(-3\sin t\,dt) + (2\sin t - 6\cos t)(2\cos t\,dt)$$

$$= \int_0^{2\pi} (-5\sin t\cos t + 18\sin^2 t - 12\cos^2 t)\,dt$$

$$= \int_0^{2\pi} \left[-\frac{5}{2}\sin 2t + 9(1-\cos 2t) - 6(1+\cos 2t)\right]dt$$

$$= \left|\frac{5}{4}\cos 2t + 9\left(t - \frac{\sin 2t}{2}\right) - 6\left(t + \frac{\sin 2t}{2}\right)\right|_0^{2\pi}$$

$$= \left|\frac{5}{4}\cos 2t + 3t - \frac{15}{2}\sin 2t\right|_0^{2\pi}$$

$$= \left(\frac{5}{4}\cos 4\pi + 6\pi\right) - \left(\frac{5}{4}\cos 0\right)$$

$$= \frac{5}{4} + 6\pi - \frac{5}{4} = 6\pi$$

5.3 LINE INTEGRAL DEPENDENT ON PATH (SAME END POINTS)

We now show that the value of a line integral $\int_C \vec{A} \cdot d\vec{r}$ in general, depends not only on the end points P_1 and P_2 of the path C but also on the geometric shape of the path C ; i.e. if we integrate from P_1 to P_2 along different paths, we in general, obtain different values of the integral.

EXAMPLE (4): **If $\vec{A} = (1 + x^2 y)\hat{i} + 2xy\hat{j}$, evaluate $\int_C \vec{A} \cdot d\vec{r}$ from (0 , 0) to (1, 1) along the following paths C :**

(i) **the straight line from (0 , 0) to (1 , 1).**

(ii) **the line segment C_1 from (0 , 0) to (1 , 0) and then the line segment C_2 from (1 , 0) to (1 , 1).**

(iii) **the line segment C_1 from (0 , 0) to (0 , 1) and then the line segment C_2 from (0 , 1) to (1 , 1).**

SOLUTION: Since integration is performed in the xy–plane, therefore $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$,

and so $\int_C \vec{A} \cdot d\vec{r} = \int_C (1 + x^2 y)\,dx + 2xy\,dy$ (1)

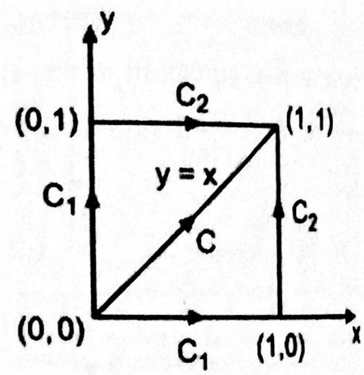

Figure (5.7)

where the path C in each case is shown in figure (5.7).

(i) Along the straight line from (0 , 0) to (1 , 1), $y = x$, $dy = dx$ while x varies from 0 to 1. The line integral (1) becomes

$$\int_C \vec{A} \cdot d\vec{r} = \int_0^1 [1 + x^2(x)]\,dx + 2x(x)\,dx = \int_0^1 (1 + 2x^2 + x^3)\,dx$$

$$= \left| x + \frac{2}{3}x^3 + \frac{1}{4}x^4 \right|_0^1 = 1 + \frac{2}{3} + \frac{1}{4} = \frac{23}{12}$$

(ii) In this case we have

$$\int_C \vec{A} \cdot d\vec{r} = \int_{C_1} \vec{A} \cdot d\vec{r} + \int_{C_2} \vec{A} \cdot d\vec{r} \qquad (2)$$

where C is the curve consisting of the line segments C_1 and C_2 as shown in figure (5.7).

Along the line segment C_1 from (0 , 0) to (1 , 0), $y = 0$, $dy = 0$, while x varies from 0 to 1.

The integral (1) over this part of the path is $\int_{C_1} \vec{A} \cdot d\vec{r} = \int_{x=0}^{1} dx = 1$

Along the line segment C_2 from $(1,0)$ to $(1,1)$, $x = 1$, $dx = 0$, while y varies from 0 to 1.

The integral (1) over this part of the path is $\int_{C_2} \vec{A}\cdot d\vec{r} = \int_{y=0}^{1} 2y\,dy = 1$

From equation (2), we get $\int_C \vec{A}\cdot d\vec{r} = 1 + 1 = 2$

(iii) Along the line segment C_1 from $(0,0)$ to $(0,1)$, $x = 0$, $dx = 0$, while y varies from 0 to 1.

The integral (1) over this part of the path is $\int_{C_1} \vec{A}\cdot d\vec{r} = \int_{x=0}^{1} 0\,dy = 0$

Along the line segment C_2 from $(0,1)$ to $(1,1)$, $y = 1$, $dy = 0$, while x varies from 0 to 1.

The integral (1) over this part of the path is $\int_{C_2} \vec{A}\cdot d\vec{r} = \int_{x=0}^{1} (1+x^2)\,dx = \left| x + \frac{x^3}{3} \right|_0^1 = \frac{4}{3}$

From equation (2), we get $\int_C \vec{A}\cdot d\vec{r} = 0 + \frac{4}{3} = \frac{4}{3}$.

NOTE: In example (4), we have seen that the value of a line integral $\int_C \vec{A}\cdot d\vec{r}$ in general, depends on the path C joining the points P_1 and P_2. We now show that for certain types of vector functions the value of the line integral will depend only on P_1 and P_2 but will not depend on the path C from P_1 to P_2. We first state the following definition.

5.4 LINE INTEGRAL INDEPENDENT OF PATH (OR CONSERVATIVE FIELD)

The line integral $\int_C \vec{A}\cdot d\vec{r}$ is said to be independent of the path C (or the vector field $\vec{A}$ is conservative) in a given region R, if the value of the line integral $\int_{P_1}^{P_2} \vec{A}\cdot d\vec{r}$ is the same for all paths C joining any two given points P_1 and P_2 in R. Thus as shown in figure (5.8), the line integral is independent of the path C if the integrals along C_1, C_2, C_3 have the same value.

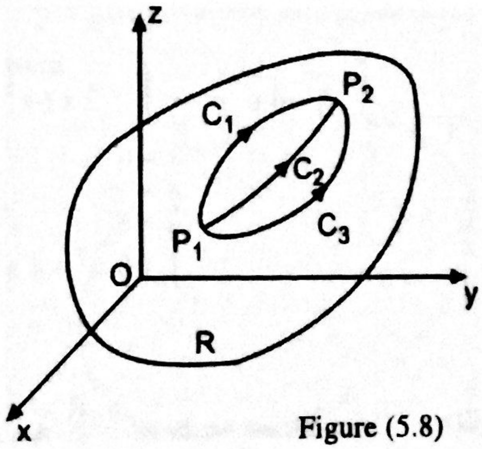

Figure (5.8)

EXAMPLE (5): If $\vec{A} = 2xy^2\,\hat{i} + 2(x^2y + y)\,\hat{j}$, evaluate $\int_C \vec{A}\cdot d\vec{r}$ from $(0,0)$ to $(2,4)$ along the following paths C:

(i) the straight line $y = 2x$

(ii) the parabola $y = x^2$

(iii) the line segment C_1 from $(0,0)$ to $(2,0)$ and then the line segment C_2 from $(2,0)$ to $(2,4)$.

SOLUTION: Since integration is performed in the xy–plane, therefore $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

Then
$$\int_C \vec{A}\cdot d\vec{r} = \int_C 2xy^2\,dx + 2(x^2y + y)\,dy \qquad (1)$$

where the path C in each case is shown in figure (5.9).

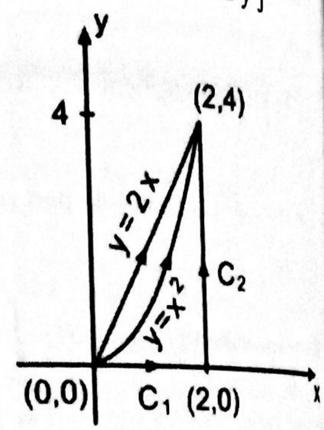

Figure (5.9)

(i) Along the straight line $y = 2x$, we have $dy = 2\,dx$ while x varies from 0 to 2.

The line integral (1) becomes

$$\int_C \vec{A}\cdot d\vec{r} = \int_{x=0}^{2} 2x(2x)^2\,dx + 2\left[x^2(2x) + 2x\right]2\,dx$$

$$= \int_0^2 (16x^3 + 8x)\,dx = \left|4x^4 + 4x^2\right|_0^2 = 64 + 16 = 80.$$

(ii) Along the parabola $y = x^2$, we have $dy = 2x\,dx$ while x varies from 0 to 2.

The line integral (1) becomes

$$\int_C \vec{A}\cdot d\vec{r} = \int_{x=0}^{2} 2x(x^2)^2\,dx + 2\left[x^2(x^2) + x^2\right]2x\,dx$$

$$= \int_0^2 (6x^5 + 4x^3)\,dx = \left|x^6 + x^4\right|_0^2 = 64 + 16 = 80.$$

(iii) In this case we have
$$\int_C \vec{A}\cdot d\vec{r} = \int_{C_1} \vec{A}\cdot d\vec{r} + \int_{C_2} \vec{A}\cdot d\vec{r} \qquad (2)$$

where C is the curve consisting of the line segments C_1 and C_2.

Along the line segment C_1, $y = 0$, therefore $dy = 0$, while x varies from 0 to 2. Thus

$$\int_{C_1} \vec{A}\cdot d\vec{r} = \int_{x=0}^{2} 0\, dx = 0$$

Along the line segment C_2, $x = 2$, therefore $dx = 0$ while y varies from 0 to 4.

$$\int_{C_2} \vec{A}\cdot d\vec{r} = \int_{y=0}^{4} 2(4y+y)\,dy = \int_{0}^{4} 10y\,dy = 5\left|y^2\right|_0^4 = 5(16) = 80$$

From equation (2), we get $\displaystyle\int_C \vec{A}\cdot d\vec{r} = 0 + 80 = 80$

5.5 THEOREMS ON LINE INTEGRALS INDEPENDENT OF PATH

THEOREM (5.1): Prove that a necessary and sufficient condition for $\displaystyle\int_{P_1}^{P_2} \vec{A}\cdot d\vec{r}$ to be independent of the path joining any two points P_1 and P_2 (i.e. $\vec{A}$ to be conservative) in a given region is that $\displaystyle\oint_C \vec{A}\cdot d\vec{r} = 0$ for all closed paths C in the region.

PROOF: Let C be any simple closed curve, and let P_1 and P_2 be any two points on C as shown in figure (5.10). Then since by hypothesis, the integral is independent of the path

(i.e. $\bar{A}$ is conservative), we have $\displaystyle\int_{P_1AP_2} \vec{A}\cdot d\vec{r} = \int_{P_1BP_2} \vec{A}\cdot d\vec{r}$

Reversing the direction of integration in the integral on the right, we have

$$\int_{P_1AP_2} \vec{A}\cdot d\vec{r} = -\int_{P_2BP_1} \vec{A}\cdot d\vec{r}$$

or

$$\int_{P_1AP_2} \vec{A}\cdot d\vec{r} + \int_{P_2BP_1} \vec{A}\cdot d\vec{r} = 0$$

or

$$\oint_C \vec{A}\cdot d\vec{r} = 0$$

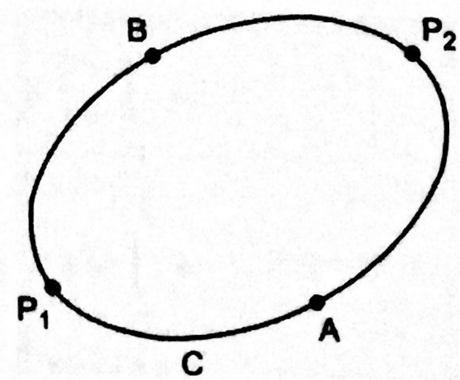

Figure (5.10)

Conversely, if $\oint_C \vec{A}\cdot d\vec{r} = 0$, then $\int_{P_1AP_2} \vec{A}\cdot d\vec{r} + \int_{P_2BP_1} \vec{A}\cdot d\vec{r} = 0$

or $\int_{P_1AP_2} \vec{A}\cdot d\vec{r} - \int_{P_1BP_2} \vec{A}\cdot d\vec{r} = 0$

or $\int_{P_1AP_2} \vec{A}\cdot d\vec{r} = \int_{P_1BP_2} \vec{A}\cdot d\vec{r}$

which shows that the line integral is independent of the path joining P_1 and P_2 as required.

SCALAR POTENTIAL FUNCTION

A scalar potential function ϕ is a single–valued function for which there exists a continuous vector field $\vec{A}$ in a simply connected region R that satisfies the relation $\vec{A} = \nabla\phi$.

THEOREM (5.2): Prove that a necessary and sufficient condition for $\int_C \vec{A}\cdot d\vec{r}$ to be independent of the path C joining any two points $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$ (i.e. $\vec{A}$ to be conservative) is that there exists a scalar function ϕ such that $\vec{A} = \nabla\phi$, where ϕ is single valued and has continuous partial derivatives.

PROOF: Let $\vec{A} = \nabla\phi$, then

$$\int_C \vec{A}\cdot d\vec{r} = \int_{P_1}^{P_2} \vec{A}\cdot d\vec{r} = \int_{P_1}^{P_2} \nabla\phi\cdot d\vec{r}$$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

If $\phi(x,y,z)$ then $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$= \int_{P_1}^{P_2} \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1) = \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

Thus the line integral depends only on points P_1 and P_2 and not on the path joining them i.e. $\vec{A}$ is conservative.

Conversely, let $\int_C \vec{A}\cdot d\vec{r}$ be independent of the path C joining any two points. We choose these points as a fixed point $P_1 \equiv (x_1, y_1, z_1)$ and a variable point $P_2 \equiv (x, y, z)$, so that the result is a function only of the coordinates (x , y , z) of the variable end point. Then

$$\phi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \vec{A}\cdot d\vec{r} = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \vec{A}\cdot\frac{d\vec{r}}{ds}\,ds$$

By differentiation, $\frac{d\phi}{ds} = \vec{A}\cdot\frac{d\vec{r}}{ds}$ —(1)

But $\frac{d\phi}{ds} = \frac{\partial\phi}{\partial s} = \nabla\phi\cdot\frac{d\vec{r}}{ds}$ —(2)

If $\phi = \int_{P_1}^{P_2} \vec{A}\cdot\frac{d\vec{r}}{ds}\,ds$

Taking derivative w.r.t. s

$\frac{d\phi}{ds} = \vec{A}\cdot\frac{d\vec{r}}{ds}$

$\nabla\phi\cdot\frac{d\vec{r}}{ds} = \left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right)\cdot\left(\frac{dx}{ds}i + \frac{dy}{ds}j + \frac{dz}{ds}k\right)$

$= \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds}$

$= \frac{d\phi}{ds}$ or $\frac{\partial\phi}{\partial s}$

From equations (1) and (2), we have $(\nabla\phi - \vec{A})\cdot\frac{d\vec{r}}{ds} = 0$

Since $\frac{d\vec{r}}{ds}$ is a unit tangent vector and $\neq \vec{0}$, therefore equation (3) implies that

$\nabla\phi - \vec{A} = 0$ or $\vec{A} = \nabla\phi$. Hence the theorem.

THEOREM (5.3): **Prove that a necessary and sufficient condition that a vector field $\vec{A}$ be conservative is that $\nabla\times\vec{A} = \vec{0}$ (i.e. $\vec{A}$ is irrotational).**

PROOF: If $\vec{A}$ is a conservative field then by theorem (5.2), we have $\vec{A} = \nabla\phi$.

Thus $\nabla\times\vec{A} = \nabla\times\nabla\phi = \vec{0}$

Conversely, if $\nabla\times\vec{A} = \vec{0}$, then

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \vec{0} \text{ and thus}$$

$$\frac{\partial A_3}{\partial y} = \frac{\partial A_2}{\partial z}, \quad \frac{\partial A_1}{\partial z} = \frac{\partial A_3}{\partial x}, \quad \frac{\partial A_2}{\partial x} = \frac{\partial A_1}{\partial y}$$

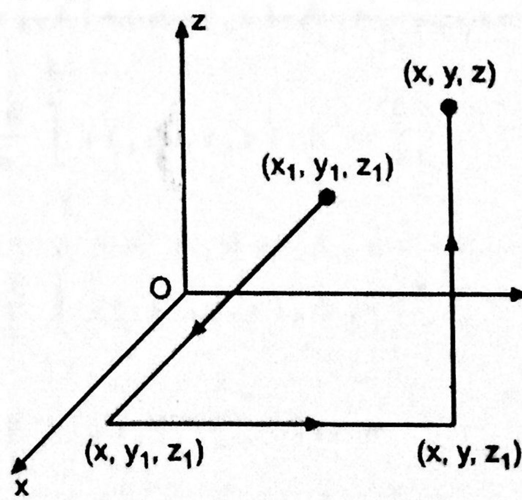

Figure (5.11)

We must prove that $\vec{A} = \nabla\phi$ follows as a consequence of this.

Now

$$\int_C \vec{A}\cdot d\vec{r} = \int_C A_1(x,y,z)\,dx + A_2(x,y,z)\,dy + A_3(x,y,z)\,dz$$

where C is a path joining (x_1, y_1, z_1) and (x , y , z).

Let us choose as a particular path, the straight line segments from (x_1, y_1, z_1) to (x, y_1, z_1) to (x, y, z_1) to (x, y, z) and call $\phi(x, y, z)$ the value of the integral along this particular path. Then omitting the integrand we have

$$\phi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x,y_1,z_1)} [\ \ ] + \int_{(x,y_1,z_1)}^{(x,y,z_1)} [\ \ ] + \int_{(x,y,z_1)}^{(x,y,z)} [\ \ ] \qquad (1)$$

(i) Along the straight line from (x_1, y_1, z_1) to (x, y_1, z_1), $y = \text{constant} = y_1$, $z = \text{constant} = z_1$ so that $dy = 0, dz = 0$, while x varies from x_1 to x.

(ii) Along the straight line from (x, y_1, z_1) to (x, y, z_1), $x = \text{constant}$, $z = \text{constant}$, so that $dx = 0, dz = 0$ while y varies from y_1 to y.

(iii) Along the straight line from (x, y, z_1) to (x, y, z), $x = \text{constant}$, $y = \text{constant}$ so that $dx = 0, dy = 0$ while z varies from z_1 to z. Thus we can write equation (1) as

$$\phi(x,y,z) = \int_{x_1}^{x} A_1(x, y_1, z_1)\,dx + \int_{y_1}^{y} A_2(x, y, z_1)\,dy + \int_{z_1}^{z} A_3(x, y, z)\,dz$$

It follows that $\dfrac{\partial \phi}{\partial z} = A_3(x, y, z)$

$$\frac{\partial \phi}{\partial y} = A_2(x, y, z_1) + \int_{z_1}^{z} \frac{\partial A_3}{\partial y}(x, y, z)\,dz$$

$$= A_2(x, y, z_1) + \int_{z_1}^{z} \frac{\partial A_2}{\partial z}(x, y, z)\,dz$$

$\because \dfrac{\partial A_3}{\partial y} = \dfrac{\partial A_2}{\partial z}$

$$= A_2(x, y, z_1) + \left| A_2(x, y, z) \right|_{z_1}^{z} = A_2(x, y, z)$$

$$\frac{\partial \phi}{\partial x} = A_1(x, y_1, z_1) + \int_{y_1}^{y} \frac{\partial A_2}{\partial x}(x, y, z_1)\,dy + \int_{z_1}^{z} \frac{\partial A_3}{\partial x}(x, y, z)\,dz$$

$\because \dfrac{\partial A_2}{\partial x} = \dfrac{\partial A_1}{\partial y}$ & $\dfrac{\partial A_3}{\partial x} = \dfrac{\partial A_1}{\partial z}$

$$= A_1(x, y_1, z_1) + \int_{y_1}^{y} \frac{\partial A_1}{\partial y}(x, y, z_1)\,dy + \int_{z_1}^{z} \frac{\partial A_1}{\partial z}(x, y, z)\,dz$$

$$= A_1(x, y_1, z_1) + \left| A_1(x, y, z_1) \right|_{y_1}^{y} + \left| A_1(x, y, z) \right|_{z_1}^{z}$$

$$= A_1(x, y_1, z_1) + A_1(x, y, z_1) - A_1(x, y_1, z_1) + A_1(x, y, z) - A_1(x, y, z_1)$$

$$= A_1(x, y, z)$$

Then $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k} = \dfrac{\partial \phi}{\partial x}\hat{i} + \dfrac{\partial \phi}{\partial y}\hat{j} + \dfrac{\partial \phi}{\partial z}\hat{k} = \nabla \phi$

and so $\bar{A}$ is a conservative field.

EXAMPLE (6): **Show that the vector field $\vec{A} = (\sin y + z)\,\hat{i} + (x\cos y - z)\,\hat{j} + (x - y)\,\hat{k}$ is conservative. Hence find the scalar potential function ϕ for which $\vec{A} = \nabla\phi$.**

SOLUTION: We know that a necessary and sufficient condition for a vector field $\vec{A}$ to be conservative is $\nabla \times \vec{A} = \vec{0}$

Now
$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x - y \end{vmatrix}$$

$$= (-1+1)\,\hat{i} + (1-1)\,\hat{j} + (\cos y - \cos y)\,\hat{k} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = \vec{0}$$

Thus the vector field $\vec{A}$ is a conservative.

Let
$$\vec{A} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = (\sin y + z)\,\hat{i} + (x\cos y - z)\,\hat{j} + (x - y)\,\hat{k}$$

Then
$$\frac{\partial\phi}{\partial x} = \sin y + z \qquad (1)$$
$$\frac{\partial\phi}{\partial y} = x\cos y - z \qquad (2)$$
$$\frac{\partial\phi}{\partial z} = x - y \qquad (3)$$

Integrating equations (1), (2), and (3), we get

$$\phi = x\sin y + xz + f(y,z)$$
$$\phi = x\sin y - yz + g(x,z)$$
$$\phi = xz - yz + h(x,y)$$

$\Rightarrow \frac{\partial\phi}{\partial y} = x\cos y + 0 + \frac{\partial f(y,z)}{\partial y}$

$x\cos y - z = x\cos y + \frac{\partial f}{\partial y} \Rightarrow \frac{\partial f}{\partial y} = -z$, $f(y,z) = -zy$

similarly $g(x,z) = xz$

$h(x,y) = x\sin y$

These agree if we choose $f(y,z) = -yz, \quad g(x,z) = xz, \quad h(x,y) = x\sin y$

so that $\phi = x\sin y + xz - yz + C$

where C is any constant.

THEOREM (5.4): **Show that a necessary and sufficient condition that $A_1\,dx + A_2\,dy + A_3\,dz$ be an exact differential is that $\nabla \times \vec{A} = \vec{0}$ where $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$.**

PROOF: Let $A_1\,dx + A_2\,dy + A_3\,dz = d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

be an exact differential of a scalar function $\phi(x,y,z)$. Then on comparing coefficients, we have

$$A_1 = \frac{\partial\phi}{\partial x}, \quad A_2 = \frac{\partial\phi}{\partial y}, \quad A_3 = \frac{\partial\phi}{\partial z}$$

and so
$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \nabla\phi$$

Then $\nabla \times \vec{A} = \nabla \times \nabla \phi = \vec{0}$

Conversely, if $\nabla \times \vec{A} = \vec{0}$, then by the theorem (5.3) $\vec{A} = \nabla \phi$ and so

$$\vec{A} \cdot d\vec{r} = \nabla \phi \cdot d\vec{r} = d\phi$$

i.e. $A_1 dx + A_2 dy + A_3 dz = d\phi$, an exact differential of a scalar function ϕ.

EXAMPLE (7): **Show that $(y^2 z^3 \cos x - 4x^3 z)\,dx + 2z^3 y \sin x\,dy + (3y^2 z^2 \sin x - x^4)\,dz$ is an exact differential of a scalar function ϕ and find ϕ.**

SOLUTION: Here $\vec{A} = (y^2 z^3 \cos x - 4x^3 z)\hat{i} + 2z^3 y \sin x\,\hat{j} + (3y^2 z^2 \sin x - x^4)\hat{k}$

and
$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 z^3 \cos x - 4x^3 z & 2z^3 y \sin x & 3y^2 z^2 \sin x - x^4 \end{vmatrix} = \vec{0}$$

so by the above theorem (5.4), we have

$$(y^2 z^3 \cos x - 4x^3 z)\,dx + 2z^3 y \sin x\,dy + (3y^2 z^2 \sin x - x^4)\,dz = d\phi$$

or
$$(y^2 z^3 \cos x\,dx + 2z^3 y \sin x\,dy + 3y^2 z^2 \sin x\,dz) - (4x^3 z\,dx + x^4\,dz) = d\phi$$

$$d(y^2 z^3 \sin x) - d(x^4 z) = d\phi$$

or
$$\phi = y^2 z^3 \sin x - x^4 z + \text{constant}.$$

EXAMPLE (8): **If $\vec{A} = x\hat{i} + 2y\hat{j} + z\hat{k}$, show that the value of the line integral $\int_C \vec{A} \cdot d\vec{r}$ is 2 for any path joining (0, 0, 0) to (1, 1, 1).**

SOLUTION: Since $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, therefore $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

and so
$$\int_C \vec{A} \cdot d\vec{r} = \int_C x\,dx + 2y\,dy + z\,dz \qquad (1)$$

Note that $x\,dx + 2y\,dy + z\,dz$ can be expressed as a total (or exact) differential

i.e. $x\,dx + 2y\,dy + z\,dz = d\left(\dfrac{x^2}{2} + y^2 + \dfrac{z^2}{2}\right)$.

Thus equation (1) can be written as

$$\int_C \vec{A} \cdot d\vec{r} = \int_{(0,0,0)}^{(1,1,1)} d\left(\frac{x^2}{2} + y^2 + \frac{z^2}{2}\right) = \left|\frac{x^2}{2} + y^2 + \frac{z^2}{2}\right|_{(0,0,0)}^{(1,1,1)} = \frac{1}{2} + 1 + \frac{1}{2} = 2$$

which shows that the result depends only on the end points (0, 0, 0) and (1, 1, 1) and is independent of the path of integration.

EXAMPLE (9): If $\phi = xy$, evaluate $\int_C \phi\, d\vec{r}$ along the following paths C :

(i) the straight line $y = x$ from $(0,0)$ and $(1,1)$ and

(ii) the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

SOLUTION: The paths C joining $(0,0)$ and $(1,1)$ by the straight line and the parabola are in figure (5.12). Since $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$, therefore

$$\int_C \phi\, d\vec{r} = \int_C xy(dx\,\hat{i} + dy\,\hat{j})$$

$$= \hat{i}\int_C xy\,dx + \hat{j}\int_C xy\,dy \qquad (1)$$

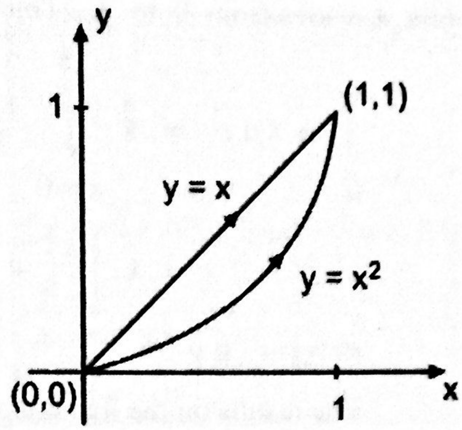

Figure (5.12)

Along the straight line $y = x$, $dy = dx$, while x varies from 0 to 1. Thus line integral (1) becomes

$$\int_C \phi\, d\vec{r} = \hat{i}\int_{x=0}^{1} x(x)\,dx + \hat{j}\int_{x=0}^{1} x(x)\,dx$$

$$= \hat{i}\int_0^1 x^2\,dx + \hat{j}\int_0^1 x^2\,dx = \hat{i}\left|\frac{x^3}{3}\right|_0^1 + \hat{j}\left|\frac{x^3}{3}\right|_0^1 = \frac{1}{3}\hat{i} + \frac{1}{3}\hat{j}$$

Along the parabola $y = x^2$, $dy = 2x\,dx$, while x varies from 0 to 1. Thus integral (1) becomes

$$\int_C \phi\, d\vec{r} = \hat{i}\int_{x=0}^{1} x(x^2)\,dx + \hat{j}\int_{x=0}^{1} x(x^2)\,2x\,dx$$

$$= \hat{i}\int_0^1 x^3\,dx + \hat{j}\int_0^1 2x^4\,dx = \hat{i}\left|\frac{x^4}{4}\right|_0^1 + \hat{j}\left|\frac{2}{5}x^5\right|_0^1 = \frac{1}{4}\hat{i} + \frac{2}{5}\hat{j}$$

EXAMPLE (10): If $\vec{A} = y\,\hat{i} + x\,\hat{j}$, evaluate $\int_C \vec{A} \times d\vec{r}$ along the curve C in the xy–plane, $y = \frac{x^3}{3}$ from $(0,0)$ to $(3,9)$.

SOLUTION: The given curve in the xy – plane is shown in figure (5.13)

$$\vec{A} \times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ y & x & 0 \\ dx & dy & 0 \end{vmatrix} = (y\,dy - x\,dx)\,\hat{k}$$

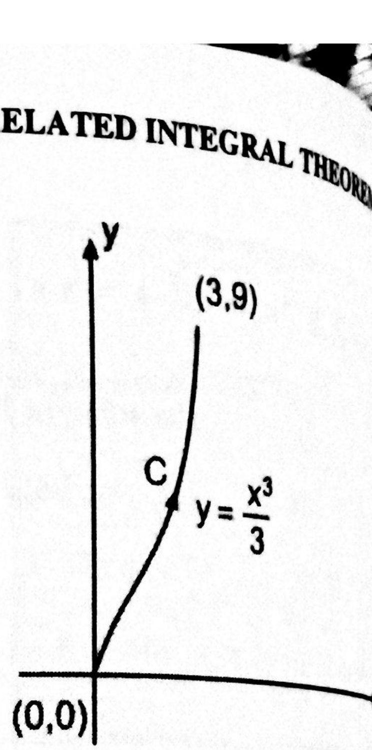

Figure (5.13)

Therefore $\int_C \vec{A} \times d\vec{r} = \hat{k} \int_C y\,dy - x\,dx$ (1)

Along the curve C, we have $y = \frac{x^3}{3}$, $dy = x^2\,dx$

while x varies from 0 to 3. Thus line integral (1) becomes

$$\int_C \vec{A} \times d\vec{r} = \hat{k} \int_{x=0}^{3} \frac{x^3}{3}(x^2)\,dx - x\,dx = \hat{k} \int_0^3 \left(\frac{1}{3}x^5 - x\right) dx$$

$$= \hat{k} \left| \frac{x^6}{18} - \frac{x^2}{2} \right|_0^3 = \hat{k} \left[\frac{81}{2} - \frac{9}{2} \right] = 36\,\hat{k}$$

SUMMARY

The results on the line integrals can be summarized as follows:

If the vector field $\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$ is conservative, then all of the following statements are equ[ivalent]
i.e. any one of them implies each of the others:

(i) $\int_{P_1}^{P_2} \vec{A} \cdot d\vec{r}$ is independent of the path joining any two points P_1 and P_2.

(ii) $\oint_C \vec{A} \cdot d\vec{r} = 0$ around all closed paths C passing through P_1 and P_2.

(iii) $\vec{A} = \nabla \phi$, where ϕ is a scalar point function.

(iv) $\nabla \times \vec{A} = \vec{0}$ identically.

(v) $\vec{A} \cdot d\vec{r} = A_1\,dx + A_2\,dy + A_3\,dz$ is an exact differential.