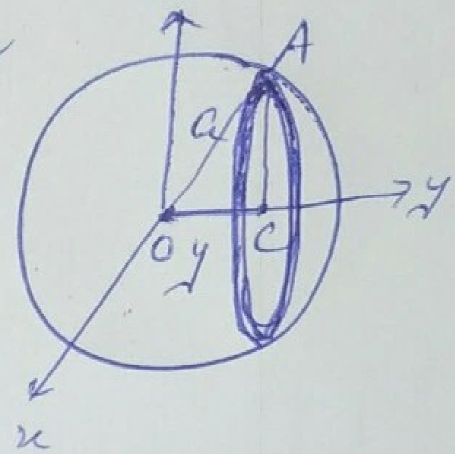
Moment of inertia for a sphere about it axis



The moment of inertia for a sphere can also be found by dividing the sphere into small discs of width Δy (if axis is along y-axis).

Let ith disc is at a distance y_i from the centre (origin) of the sphere of radius a. using pythagorus Theorem we can write

$$a^2 = y^2 + (AC)^2$$

$$(AC)^2 = a^2 - y^2 \Rightarrow (AC) = \sqrt{a^2 - y_i^2}$$

radius of the di

So area of the ith disc $= \pi (AC)^2$

$$= \pi (a^2 - y_i^2)$$

its volume $= \pi (a^2 - y_i^2)\, \Delta y_i$

Hence $\Delta m_i = \rho \Delta V_i = \rho \pi (a^2 - y_i^2) \Delta y_i$

For m.o.i using

$$I = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta m_i\, y_i^2$$

$$= \int_{-a}^{a} y^2 \rho \pi (a^2 - y^2)\, dy$$

$$= \pi\rho\int_{-a}^{a}(a^2y^2 - y^4)\,dy$$

$$= \pi\rho\left|a^2\frac{y^3}{3} - \frac{y^5}{5}\right|_{-a}^{a}$$

$$= \pi\rho\left\{\left(\frac{a^5}{3}-\frac{a^5}{5}\right) - \left(\frac{-a^5}{3}+\frac{a^5}{5}\right)\right\}$$

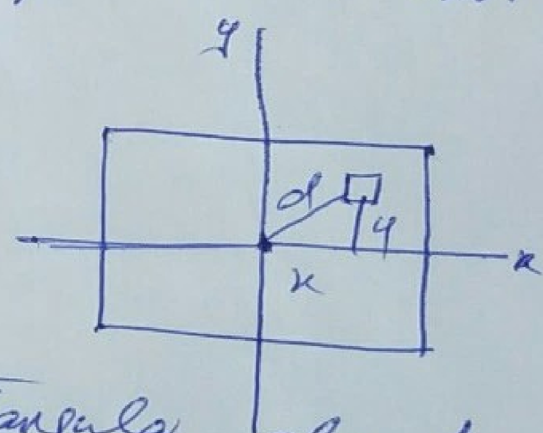
$$= \rho\pi\left(\frac{2a^5}{3}-\frac{2a^5}{5}\right) = \rho\pi\left(\frac{4a^5}{15}\right)$$

Now using $\rho = \frac{M}{\frac{4}{3}\pi a^3} = \frac{3M}{4\pi a^3}$ we get

$$I = \frac{3M}{4\pi a^3}\times\pi\times\frac{4a^5}{15} = \frac{1}{5}Ma^2$$ is

moment of inertia of the sphere

To find moment of inertia of a rectangular plate about an axis through its centre and perpendicular to the plate without using perpendicular axis theorem.



Let $2a$ and $2b$ be the length and width of the plate. Let us consider a rectangular element of the plate of dimension $\Delta x\times\Delta y$

Then $\Delta m_i = \rho\,\Delta A_i$

$\Delta m_i = \rho\,\Delta x_i\,\Delta y_i$

If d_i is $\perp$ distance of the element from perp. axis (z-axis), then

$$d_i^2 = x_i^2 + y_i^2$$

Hence $$I_{zz} = \sum m_i d_i^2$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} (x_i^2 + y_i^2) \Delta m_i$$

$$= \int_{-b}^{b}\int_{-a}^{a} (x^2 + y^2)\, \rho \, dx\, dy$$

$$= \rho \int_{-b}^{b}\int_{-a}^{a} (x^2 + y^2)\, dx\, dy$$

$$= \rho \int_{-b}^{b} \left(\frac{x^3}{3} + xy^2\right)\Big|_{-a}^{a} dy$$

$$= \rho \int_{-b}^{b} \left(\frac{2a^3}{3} + 2ay^2\right) dy$$

$$= \rho \left| \frac{2a^3}{3} y + \frac{2a}{3} y^3 \right|_{-b}^{b}$$

$$= \rho \left[\frac{2a^3}{3}(2b) + \frac{2a}{3}(2b^3) \right]$$

$$= \rho \left[\frac{4a^3 b}{3} + \frac{4ab^3}{3} \right]$$

$$= \rho \frac{(4ab)}{3} (a^2 + b^2)$$

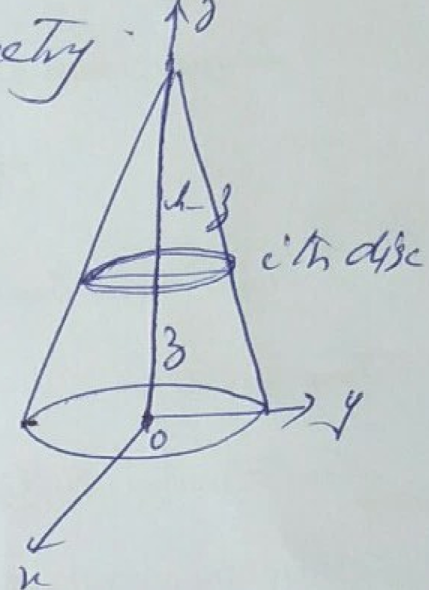
$$= \frac{M}{(2a)(2b)} = \frac{4ab}{3}(a^2 + b^2)$$

$$= \frac{1}{3} M (a^2 + b^2)$$

is m.o.i about $\perp$ axis

To find find m.o.i about x-axis or y-axis, This technique can not be used. See page-27 (notes)

Moment of inertia of a right circular cone about its axis of ~~system~~ symmetry.



Let h be height and a radius of the base of the cone. Let axis of symmetry is along z axis.

A cone can be considered as an infinite collection of small discs of varying radius r. Since m.o.i of a disc about a $\perp$ axis is

$$I = \frac{1}{2} Ma^2 \text{ (See previous pages)}$$

So for collection of varying discs, we can write

$$\delta I = \frac{1}{2} \delta m\, r^2 \quad \text{ie} \quad \Delta I = \frac{1}{2} \Delta m\, r^2.$$

If we take an ith disc at a distance z from origin O. From similar Δs

$$\frac{r}{a} = \frac{h-z}{h} \Rightarrow r = \left(\frac{h-z}{h}\right)a$$

and $\Delta m = \rho \Delta V = \rho\pi\left(\left(\frac{h-z}{h}\right)a\right)^2 \Delta z$.

Hence from $\Delta I = \frac{1}{2} \Delta m\, r^2$

$$\Rightarrow I = \frac{1}{2}\int_0^h \rho\pi\left(\frac{h-z}{h}\right)^2 a^2 dz \left(\frac{h-z}{h}\right)^2 a^2$$

$$= \frac{1}{2}\frac{\rho\pi a^4}{h^4}\int_0^h (h-z)^4\,dz$$

$$= \frac{\rho\pi a^4}{2h^4}\left|-\frac{(h-z)^5}{5}\right|_0^h$$

$$= \frac{\rho\pi a^4}{2h^4}\left[0+\frac{h^5}{5}\right] = \frac{\rho\pi a^4 h^5}{10h^4}$$

$$= \frac{M}{\frac{1}{3}\pi a^2 h}\times\frac{\pi a^4 h}{10} = \frac{3}{10}Ma^2$$

Also find moment of inertia about x-axis & y-axis passing through its base.

Ass

i) Solve any other five problem for finding moment of inertia

ii) Give the progress of mathematics (specially ~~in~~ mechanics) in the 19th century

i) Write a brief note on the work of Archemedes / Newton.

ii) Give the progress on Mathematics (Mechanics in 20th century.