$e^{|\Omega_2|t}$ which gets larger and larger with the passage of time and therefore the motion is unstable.

The above analysis is applicable to any rigid body which is rotating about any one of the three principal axes (with distinct moments of inertia) at a point fixed in it. Therefore we conclude that the rotation around the principal axis corresponding to either the largest or the smallest M.I. is stable and that around the principal axis with intermediate M.I. is unstable.

If two of the M.I. say, I_1 and I_2 are equal, then from (9.6.5) we find that $\dot{\mu} = 0$) and therefore $\mu = \mu_0$, a constant. The equation (9.6.4) for λ now becomes

$$\frac{d\lambda}{dt} = \mu_1 \mu_0 \qquad (9.6.12)$$

where $\mu_1 = (I_3 - I_1)/I_1$. The solution of (9.6.12) is $\lambda = \mu_1 \mu_0 t + \mu_3$.

This shows that the perturbation will increase linearly with time and the rotation around the OX- axis is unstable. We can obtain a similar result for rotation about the OY-axis. It can be further shown that the motion will be stable only when the rigid body is rotating about the OZ-axis irrespective of whether I_3 is greater than or less than $I_1 = I_2$.

9.7 Euler's Angles and Rigid Body Motion

A rigid body constrained to rotate about a fixed point has only three degrees of freedom. Therefore we require three parameters to specify the configuration of such a body. Euler's angle are three angular coordinates which are used to specify the configuration (orientation) of a rigid body. The Euler angles are usually denoted by θ, ϕ, ψ. Note that there is no universally agreed notation, neither is there agreed convention about their signs.

Let the fixed point about which the body is rotating be O. To define the Euler angles we consider a coordinate system (or a frame of reference) $OX_0Y_0Z_0$ fixed in space, and another coordinate system $OXYZ$ fixed in the body and rotating with it. The first coordinate system is usually referred to as *space* or *fixed* or *inertial* coordinate system, whereas the second coordinate system is referred to as *body* or *moving* or *rotating* coordinate system. We suppose that the two coordinate systems are initially (i.e. at $t = 0$) coincident and define the the Eulerian angles θ, ϕ, ψ in relation to the orientation of the axes of the rotating coordinate system, as follows.

θ = angle between the axes OZ_0 and OZ. It varies from 0 to π.

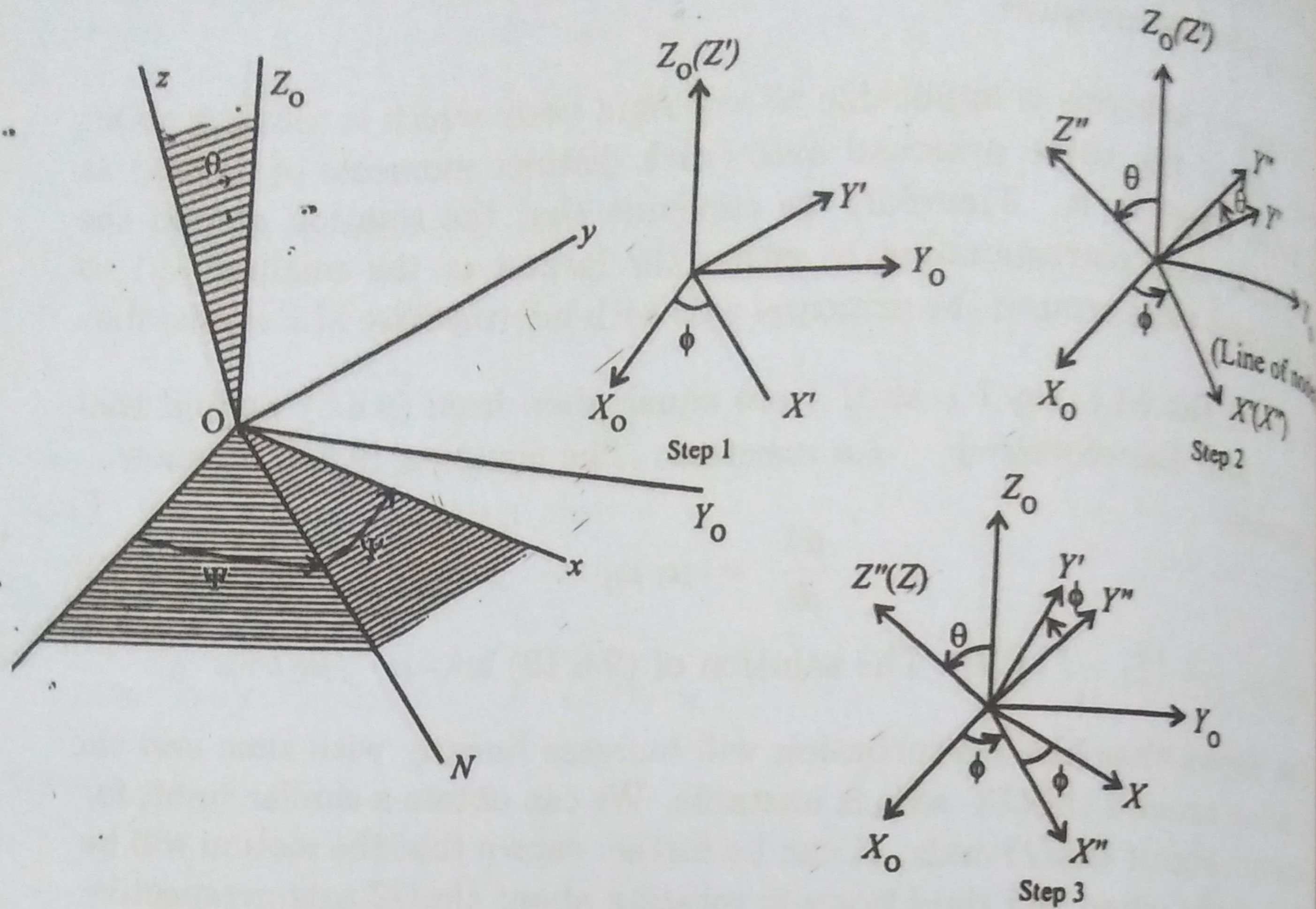

Figure 9.5: Steps in the determination of the Euler angles.

ϕ = angle between the fixed axis OX_0 and the line ON. The line ON is the line of intersection of the planes OX_0Y_0 and OXY, and is called the *line of nodes*. The angle ϕ can also be regarded as the angle between the planes OZ_0Z and OX_0Z_0. It varies from 0 to 2π.

ψ = angle between the body axis OX and the line of nodes ON. It varies from 0 to 2π.

As the body rotates the Euler angles θ, ϕ, ψ vary with time and their derivatives $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$ represent angular speeds about certain axes.

Next we discuss the transformation from the space coordinate system $OX_0Y_0Z_0$ to the body coordinate system $OXYZ$, and find the corresponding rotation matrix. In order to obtain the desired rotation matrix, we introduce two other coordinate systems $OX'Y'Z'$ and $OX''Y''Z''$ and perform the following sequence of rotations:

(1) $OX_0Y_0Z_0 \rightarrow OX'Y'Z'$, (2) $OX'Y'Z' \rightarrow OX''Y''Z''$
(3) $OX''Y''Z'' \rightarrow OXYZ$

1. The first rotation which we perform, through an angle ϕ, is in the counterclockwise direction, in the X_0Y_0-plane (i.e. XY- plane of the fixed coordinate system), about the axis OZ_0. This rotation can be represented by

the matrix

$$R_\phi = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (9.7.1)$$

The angle ϕ is called *precession angle*. After applying this transformation, the new coordinate system is denoted by $OX'Y'Z'$, and the relation between the coordinates is given by

$$\mathbf{X}' = R_\phi \mathbf{X}_0 \quad (9.7.2)$$

where $\mathbf{X}_0$ denotes the column vector of coordinates i.e. $[x_0,\ y_0,\ z_0]^t$. The column vector $\mathbf{X}'$ has a similar definition.

2. The second rotation takes place in the $Y'Z'$-plane, in the counterclockwise direction about the OX'-axis through an angle θ. The rotation matrix in this case is given by

$$R_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad (9.7.3)$$

The angle θ is called *nutation* angle. The new coordinate system is now denoted by $OX''Y''Z''$, and the coordinates are related by

$$\mathbf{X}'' = R_\theta \mathbf{X}' \quad (9.7.4)$$

3. The third rotation takes place in the $OX''Y''$-plane in the counterclockwise direction through an angle ψ about the OZ''-axis. This transformation brings us to the body coordinate system $OXYZ$. The rotation matrix in this case is given by

$$R_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (9.7.5)$$

and the corresponding coordinate vectors are related by

$$\mathbf{X} = R_\psi \mathbf{X}'' \quad (9.7.6)$$

The angle ψ is called the *body angle*. The transformation from the fixed coordinate system $OX_0Y_0Z_0$ to the body coordinate system $OXYZ$ (see figure 9.5) is given by the rotation matrix $R = R_\psi R_\theta R_\phi$, which when written in full becomes

$$R = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

The elements of the product matrix $R = (r_{ij})$ are given by

$$r_{11} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \end{bmatrix} \begin{bmatrix} \cos\phi \\ -\cos\theta\sin\phi \\ \sin\theta\sin\phi \end{bmatrix} = \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi$$

$$r_{12} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \end{bmatrix} \begin{bmatrix} \sin\phi \\ -\cos\theta\cos\phi \\ -\sin\theta\cos\phi \end{bmatrix} = \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi$$

$$r_{13} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \sin\theta \\ \cos\theta \end{bmatrix} = \sin\psi\sin\theta.$$

$$r_{21} = \begin{bmatrix} -\sin\psi & \cos\psi & 0 \end{bmatrix} \begin{bmatrix} \cos\phi \\ -\cos\theta\sin\phi \\ \sin\theta\sin\phi \end{bmatrix} = -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi.$$

$$r_{22} = \begin{bmatrix} -\sin\psi & \cos\psi & 0 \end{bmatrix} \begin{bmatrix} \sin\phi \\ \cos\theta\cos\phi \\ -\sin\theta\sin\phi \end{bmatrix} = -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi.$$

$$r_{23} = \begin{bmatrix} -\sin\psi & \cos\psi & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \sin\theta \\ \cos\theta \end{bmatrix} = \cos\psi\ \sin\theta.$$

$$r_{31} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi \\ -\cos\theta\sin\phi \\ \sin\theta\sin\phi \end{bmatrix} = \sin\theta\sin\phi.$$

$$r_{32} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\phi \\ \cos\theta\cos\phi \\ -\sin\theta\cos\phi \end{bmatrix} = -\sin\theta\cos\phi$$

$$r_{33} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \sin\theta \\ \cos\theta \end{bmatrix} = \cos\theta$$

An infinitesimal rotation can be represented as a vector. (Recall that such a rotation about an axis specified by the unit vector **e** is given by $\delta\vec{\theta} = \delta\theta\mathbf{e}$). When the body is rotating about an instantaneous axis through the fixed point O, its angular velocity can be expressed in terms of the time derivatives $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$ of the Euler angles. We note that

(i) $\dot{\phi}$ is along the axis OZ_0.

(ii) $\dot{\theta}$ is along the line of nodes, which is the line of intersection of the planes OX_0Y_0 and OXY.

(iii) $\dot{\psi}$ is along the axis OZ.

It is not convenient to use these components of angular velocity $\vec{\omega}$ of the rigid body. Instead we use the body coordinate system $OXYZ$ and express the angular velocity components $(\omega_1, \omega_2, \omega_3)$ w.r.t. this system in terms of $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$. For this purpose we consider these components along the three body axes. Remembering that the general infinitesimal rotation associated with ω can be regarded as consisting of three successive infinitesimal rotations with angular velocities $\vec{\omega}_\phi$, $\vec{\omega}_\theta$ and $\vec{\omega}_\psi$ with their magnitudes equal to $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ respectively. Therefore the vector $\vec{\omega}$ can be expressed as $\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi$

We note that $\vec{\omega}_\phi$ is along the OZ_0-axis, $\vec{\omega}_\theta$ along OX'-axis (or along ON, the line of nodes) and $\vec{\omega}_\psi$ along OZ-axis. We will now use the orthogonal transformation given in (9.7.2), (9.7.4), and (9.7.6) to obtain the components of $\vec{\omega}$ along the set of axes we desire.

The body system of axes is the most useful for discussing the equations of motion. Therefore we will obtain components of $\vec{\omega}$ in this system.

Now since $\vec{\omega}_\phi = (0, 0, \dot{\phi}) = [0, 0, \dot{\phi}]^t$ in $OX_0Y_0Z_0$ coordinate system, and the vectors in space and body coordinate systems are connected by $X = RX_0 = R_\psi R_\theta R_\phi \mathbf{X}_0$

$$\vec{\omega}_\phi(b) = R_\psi R_\theta R_\phi \vec{\omega}_\theta = R_\psi R_\theta \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$= R_\psi R_\theta \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} = R_\psi \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\phi}\sin\theta \\ \dot{\phi}\cos\theta \end{bmatrix} = \begin{bmatrix} \dot{\phi}\sin\theta\sin\psi \\ \dot{\phi}\sin\theta\cos\psi \\ \dot{\phi}\cos\theta \end{bmatrix}$$

where $\vec{\omega}_\theta(b)$ denotes the contribution to angular velocity in the body system due to rotation through angle ϕ about OZ_0-axis.

The rotation through angle θ is about the axis OX' and therefore the corresponding angular velocity vector $\vec{\omega}_\theta$ is directed along OX' axis. The vector $\vec{\omega}_\theta$ therefore is represented by the column vector $[\dot{\phi},0,0]^t$ in the coordinate system $OX'Y'Z'$. Its transform in the body coordinate system is $\vec{\omega}_\theta(b)$ and is related by

$$\vec{\omega}_\theta(b) = R_\psi R_\theta \vec{\omega}_\theta = R_\psi \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{bmatrix}$$

The rotation about OZ'' axis through angle ψ is the same as rotation about OZ through the same angle. Hence $\vec{\omega} = \dot{\psi}\mathbf{k}'' = \dot{\psi}\mathbf{k}$. Now

$$\vec{\omega} = \vec{\omega}_\theta(b) + \vec{\omega}_\theta(b) + \vec{\omega}_\psi(b)$$

or

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \dot{\phi}\sin\theta\sin\psi \\ \dot{\phi}\sin\theta\cos\psi \\ \dot{\phi}\cos\theta \end{bmatrix} + \begin{bmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

which gives

$$\omega_1 = \dot{\phi}\sin\theta\sin\phi + \dot{\phi}\cos\psi \qquad (9.7.7a.)$$

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \qquad (9.7.7b.)$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} \qquad (9.7.7c.)$$

These equations are called *Euler's geometrical equations*. They describe rigid body motion relative to the body coordinate system.

Note on Notation

The notation used by the British authors is different from the one used here. We have adopted the notation used by American authors, (Goldstein in particular). Our ϕ and ψ are equal to $\phi+\pi/2$ and $\pi/2-\psi$ in the British notation. Some other authors use ϕ instead of ψ and vice versa.