4.11 DIVERGENCE OF A VECTOR POINT FUNCTION

Let $\vec{A}(x,y,z)$ be a differentiable vector point function in a certain region of space. Then the divergence of $\vec{A}$ written $\nabla . \vec{A}$ or div $\vec{A}$ is defined by

$$\nabla . \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) . (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Note that $\nabla . \vec{A}$ is a scalar quantity. Also note that $\nabla . \vec{A} \neq \vec{A} . \nabla$.

If $\vec{A}$ is a constant vector, then $\nabla . \vec{A} = 0$. If $\nabla . \vec{A} = 0$ everywhere in some region R, then $\vec{A}$ is called a solenoidal vector point function in that region.

EXAMPLE (5): If $\vec{A} = x^2 z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$, find $\nabla . \vec{A}$ at the point $(1,-1,1)$.

SOLUTION:

$$\nabla . \vec{A} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2 z)$$

$$= 2xz - 6y^2z^2 + xy^2$$

Therefore, $(\nabla . \vec{A})_{(1,-1,1)} = -3$

4.12 PROPERTIES OF THE DIVERGENCE

THEOREM (4.5): If $\vec{A}$ and $\vec{B}$ are differentiable vector point functions, and ϕ is a differentiable scalar point function, then prove that

(i) $\nabla . (\vec{A} + \vec{B}) = \nabla . \vec{A} + \nabla . \vec{B}$

(ii) $\nabla . (\phi\vec{A}) = \phi(\nabla . \vec{A}) + \vec{A} . (\nabla\phi)$

PROOF:

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

(i) $\vec{A} + \vec{B} = (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$

Hence

$$\nabla . (\vec{A} + \vec{B}) = \frac{\partial}{\partial x}(A_1 + B_1) + \frac{\partial}{\partial y}(A_2 + B_2) + \frac{\partial}{\partial z}(A_3 + B_3)$$

$$= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)$$

$$= \nabla . \vec{A} + \nabla . \vec{B}$$

(ii) We have $\phi\vec{A} = \phi A_1\hat{i} + \phi A_2\hat{j} + \phi A_3\hat{k}$

Hence $\nabla.(\phi\vec{A}) = \frac{\partial}{\partial x}(\phi A_1)+\frac{\partial}{\partial y}(\phi A_2)+\frac{\partial}{\partial z}(\phi A_3)$

$$= \phi\frac{\partial A_1}{\partial x}+\frac{\partial \phi}{\partial x}A_1+\phi\frac{\partial A_2}{\partial y}+\frac{\partial \phi}{\partial y}A_2+\phi\frac{\partial A_3}{\partial z}+\frac{\partial \phi}{\partial z}A_3$$

$$= \phi\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)+\left(\frac{\partial \phi}{\partial x}A_1+\frac{\partial \phi}{\partial y}A_2+\frac{\partial \phi}{\partial z}A_3\right)$$

$$= \phi(\nabla.\vec{A})+\left(\frac{\partial \phi}{\partial x}\hat{i}+\frac{\partial \phi}{\partial y}\hat{j}+\frac{\partial \phi}{\partial z}\hat{k}\right).(A_1\hat{i}+A_2\hat{j}+A_3\hat{k})$$

$$= \phi(\nabla.\vec{A})+(\nabla\phi).\vec{A}$$

$$= \phi(\nabla.\vec{A})+\vec{A}.\nabla\phi$$

Note that if ϕ is constant, then $\nabla.(\phi\vec{A}) = \phi\,\nabla.\vec{A}$.

THEOREM (4.6): **Prove that**

(i) $\nabla.\vec{r} = 3$ (ii) $\nabla.[f(r)\vec{r}] = 3f(r)+rf'(r)$

(iii) $\nabla.(r^n\vec{r}) = (n+3)r^n$ (iv) $\nabla.\left(\frac{\vec{r}}{r^3}\right) = 0$

PROOF: Let $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, then

(i) $\nabla.\vec{r} = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z) = 1+1+1 = 3$.

(ii) Using theorem (4.5) part (ii), we have

$$\nabla.[f(r)\vec{r}] = f(r)(\nabla.\vec{r})+\vec{r}.[\nabla f(r)]$$

$$= 3f(r)+\vec{r}.\frac{f'(r)\vec{r}}{r} \quad \text{[using theorem (4.2)]}$$

$$= 3f(r)+\frac{f'(r)}{r}\vec{r}.\vec{r}$$

$$= 3f(r)+rf'(r) \quad (\text{since } \vec{r}.\vec{r} = r^2)$$

(iii) Setting $f(r) = r^n$ in part (ii), we have

$$\nabla.(r^n\vec{r}) = 3r^n+rnr^{n-1} = 3r^n+nr^n = (n+3)r^n$$

(iv) Let $n = -3$ in part (iii), then $\nabla.\left(\frac{\vec{r}}{r^3}\right) = (-3+3)r^{-3} = 0$

EXAMPLE (6): **Find the most general differentiable function $f(r)$ so that $f(r)\vec{r}$ is solenoidal**

SOLUTION: If $f(r)\vec{r}$ is solenoidal, then $\nabla.[f(r)\vec{r}] = 0$

$$[\nabla f(r)] \cdot \bar{r} + f(r)\, \nabla \cdot \bar{r} = 0 \qquad (1)$$

But $\nabla f(r) = \dfrac{f'(r)\,\bar{r}}{r}$, therefore equation (1) becomes

$$\frac{f'(r)\,\bar{r} \cdot \bar{r}}{r} + 3\, f(r) = 0 \quad \text{or} \quad r f'(r) + 3 f(r) = 0$$

or
$$f'(r) = -\frac{3 f(r)}{r}$$

or
$$\frac{f'(r)}{f(r)} = -\frac{3}{r}$$

Integrating we get $\ln f(r) = -3 \ln r + \ln C = \ln \dfrac{C}{r^3}$

or $f(r) = \dfrac{C}{r^3}$ where C is an arbitrary constant.

4.13 PHYSICAL INTERPRETATION OF THE DIVERGENCE

Consider the motion of an incompressible fluid (i.e. fluid with constant density say oil or water). Figure (4.8) shows an imaginary small rectangular parallelopiped of dimensions Δx, Δy, Δz with edges parallel to the coordinate axes and having centre at the point $P(x, y, z)$. Let $\bar{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$ be the velocity of the fluid at P.

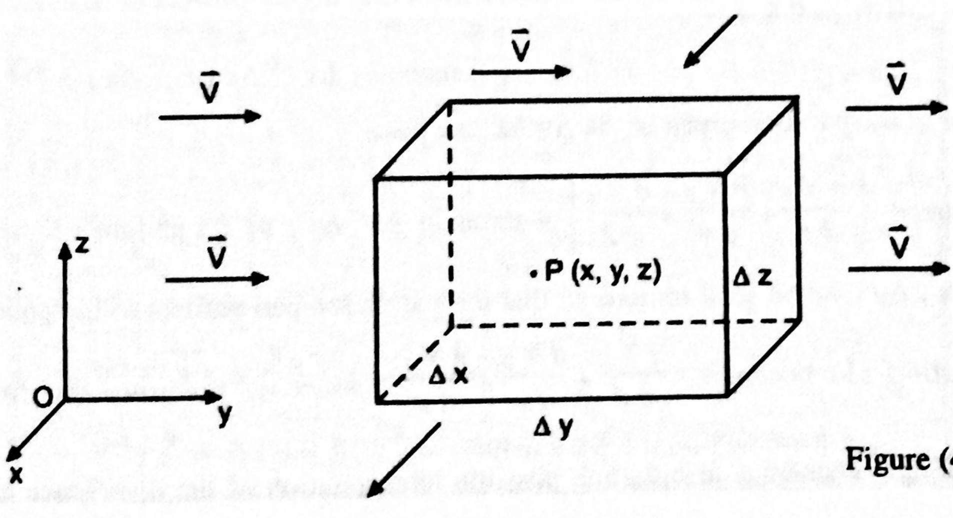

Figure (4.8)

We start by considering the flux (i.e. the amount of fluid crossing in unit time) through opposite faces of the parallelepiped. As the density is constant, either volume or mass can be used as a measure of the amount. For simplicity, volume will be considered here.

Consider first the flux across the faces which are perpendicular to the x–axis. These faces are of area $\Delta y\, \Delta z$ and the flux will only depend on V_1, this being the component of $\bar{V}$ in the x–direction. Thus flux into the parallelopiped through the back face $= V_1\left(x - \frac{1}{2}\Delta x, y, z\right) \Delta y\, \Delta z$

and the flux out of the parallelopiped through the front face $= V_1\left(x + \frac{1}{2}\Delta x, y, z\right) \Delta y\, \Delta z$.

Hence the net flux out of the parallelopiped through these two faces

$$= \left[V_1\left(x+\frac{1}{2}\Delta x , y , z\right) - V_1\left(x-\frac{1}{2}\Delta x , y , z\right) \right] \Delta y \, \Delta z$$

Using Taylor's series, we have

$$V_1\left(x+\frac{1}{2}\Delta x , y , z\right) = V_1(x , y , z) + \frac{1}{2}\Delta x \frac{\partial}{\partial x} V_1(x , y , z) + \ldots\ldots$$

and $$V_1\left(x-\frac{1}{2}\Delta x , y , z\right) = V_1(x , y , z) - \frac{1}{2}\Delta x \frac{\partial}{\partial x} V_1(x , y , z) + \ldots\ldots$$

Therefore, $$V_1\left(x+\frac{1}{2}\Delta x , y , z\right) - V_1\left(x-\frac{1}{2}\Delta x , y , z\right) = \Delta x \frac{\partial}{\partial x} V_1(x , y , z) + \ldots\ldots$$

Thus net flux in the x – direction $= \frac{\partial V_1}{\partial x}\Delta x \, \Delta y \, \Delta z + \ldots\ldots$ $\left[\text{by writing } \frac{\partial V_1}{\partial x} \text{ for } \frac{\partial}{\partial x} V_1(x , y , z) \right]$

Similarly, the net flux in the y – direction $= \frac{\partial V_2}{\partial y}\Delta x \, \Delta y \, \Delta z + \ldots\ldots$

and the net flux in the z – direction $= \frac{\partial V_3}{\partial z}\Delta x \, \Delta y \, \Delta z + \ldots\ldots$

Adding these three contributions, total flux out of the parallelopiped

$$= \left\{ \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \right\} \Delta x \, \Delta y \, \Delta z + \text{terms involving higher powers of } \Delta x , \Delta y , \Delta z$$

such as $(\Delta x)^2 \Delta y \, \Delta z$, $\Delta x (\Delta y)^3 \Delta z$ etc.

Now as the volume of the parallelopiped is $\Delta x \, \Delta y \, \Delta z$, we have

$$\text{Flux per unit volume} = \left\{ \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial Z} \right\} + \text{terms in } \Delta x , \Delta y , \text{ or } \Delta z \text{ and their powers.} \qquad (1)$$

Finally, we let Δx, Δy, Δz all tend to zero so that the parallelopiped shrinks to the point P. The right hand side of equation (1) becomes $\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$, which is precisely the divergence of the vector $\vec{V}$. Thus Fluid Mechanics affords one possible interpretation of the divergence as the amount of outward flux of the velocity field $\vec{V}$ per unit volume. Note that the divergence of $\vec{V}$ measures the expansion of the fluid at the point.

4.14 LAPLACIAN

If $\phi(x , y , z)$ is a scalar point function, then the divergence of the gradient of ϕ written as $\nabla . \nabla \phi = \nabla^2 \phi$ is called the Laplacian of ϕ, and the equation $\nabla^2 \phi = 0$ is called Laplace's equation. If a scalar function ϕ satisfies the Laplace's equation $\nabla^2 \phi = 0$ in a certain region R, then ϕ is said to be a **harmonic** function in that region.

THEOREM (4.7): If ϕ is a differentiable scalar point function, then show that

$$\nabla . \nabla \phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a Laplacian operator.

PROOF: We have

$$\nabla . \nabla \phi = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) . \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial z} \right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi = \nabla^2 \phi$$

EXAMPLE (7): Find $\nabla^2 \phi$ if $\phi = 2x^3 y^2 z^4$.

SOLUTION: We know that

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \qquad (1)$$

Now $\frac{\partial \phi}{\partial x} = 6x^2 y^2 z^4, \quad \frac{\partial \phi}{\partial y} = 4x^3 y z^4 \quad \frac{\partial \phi}{\partial z} = 8x^3 y^2 z^3$

and $\frac{\partial^2 \phi}{\partial x^2} = 12 x y^2 z^4, \quad \frac{\partial^2 \phi}{\partial y^2} = 4x^3 z^4, \quad \frac{\partial^2 \phi}{\partial z^2} = 24 x^3 y^2 z^2$

Thus from equation (1) $\nabla^2 \phi = 12 x y^2 z^4 + 4x^3 z^4 + 24 x^3 y^2 z^2$

THEOREM (4.8): Prove that

(i) $\nabla^2 f(r) = \frac{2}{r} f'(r) + f''(r)$

(ii) $\nabla^2 r^n = n(n+1) r^{n-2}$, where n is a real constant.

(iii) $\nabla^2 \left(\frac{1}{r} \right) = 0$

PROOF:

(i) $\nabla^2 f(r) = \nabla . \nabla f(r) = \nabla . \left[\frac{f'(r) \bar{r}}{r} \right]$

$$= \frac{f'(r)}{r} \nabla . \bar{r} + \bar{r} . \nabla \left[\frac{f'(r)}{r} \right] \quad \text{[Theorem (4.5) (ii)]}$$

$$= \frac{f'(r)}{r} (3) + \bar{r} . \frac{1}{r} \frac{d}{dr} \left[\frac{f'(r)}{r} \right] \bar{r} \quad \text{[Theorem (4.2)]}$$

$$= \frac{3}{r} f'(r) + \frac{1}{r} \left[\frac{r f''(r) - f'(r)}{r^2} \right] \bar{r} . \bar{r}$$

$$= \frac{3}{r} f'(r) + \frac{1}{r}\left[\frac{r f''(r) - f'(r)}{r^2}\right] r^2$$

$$= \frac{3}{r} f'(r) - \frac{1}{r} f'(r) + f''(r)$$

$$= \frac{2}{r} f'(r) + f''(r)$$

(ii) Setting $f(r) = r^n$ in part (i), we get

$$\nabla^2 r^n = \frac{2}{r} n r^{n-1} + n(n-1) r^{n-2}$$

$$= 2n r^{n-2} - n r^{n-2} + n^2 r^{n-2}$$

$$= n(n+1) r^{n-2}$$

(iii) Let $n = -1$ in part (ii), then

$$\nabla^2\left(\frac{1}{r}\right) = -1(-1+1) r^{-3} = 0$$

Any function ϕ which satisfies the Laplace's equation $\nabla^2 \phi = 0$ is called a solution of this equation

It follows that $\phi = \frac{1}{r}$ is a solution of Laplace's equation and is therefore **harmonic**.

EXAMPLE (8): **Show that**

(i) $\nabla^2(\ln r) = \frac{1}{r^2}$ **(ii)** $\nabla^2(e^r) = e^r\left(1 + \frac{2}{r}\right)$

(iii) $\nabla^4(e^r) = \nabla^2(\nabla^2 e^r) = e^r\left(1 + \frac{4}{r}\right)$

SOLUTION: We know that $\nabla^2 f(r) = \frac{2}{r} f'(r) + f''(r)$ (1)

(i) Let $f(r) = \ln r$, then $f'(r) = \frac{1}{r}$, $f''(r) = -\frac{1}{r^2}$ and from equation (1), we get

$$\nabla^2(\ln r) = \frac{2}{r}\left(\frac{1}{r}\right) - \frac{1}{r^2} = \frac{2}{r^2} - \frac{1}{r^2} = \frac{1}{r^2}$$

(ii) Let $f(r) = e^r$, then $f'(r) = e^r$, $f''(r) = e^r$ and from equation (1), we get

$$\nabla^2(e^r) = \frac{2}{r} e^r + e^r = e^r\left(1 + \frac{2}{r}\right) \quad (2)$$

(iii) $\nabla^4(e^r) = \nabla^2(\nabla^2 e^r) = \nabla^2\left[e^r\left(1 + \frac{2}{r}\right)\right]$ [using equation (2)]

Here $f(r) = e^r\left(1 + \frac{2}{r}\right)$

$$f'(r) = e^r\left(-\frac{2}{r^2}\right) + \left(1 + \frac{2}{r}\right) e^r = e^r\left(1 + \frac{2}{r} - \frac{2}{r^2}\right)$$

$$f''(r) = e^r\left(-\frac{2}{r^2} + \frac{4}{r^3}\right) + \left(1 + \frac{2}{r} - \frac{2}{r^2}\right) e^r = \left(1 + \frac{2}{r} - \frac{4}{r^2} + \frac{4}{r^3}\right) e^r$$

Thus from equation (1), we get

$$\nabla^4(e^r) = \nabla^2(\nabla^2 e^r) = \frac{2}{r}\left(1+\frac{2}{r}-\frac{2}{r^2}\right)e^r + \left(1+\frac{2}{r}-\frac{4}{r^2}+\frac{4}{r^3}\right)e^r$$

$$= \left(\frac{2}{r}+\frac{4}{r^2}-\frac{4}{r^3}+1+\frac{2}{r}-\frac{4}{r^2}+\frac{4}{r^3}\right)e^r$$

$$= e^r\left(1+\frac{4}{r}\right)$$

EXAMPLE (9): **Find the function $f(r)$ such that $\nabla^2 f(r) = 0$.**

SOLUTION: We have $\nabla^2 f(r) = 0$

or $f''(r)+\frac{2}{r}f'(r) = 0$ [using theorem (4.8)]

or $\frac{f''(r)}{f'(r)} = -\frac{2}{r}$

Integrating, we get $\ln f'(r) = -2\ln r + \ln C$

$$= \ln\frac{C}{r^2}$$

where C is the constant of integration.

Taking antilog, we have $f'(r) = \frac{C}{r^2}$

Integrating again, $f(r) = -\frac{C}{r} + D = A + \frac{B}{r}$

where $A = D$ and $B = -C$ are arbitrary constants.

4.15 CURL OF

Another equation in a and b ...

i.e. b = 1 (2)

Solving equations (1) and (2), we find that $a = \frac{5}{2}$, $b = 1$

PROBLEMS ON THE DIVERGENCE

PROBLEM (17): **Determine the constant a so that $\vec{V} = (x+3y)\hat{i} + (y-2x)\hat{j} + (x+az)$ is solenoidal .**

SOLUTION: A vector $\vec{V}$ is solenoidal if its divergence is zero . Now

$$\nabla.\vec{V} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2x) + \frac{\partial}{\partial z}(x+az) = 1+1+a$$

Then $\nabla.\vec{V} = a+2 = 0$ implies $a = -2$

PROBLEM (18): **If ϕ and ψ are scalar point functions , show that**

$$\nabla.(\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$$

SOLUTION: We know that

$$\nabla.(\phi\vec{A}) = \phi(\nabla.\vec{A}) + \vec{A}.\nabla\phi \quad (1)$$

Let $\vec{A} = \nabla\psi$ in equation (1) , we get

$$\nabla.(\phi\nabla\psi) = \phi(\nabla.\nabla\psi) + \nabla\psi.\nabla\phi = \phi\nabla^2\psi + \nabla\psi.\nabla\phi \quad (2)$$

Interchanging ϕ and ψ yields.

$$\nabla.(\psi\nabla\phi) = \psi\nabla^2\phi + \nabla\phi.\nabla\psi \quad (3)$$

Subtracting equation (3) from equation (2) , we get

$$\nabla.(\phi\nabla\psi) - \nabla.(\psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$$

or

$$\nabla.(\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$$

PROBLEM (19): **For a constant vector $\vec{A}$, show that $\nabla.[(\vec{A}.\vec{r})\vec{r}] = 4(\vec{A}.\vec{r})$**

SOLUTION:

$$\nabla.[(\vec{A}.\vec{r})\vec{r}] = (\vec{A}.\vec{r})(\nabla.\vec{r}) + \vec{r}.\nabla(\vec{A}.\vec{r})$$

$$= 3(\vec{A}.\vec{r}) + \vec{r}.\vec{A} \quad [\text{since } \nabla(\vec{A}.\vec{r}) = \vec{A}]$$

$$= 4(\vec{A}.\vec{r})$$

PROBLEM (20): If $\vec{A} = \vec{A}(x, y, z)$, show that $(d\vec{r} \cdot \nabla)\vec{A} = d\vec{A}$.

SOLUTION: Since $\vec{A} = \vec{A}(x, y, z)$, therefore

$$d\vec{A} = \frac{\partial \vec{A}}{\partial x}dx + \frac{\partial \vec{A}}{\partial y}dy + \frac{\partial \vec{A}}{\partial z}dz$$

$$= \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)\vec{A}$$

$$= \left[(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) \cdot \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\right]\vec{A}$$

$$= (d\vec{r} \cdot \nabla)\vec{A}$$

PROBLEM (21): If $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, show that $\nabla \cdot \vec{A} = \nabla A_1 \cdot \hat{i} + \nabla A_2 \cdot \hat{j} + \nabla A_3 \cdot \hat{k}$

SOLUTION: Since $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, therefore

$$\nabla \cdot \vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \qquad (1)$$

Now $\nabla A_1 \cdot \hat{i} = \left(\frac{\partial A_1}{\partial x}\hat{i} + \frac{\partial A_1}{\partial y}\hat{j} + \frac{\partial A_1}{\partial z}\hat{k}\right) \cdot \hat{i} = \frac{\partial A_1}{\partial x}$

Similarly $\nabla A_2 \cdot \hat{j} = \frac{\partial A_2}{\partial y}$ and $\nabla A_3 \cdot \hat{k} = \frac{\partial A_3}{\partial z}$

Thus equation (1) becomes

$$\nabla \cdot \vec{A} = \nabla A_1 \cdot \hat{i} + \nabla A_2 \cdot \hat{j} + \nabla A_3 \cdot \hat{k}$$

PROBLEM (22): Prove that $\nabla \cdot \left[\frac{f(r)\,\vec{r}}{r}\right] = \frac{2}{r}f(r) + f'(r)$.

SOLUTION:

$$\nabla \cdot \left[\frac{f(r)\,\vec{r}}{r}\right] = \nabla \cdot \left[\frac{f(r)}{r}\vec{r}\right]$$

$$= \frac{f(r)}{r}(\nabla \cdot \vec{r}) + \vec{r} \cdot \nabla\left[\frac{f(r)}{r}\right] \qquad [\text{ Theorem (4.5) (ii)}]$$

$$= \frac{f(r)}{r}(3) + \vec{r} \cdot \frac{1}{r}\frac{d}{dr}\left[\frac{f(r)}{r}\right]\vec{r} \qquad [\text{ Theorem (4.2)}]$$

$$= \frac{3}{r}f(r) + \vec{r} \cdot \frac{1}{r}\left[\frac{r f'(r) - f(r)}{r^2}\right]\vec{r}$$

$$= \frac{3}{r}f(r) + \frac{1}{r}\left[\frac{r f'(r) - f(r)}{r^2}\right](\vec{r} \cdot \vec{r})$$

$$= \frac{3}{r}f(r) + \frac{1}{r}\left[\frac{r f'(r) - f(r)}{r^2}\right]r^2$$

$$= \frac{3}{r} f(r) + f'(r) - \frac{1}{r} f(r)$$

$$= \frac{2}{r} f(r) + f'(r)$$

PROBLEM (23): **Show that**

(i) $\nabla . \left[r \nabla \left(\frac{1}{r^3} \right) \right] = \frac{3}{r^4}$ **(ii)** $\nabla . \left[\frac{1}{r} \nabla \left(\frac{1}{r} \right) \right] = \frac{1}{r^4}$

(iii) $\nabla \left[\nabla . \left(\frac{\vec{r}}{r} \right) \right] = -\frac{2 \vec{r}}{r^3}$

SOLUTION: We have

(i) $\nabla . \left[r \nabla \left(\frac{1}{r^3} \right) \right] = \nabla . [r (-3 r^{-5} \vec{r})]$ [using theorem (4.2)]

$= -3 \nabla . (r^{-4} \vec{r})$

$= -3 (-4 + 3) r^{-4} = \frac{3}{r^4}$ [using theorem (4.6)]

(ii) $\nabla . \left[\frac{1}{r} \nabla \left(\frac{1}{r} \right) \right] = \nabla . \left[\frac{1}{r} \left(-\frac{\vec{r}}{r^3} \right) \right]$ [using theorem (4.2)]

$= -\nabla . (r^{-4} \vec{r})$

$= -(-4 + 3) r^{-4} = \frac{1}{r^4}$ [using theorem (4.6)]

(iii) Using theorem (4.6) , we have $\nabla . \left(\frac{\vec{r}}{r} \right) = \frac{2}{r}$, therefore

$\nabla \left[\nabla . \left(\frac{\vec{r}}{r} \right) \right] = \nabla \left(\frac{2}{r} \right)$

$= -2 r^{-3} \vec{r} = -\frac{2 \vec{r}}{r^3}$ [using theorem (4.2)]